Calculation of the sun's acoustic impulse response by multi-dimensional spectral factorization

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Abstract. Calculation of time-distance curves in helioseismology can be formulated as a blind-deconvolution (or system identification) problem. A classical solution in one-dimensional space is Kolmogorov's Fourier domain spectral-factorization method. The helical coordinate system maps two-dimensions to one. Likewise a three-dimensional volume is representable as a concatenation of many one-dimensional signals. Thus concatenating a cube of helioseismic data into a very long 1-D signal and applying Kolmogorov's factorization, we find we can construct the three-dimensional causal impulse response of the sun by deconcatenating the Kolmogorov result.

Time-distance curves calculated in this way have the same spatial and temporal bandwidth as the original data, rather than the decreased bandwidth obtained obtained by cross-correlating traces. Additionally, the spectral factorization impulse response is minimum phase, as opposed to the zero phase time-distance curves produced by cross-correlation.

Keywords: Kolmogorov spectral factorization, impulse response, cross-correlation

1. Introduction

Time-distance helioseismology is based upon cross-correlating oscillatory dopplergram traces from different locations on the surface of the sun (Duvall et al., 1993). This allows helioseismologists to study acoustic waves traveling between the trace locations, facilitating a family of techniques that are proving very successful for studying a range of solar phenomena at a large range of scales. For example, time-distance measurements can be used to estimate both near surface flow velocities associated with super-granulation (e.g. Kosovichev and Duvall, 1997), and meridional circulation deep within the convective zone (Giles et al., 1997).

The process of picking traveltimes from time-distance curves is a critical element of these studies. Both signal-to-noise levels and signal bandwidth can limit the resolution of traveltime picks. Unfortunately, however, the cross-correlation process reduces the spatial and temporal bandwidth of the data, by essentially squaring the (ω, k_x, k_y) amplitude spectrum.

One-dimensional spectral factorization algorithms are commonly used in signal processing applications. We extend the concept of spectral factorization to three dimensions, and produce a three-dimensional minimum phase time-distance impulse response with the same spectra as the original data.

2. Spectral factorization

Under an assumption of translational invariance, we can model the acoustic oscillation recorded in solar dopplergrams as a source function convolved with an impulse response. After a three-dimensional Fourier transform, the convolution becomes a simple multiplication:

$$D(k_x, k_y, \omega) = S(k_x, k_y, \omega)G(k_x, k_y, \omega), \tag{1}$$

where D is the observed data, S is the source function, and G is the impulse response. We are interested in the three-dimensional time-space acoustic impulse response, $g(x, y, t)$. As it stands, however, equation (1) has many more unknowns than knowns, so additional assumptions are required before we can estimate G.

Secondly, we assume $s(x, y, t)$ is white in space and time, or equivalently, $S\bar{S} = 1$, where $\bar{S}$ denotes the complex conjugate of S. If this is not true in practice, spectral color from the source function will leak into the derived impulse response. Under this assumption, equation (1) reduces to the statement that the power spectrum of the impulse response equals the power spectrum of the data,

$$|D|^2 = |G|^2 . \tag{2}$$

While defining the amplitude spectrum of G, this equation places no constraints on its phase, and so we need an additional assumption to ensure a unique solution. Without justification, we will assume G is a minimum phase function, where a minimum phase function is defined a causal function with a causal convolutional inverse.

If this model holds true, then estimating the impulse response reduces to estimating a minimum-phase function with the same (w, k_x, k_y) spectrum as the original data: or equivalently, multi-dimensional spectral factorization.

2.1. Kolmogorov review

Kolmogorov (1939) spectral factorization provides a highly efficient Fourier method for calculating a minimum phase time domain function with a given power spectrum.

Following Claerbout (1992), we will describe the method briefly with Z transform notation. In this notation, $Z = e^{i\omega\Delta t}$ is the unit delay operator, and functions can be evaluated either in the frequency domain as functions of ω, or in the time domain as the coefficients of the polynomial in Z. Causal functions can, therefore, be written as polynomials with non-negative powers of Z, whereas anti-causal functions contain non-positive powers of Z.

The spectral factorization problem can be summarized as given a power spectrum, $S(Z)$, we must find a minimum phase function such that

$$\bar{B}(1/Z)B(Z) = S(Z). \tag{3}$$

Since $S(Z)$ is a power spectrum, it is non-negative by definition for all ω; however, the Kolmogorov process has the additional requirement that it contains no zeros. If this is the case, then we can safely take its logarithm,

$$U(\omega) = \ln\left[S(\omega)\right]. \tag{4}$$

Since $U(\omega)$ is real and even, its time domain representation is also real and even. We can therefore isolate its causal part, $C(Z)$, and its anti-causal part, $\bar{C}(1/Z)$:

$$U(Z) = \bar{C}(1/Z) + C(Z). \tag{5}$$

Once we have $C(Z)$, we can easily obtain $B(Z)$ through

$$B(\omega) = e^{C(\omega)}. \tag{6}$$

To verify that $B(Z)$ of this form does indeed satisfy equation (3), consider

$$\begin{aligned} \bar{B}(1/Z)B(Z) &= e^{\bar{C}(1/Z)}e^{C(Z)} && (7)\\ &= e^{\bar{C}(1/Z)+C(Z)} && (8)\\ &= e^{U(Z)} && (9)\\ &= S(Z). && (10) \end{aligned}$$

$B(Z)$ will be causal since $C(Z)$ was causal, and a power series expansion proves that the exponential of a causal function is also causal. It is also clear that $1/B(Z) = e^{-C(Z)}$ will also be causal in the time domain. Therefore, $B(Z)$ will be causal, and will have a causal inverse. Hence $B(Z)$ satisfies the definition of minimum phase given above.

2.2. Multi-dimensional factorization

Kolmogorov spectral factorization, as described above, is a purely one-dimensional theory. The real contribution of this paper is to link the one-dimensional theory to the three-dimensional world. We do this by applying helical boundary conditions (Claerbout, 1998) to map a three-dimensional stochastic dopplergram into an equivalent one-dimensional dataset, and factorizing the entire cube with Kolmogorov.

The concept of helical boundary conditions is demonstrated in Figure 1, which shows the mapping of small five-point two-dimensional filter into one dimension. For the spectral factorization application, however, rather than map a two-dimensional function, we map the entire three-dimensional MDI dataset into one dimension, and apply Kolmogorov spectral factorization on the entire super-trace.

Therefore, we perform the spectral factorization in three steps. Firstly, we transform the cube of data to an equivalent one-dimensional super-trace via helical boundary conditions. Secondly, we perform one-dimensional spectral

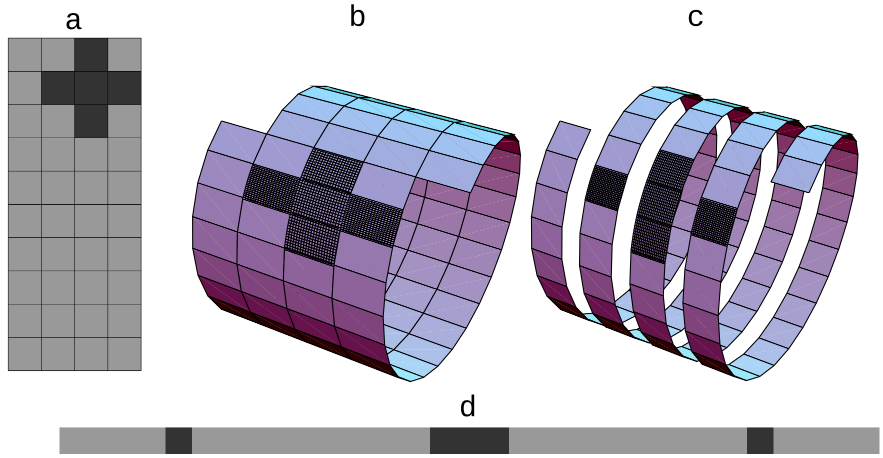

Figure 1. Illustration of helical boundary conditions mapping a two-dimensional function (a) onto a helix (b), and then unwrapping the helix (c) into an equivalent one-dimensional function (d). (Figure by Sergey Fomel).

factorization with Kolmogorov's frequency domain method. Finally, we remap the impulse response back to three-dimensional space.

The spatial axes need to be padded to reduce wrap-around effects. This spatial wrap-around is not an artifact of the Fourier transform, but rather it is an artifact of the helical boundary conditions. In this respect, there would be little advantage to choosing a time-domain spectral factorization algorithm (e.g. Wilson, 1969) over Kolmogorov.

2.3. Theoretical comparison between time-distance functions

The Kolmogorov impulse response is essentially a large impulse at zero lag (in time and space) with a small amplitude signal corresponding to the diving waves. Both components are band-limited, so we can write

$$B(Z) = W(Z)\,[1 + \epsilon F(Z)]\,, \tag{11}$$

where $F(Z)$ is the causal function of interest, ϵ is simply a scalar indicating the small amplitude of that term, and $W(Z)$ is a minimum phase band-limited seismic wavelet.

The cross-correlation process produces the auto-correlation of equation (11):

$$\bar{B}B = \bar{W}W\left[1 + \epsilon F + \epsilon \bar{F} + \epsilon^2 \bar{F}F\right]. \tag{12}$$

This function contains ϵF, the function we are interested in studying; however there are two major differences.

Firstly, $\bar{B}B$ also contains the additional terms $\epsilon\bar{F}$ and $\epsilon^2\bar{F}F$. We can discard the first of these terms, $\epsilon\bar{F}$ since it is anti-causal, and the second term contains ϵ^2 so will be much smaller than the signal of interest.

The second difference between equations (11) and (12) is the wavelet. The Kolmogorov wavelet is minimum phase, whereas the cross-correlation wavelet $\bar{W}W$ is zero-phase. The amplitude spectrum of the cross-correlation wavelet will also be the square of the Kolmogorov wavelet.

Thus the principle advantage of the Kolmogorov result is that it has a broader bandwidth than the cross-correlation. Whereas the Kolmogorov result has the same amplitude spectrum as the original data, the amplitude spectrum of the cross-correlation impulse response is equal to the *power* spectrum of the original data.

2.4. On the assumption of translational invariance

The justification for spectral factorization rests upon an assumption of translational invariance. This assumption runs counter to many applications of time-distance helioseismology, where the interest comes in three-dimensional structure. The assumption of translation invariance may be partly overcome by working with patches of data with small spatial extent.

Time-distance measurements by cross-correlation may not seem to have this perceived disadvantage; however, studies (e.g. Kosovichev and Duvall, 1997) have shown that significant amounts of averaging are required to produce signal-to-noise levels high enough to make reliable measurements. There is an implied assumption of invariance in this averaging procedure.

3. Application to SOHO/MDI dataset

Figure 2 shows a time slice through a cube of raw velocity data from the MDI instrument. The data has been transformed to Cartesian coordinates by projecting high-resolution data from an area approximately 18° square onto a tangent plane. The object in the center of the time-slice (top of cube) is a sunspot. The sampling spacing is 1 minute on the time-axis and approximately 825 km on the two spatial axes.

Time-variable features of Figure 2 fall into two distinct spectral windows. The low temporal frequency events (<1.5 mHz) are related to solar convection, while the higher frequency events are related to acoustic wave propagation. We were interested in studying acoustic wave phenomena; so as a preprocessing step, we removed the lower frequency spectral window by applying a $\frac{1-Z}{1-\rho Z}$ low-cut filter to the data.

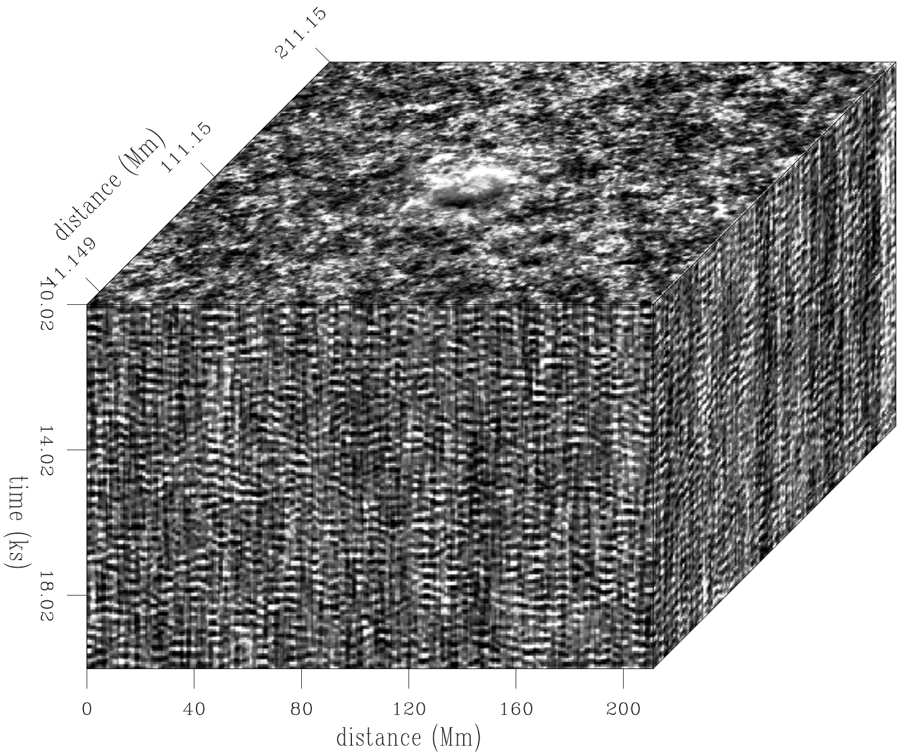

Figure 2. Cube of raw data from MDI instrument. The object in the center is a sun-spot.

3.1. Time-distance functions compared

Figure 3 shows a comparison between the impulse response derived from Kolmogorov spectral factorization, and the impulse response derived by cross-correlation.

The raw MDI data has a narrow temporal bandwidth with most of its energy having a period of about five minutes: squaring the amplitude spectrum reduces this bandwidth even more resulting in the monochromatic appearance of the left panel in Figure 3. Moreover, it is not just the temporal bandwidth that is decreased by cross-correlating traces; but the spatial bandwidth is reduced as well. The steep dips associated with the f- and low-n p modes are clearly visible near the origin in the right panel of Figure 3 are very heavily attenuated in the cross-correlation impulse response (left panel).

This difference in spatial bandwidth can be also be seen in the amplitude spectra of Figure 4. The amplitude of f- and low-n p modes are much lower in the auto-correlation result than in the Kolmogorov result.

Whereas, the temporal bandwidth may be broadened relatively simply by conventional deconvolution, recovering the full spatial bandwidth that is present in the original data would be more difficult.

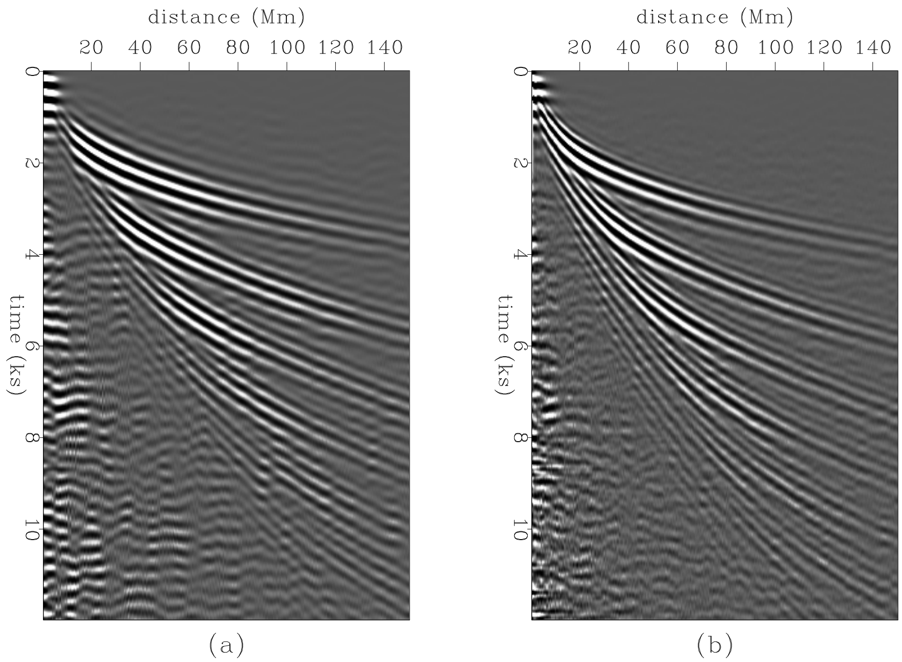

Figure 3. Time-distance impulse responses computed by (a) three-dimensional auto-correlation, and (b) three-dimensional Kolmogorov spectral factorization. Traces have been binned as a function of radius from impulse.

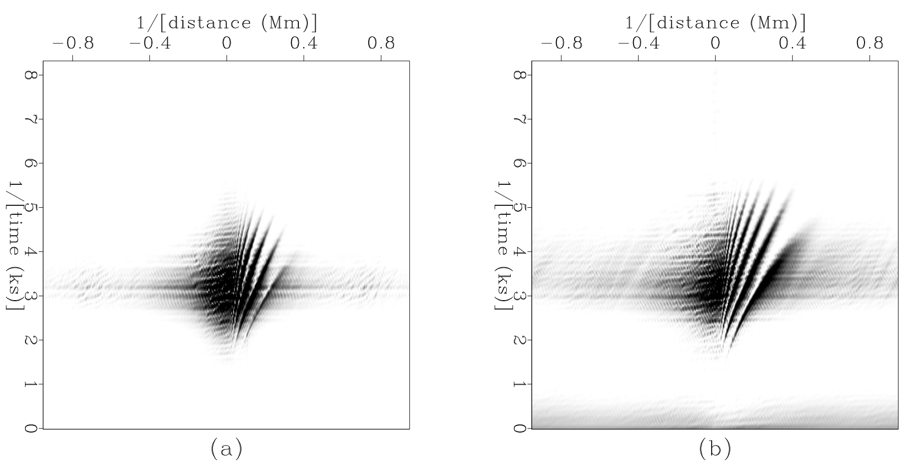

Figure 4. Two-dimensional amplitude spectra of impulse responses shown in Figure 3 above. The impulse responses were computed by (a) three-dimensional auto-correlation, and (b) three-dimensional Kolmogorov spectral factorization.

4. Conclusions

Under assumptions of translational invariance, a white source function, and a minimum-phase impulse response, we have shown that estimation of the sun's acoustic time-distance impulse response amounts to multi-dimensional spectral factorization.

We performed the spectral factorization by transforming the cube of acoustic oscillations into a one-dimensional super-trace which we could factor efficiently with Kolmogorov's frequency domain method.

We have shown that time-distance curves obtained by spectral factorization have a broader temporal and spatial bandwidth than equivalent curves calculated by cross-correlation.

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