Homework 11: Numerical Solution of ODE: Multistep Methods (due on April 30)

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1. Derive a *variable-step* Adams-Bashforth method of the form

$$y_{k+1} = y_k + h_k\,A_k\,f(x_k,y_k) + h_k\,B_k\,f(x_{k-1},y_{k-1})\,, \tag{1}$$

where $x_{k+1} = x_k + h_k$, $h_k = \alpha_k\,h_{k-1}$, and y_k approximates $y(x_k)$ – the solution of the initial-value problem

$$\left\{\begin{array}{rcl} y'(x) & = & f(x,y) \\ y(x_0) & = & y_0 \end{array}\right. \tag{2}$$

Considering α_k and h_0 as given, derive appropriate expressions for the coefficients A_k and B_k and determine the order of the method.

2. Applying numerical differentiation to approximate the derivative $y'(x_{k+1})$ in the ordinary differential equation

$$y'(x) = f(x,y)\,, \tag{3}$$

we can arrive at the *backward differentiation formula* (BDF)

$$\sum_{i=0}^{n} A_i\,y_{k+1-i} \approx h\,f(x_{k+1},y_{k+1})\;. \tag{4}$$

Derive a BDF method of the form

$$A\,y_{k+1} + B\,y_k + C\,y_{k-1} = h\,f(x_{k+1},y_{k+1})\;, \tag{5}$$

find its error and compare it with the error of the Adams-Moulton method of the same order.

3. In Homework 10, we found that Euler's method can be unstable when applied to the initial-value problem

$$\left\{\begin{array}{rcl} y''(x) & = & y(x) \\ y(0) & = & y_0 \\ y'(0) & = & -y_0 \end{array}\right. \tag{6}$$

Prove that both the backward Euler method (Adams-Moulton of order 1) and the trapezoidal method (Adams-Moulton of order 2) are unconditionally stable in this case for the positive step h.

Adams-Moulton methods have the general form

$$y_{k+1} = y_k + h\sum_{i=0}^{n} A_i\,f(x_{k+1-i},y_{k+1-i})\;. \tag{7}$$

4. (Programming) In Homework 9, we approximated the number π by computing the integral

$$\int\limits_{-1}^{1} \frac{2\,dx}{1+x^2} \tag{8}$$

A similar solution can be obtained from the initial-value problem

$$\begin{cases} y''(x) &= -\left[y'(x)\right]^2 x \\ y(-1) &= 0 \\ y'(-1) &= 1 \end{cases} \tag{9}$$

Solve this problem numerically on the interval $x \in [-1,1]$ using the following methods:

(a) Adams-Bashforth of order 2

(b) Adams-Bashforth of order 2 as predictor followed by one correction step of Adams-Moulton of order 2

(c) Adams-Moulton of order 2 solving the quadratic equation for y_{k+1}

Initialize each of these methods with the midpoint step (Runge-Kutta of order 2).

Output the absolute error in approximating $\pi = y(1)$ with the step sizes $h = 0.1$, $h = 0.01$ and $h = 0.001$.

5. (Programming) In Homework 3, we studied the motion of a planet. A more direct approach to the same problem involves the equations of Newton's mechanics for the two-body problem:

$$x''(t) = -\frac{G\,M\,x(t)}{\left[x^2(t)+y^2(t)\right]^{3/2}}\,, \quad y''(t) = -\frac{G\,M\,y(t)}{\left[x^2(t)+y^2(t)\right]^{3/2}}\,, \tag{10}$$

where t is time, x and y are the planet coordinates with respect to the star, G is the gravitational constant, and M is the mass of the star. In terms of the orbit characteristics, $G\,M = \omega^2 a$, where ω is angular frequency, and a is the major semi-axis of the orbit. The appropriate initial conditions (for the planet position "in July") are

$$x(0) = a\,(1-\epsilon)\,, \quad x'(0) = 0 \tag{11}$$

$$y(0) = 0\,, \quad y'(0) = a\,\omega\sqrt{\frac{1+\epsilon}{1-\epsilon}}\,, \tag{12}$$

where ϵ is the orbit eccentricity.

Solve the initial-value problem numerically using the following methods:

(a) Adams-Bashforth of order 2

(b) Adams-Bashforth of order 2 as predictor followed by one correction step of Adams-Moulton of order 2

Take $\epsilon = 0.6$, $a = 1\,\text{AU}$, $\omega = 2\pi\,\frac{1}{\text{year}}$, the step size $h = \frac{1}{360}$ year.

Plot the solution (planet trajectory) for $t \in [0,1]$. Does the periodicity hold? Output the error $x(k) - x(0)$ and $y(k) - y(0)$ for $k = 1,2,3$.