Final Exam: Sample Questions

Math 128A Spring 2002
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Your Name: ____________________________________________

- Time: 180 minutes.
- Answer ALL questions.
- Please read carefully every question before answering it.
- If you need extra space, use the other side of the page.
1. (X points) Consider the iteration
\[ c_{k+1} = c_k + \alpha f(c_k) , \]
where \( \alpha \) is a constant that does not change with \( k \), and \( f(x) \in C^\infty \).

   a. What is the condition on \( \alpha \) for this iteration to converge to a solution of \( f(x) = 0 \)?

   b. What is the convergence rate?

   c. What value of \( \alpha \) is required for the quadratic convergence rate?
2. (X points) Neville’s interpolation algorithm

\begin{verbatim}
NEVILLE(x, x_1, x_2, \ldots, x_n, f_1, f_2, \ldots, f_n)
  1 for i ← 1, 2, \ldots, n
  2 do
  3     N_{i,1} ← f_i
  4     d_i ← x - x_i
  5 for k ← 2, 3, \ldots, n
  6 do
  7     for i ← k, k + 1, \ldots, n
  8     do
  9         N_{i,k} ← N_{i,k-1} + d_i (N_{i,k-1} - N_{i-1,k-1}) / (x_i - x_{i-k+1})
10 return N_{n,n}
\end{verbatim}

assumes that the data points \{x_1, f_1\}, \{x_2, f_2\}, \ldots, \{x_n, f_n\} are known in advance. Modify the algorithm so that it processes the input data point by point.

*Hint:* loop by rows in the outer loop.
3. (X points) A two-dimensional function $f(x, y)$ is defined on a triangulated mesh.

a. Find an approximation of the form

$$f(x, y) \approx f(A) \phi_A(x, y) + f(B) \phi_B(x, y) + f(C) \phi_C(x, y),$$

where $A, B, C$ are the corners of a triangle, the point $\{x, y\}$ is inside the triangle, and the functions $\phi_A(x, y), \phi_B(x, y),$ and $\phi_C(x, y)$ are linear in $x$ and $y$. 

*Hint:* The area of triangle $ABC$ is equal to

$$S_{ABC} = \frac{1}{2} (x_A y_B + x_B y_C + x_C y_A - x_B y_A - x_C y_B - x_A y_C).$$
b. Find an approximation of the first partial derivatives of the form

\[
\frac{\partial f}{\partial x} \approx \alpha_A f(A) + \alpha_B f(B) + \alpha_C f(C) .
\]

\[
\frac{\partial f}{\partial y} \approx \beta_A f(A) + \beta_B f(B) + \beta_C f(C) .
\]
4. (X points) Find the first three polynomials orthogonal on the interval [0, 1] with respect to the inner product

\[
\langle f, g \rangle = \int_{0}^{1} \frac{f(x) g(x)}{\sqrt{4 - (x + 1)^2}} \, dx .
\]
5. (X points)

a. Derive a quadrature rule of the form

\[ \int_{a}^{b} f(x) \, dx = \alpha f\left(\frac{2a + b}{3}\right) + \beta f\left(\frac{a + 2b}{3}\right). \]
b. Determine its error assuming $f(x) \in C^2$. 
6. (X points) What is the result of approximating the integral

\[ \int_0^1 x^2 \, dx \]

with the composite trapezoidal rule defined on \( n \) equal subintervals? Your answer should be in closed form and should not include the sum symbol.
8. (X points) Consider the initial-value problem

\[
\begin{align*}
y''(x) &= -[y'(x)]^2 x \\
y(-1) &= 0 \\
y'(-1) &= 1
\end{align*}
\]

Using the step-size \( h = 1 \), find the output of one step of the midpoint method followed by one step of the second-order Adams-Bashforth method.
9. (X points)

a. How many floating-point operations are required to multiply $n \times n$ matrices $A$ and $B$?

b. How many floating-point operations are required to compute the matrix $C = uu^T$, where $u$ is a column vector of length $n$?

c. How many floating-point operations are required to compute the product $AC$, where $A$ is $n \times n$ matrix, and $C$ is the matrix defined above?
10. (X points) Find the inverse of the matrix

\[ A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix} \]

using Gaussian elimination. Show all steps of the computation.