Final Exam

Math 128A Spring 2002
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Your Name: ____________________

- Time: 180 minutes.
- Answer ALL questions.
- Please read carefully every question before answering it.
- If you need extra space, use the other side of the page.

1. (12 points) Consider the iteration

$$c_{k+1} = c_k - \frac{\alpha\, f(c_k)^2}{f(c_k + \alpha\, f(c_k)) - f(c_k)},$$

where α is a constant that does not change with k. Assume that $f(x) \in C^3$ and that the iteration converges to c such that $f(c) = 0,\ f'(c) \neq 0$.

a. What is the order of convergence?

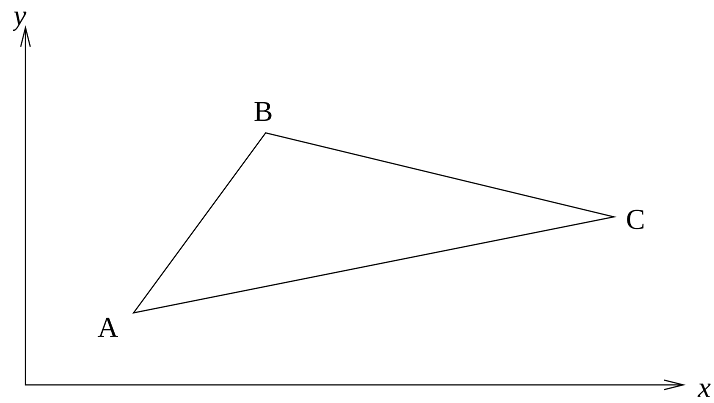
b. What value of α is required for the highest convergence rate?

2. (8 points) The following version of Neville's interpolation algorithm

NEVILLE$(x, x_1, x_2, \ldots, x_n, f_1, f_2, \ldots, f_n)$
for $i \leftarrow 1, 2, \ldots, n$
do
    $N_{i,1} \leftarrow f_i$
    $d_i \leftarrow x - x_i$
    for $k \leftarrow 2, 3, \ldots, i$
    do
        $N_{i,k} \leftarrow N_{i,k-1} + d_i \left(N_{i,k-1} - N_{i-1,k-1}\right) / (x_i - x_{i-k+1})$
return $N_{n,n}$

assumes that the data points $\{x_1, f_1\}, \{x_2, f_2\}, \ldots, \{x_n, f_n\}$ are added one by one. Modify the algorithm so that only one row of length n is stored in memory instead of the whole matrix.

3. (14 points) A two-dimensional function $f(x,y)$ is defined on a triangulated mesh.

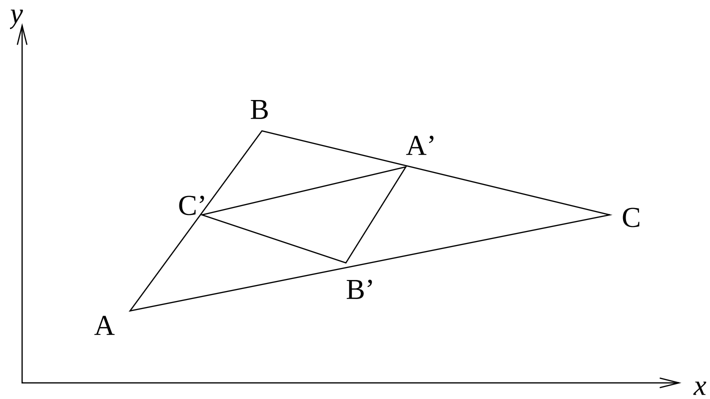


a. Find a two-dimensional quadrature rule of the form

$$\iint_{\Delta} f(x,y) \approx \alpha_A\, f(A) + \alpha_B\, f(B) + \alpha_C\, f(C)\,,$$

where the integration is taken over a triangle region, and A,B,C are the corners of the triangle. *Hint*: The area of triangle ABC is equal to

$$S_{ABC} = 1/2\,(x_A\, y_B + x_B\, y_C + x_C\, y_A - x_B\, y_A - x_C\, y_B - x_A\, y_C)\,.$$



b. Find a two-dimensional composite quadrature rule of the form

$$\iint\limits_{\Delta} f(x,y)\,dx\,dy \approx \alpha_A f(A) + \alpha_B f(B) + \alpha_C f(C) + \beta_A f(A') + \beta_B f(B') + \beta_C f(C'),$$

where A', B', C' are the midpoints of the triangle sides.

4. (8 points) Find a recursion formula for computing the Chebyshev polynomials of the second kind defined as

$$U_k(x) = \frac{1}{k+1}\, T'_{k+1}(x)\,, \quad k = 0, 1, 2, \ldots$$

where $T_k(x)$ are Chebyshev polynomials.

5. (12 points)

a. Derive a quadrature rule of the form

$$\int_{-1}^{1} f(x)\,dx = \frac{2}{3} f(-\alpha) + \frac{2}{3} f(0) + \frac{2}{3} f(\alpha)$$

b. Determine its error assuming $f(x) \in C^4[-1,1]$ and compare it with the error of Simpson's method.

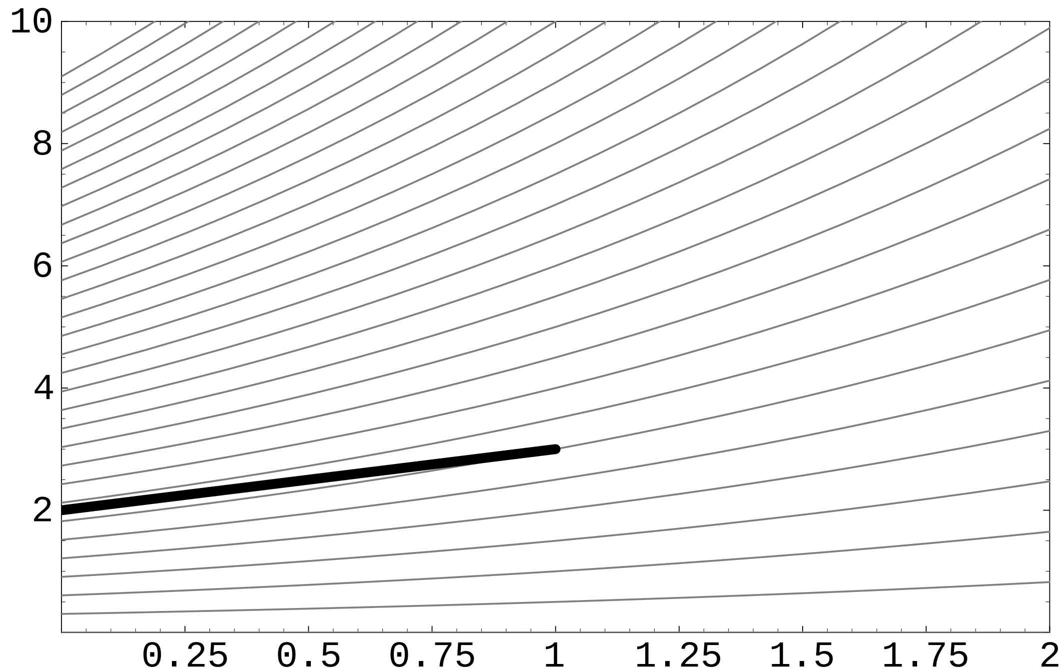
6. (8 points) What is the result of approximating the integral

$$\int_a^b e^x \, dx$$

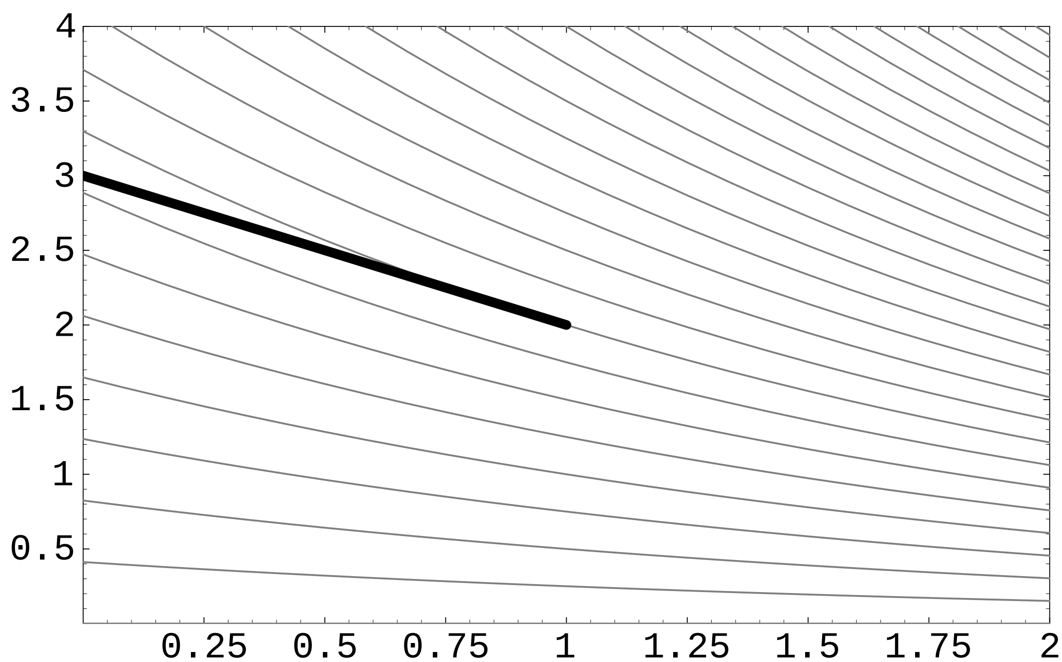
with the composite trapezoidal rule defined on n equal subintervals? Your answer should be in closed form and should not include the sum symbol.

7. (8 points) Each figure shows the solution lines of an ordinary differential equation and the fist step for the numerical solution of an initial-value problem using either Euler's method of backward Euler's method. In each figure, identify which of the two methods was used and sketch the next step.

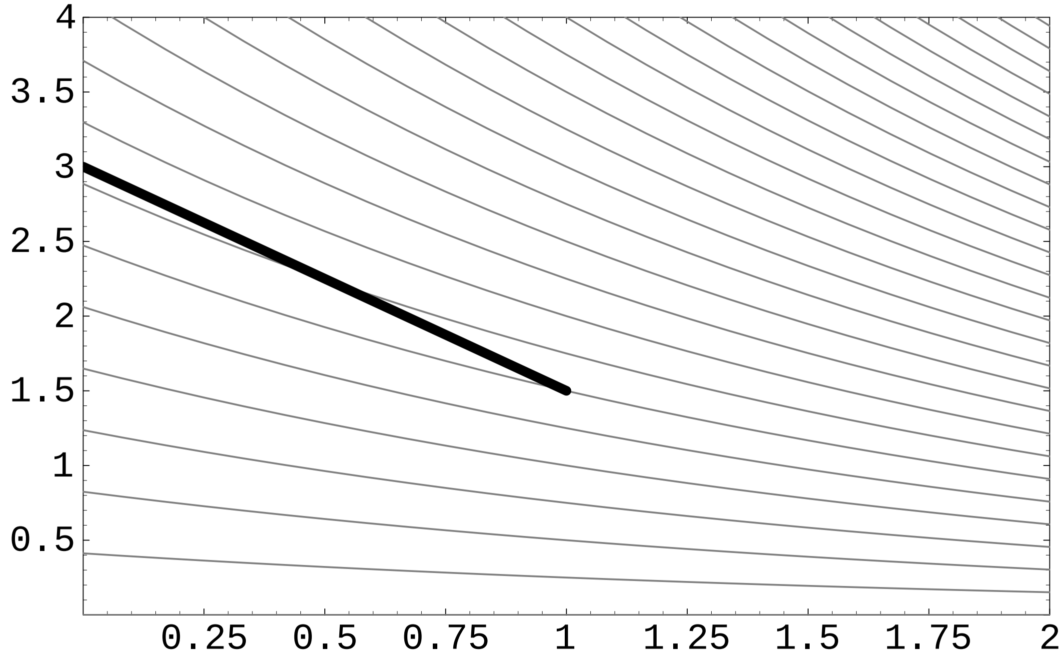
a.



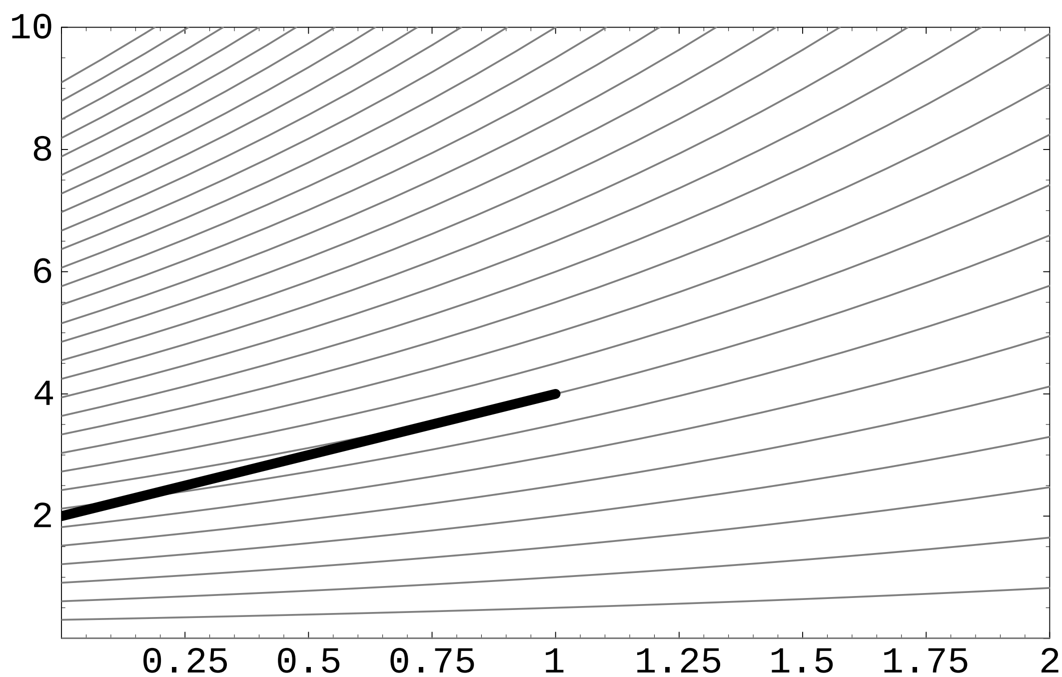
b.



c.


4
3.5
3
2.5
2
1.5
1
0.5
0.25
0.5
0.75
1
1.25
1.5
1.75
2

d.


10
8
6
4
2
0.25
0.5
0.75
1
1.25
1.5
1.75
2

8. (8 points) Consider the initial-value problem

$$\begin{cases} y''(x) &= -\left[y'(x)\right]^2 x \\ y(-1) &= 0 \\ y'(-1) &= 1 \end{cases}$$

Using the step-size $h = 1$, find the output of one step of the modified Euler method (second-order Runge-Kutta) followed by one step of the second-order Adams-Bashforth-Moulton method (prediction by Adams-Bashforth and one step of correction by Adams-Moulton).

9. (10 points) Determine the number of floating-point operations required to multiply three matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ of the sizes $n \times m$, $m \times p$ and $p \times q$ correspondingly

a. in the order $(\mathbf{AB})\mathbf{C}$;

b. in the order $\mathbf{A}(\mathbf{BC})$.

Suggest an example where the second calculation is significantly faster.

10. (12 points) Apply the method of least squares and Cholesky factorization to solve the approximate system of equations

$$\begin{cases} x+y+z & \approx & 1 \\ x+y+z & \approx & 2 \\ x & \approx & 3 \\ x+2z & \approx & 4 \end{cases}$$

Show all steps of the computation.