Homework 3: Nonlinear Equations: Newton, Steffensen, and Others
(due on February 14)

1. Prove that the sequence

\[ c_0 = 3; \quad c_{n+1} = c_n - \tan c_n, \quad n = 1, 2, \ldots \]  

converges. Find the convergence limit and the order of convergence.

2. Prove that if \( g(x) \in C^m \) for some \( m > 1 \) (continuous together with its derivatives to the order \( m \)), \( g(c) = c, \ g'(c) = g''(c) = \ldots = g^{(m-1)}(c) = 0, \ g^{(m)}(c) \neq 0 \), and the fixed-point iteration

\[ c_{n+1} = g(c_n) \]  

converges to \( c \), then the order of convergence is \( m \).

*Hint:* Use the Taylor series of \( g(x) \) around \( x = c \).

3. Determine the order of convergence for the following methods:

   (a) The *modified* Newton’s method

\[ c_{n+1} = c_n - m \frac{f(c_n)}{f'(c_n)} \]  

under the conditions \( f(x) \in C^{m+1} \) (\( m \geq 1 \)), \( f(c) = f'(c) = f''(c) = \ldots = f^{(m-1)}(c) = 0 \), and \( f^{(m)}(c) \neq 0 \).

   (b) Olver’s method

\[ c_{n+1} = c_n - \frac{f(c_n)}{f'(c_n)} - \frac{1}{2} \left( \frac{f''(c_n)f(c_n)^2}{f'(c_n)^2} \right) \]  

under the conditions \( f(x) \in C^4 \), \( f(c) = 0 \), and \( f'(c) \neq 0 \).

   (c) Steffensen’s method

\[ c_{n+1} = c_n - \frac{f(c_n)^2}{f[c_n + f(c_n)] - f(c_n)} \]  

under the conditions \( f(x) \in C^2 \), \( f(c) = 0 \), and \( f'(c) \neq 0 \).
4. (Programming) In this assignment, you will study the convergence of different methods experimentally using graphical tools. Note that the convergence limit

\[
\lim_{n \to \infty} \frac{|c - c_{n+1}|}{|c - c_n|^p} = z
\]

(6)
corresponds to the linear function

\[
y = \log z + p \, x
\]

(7)
in logarithmic coordinates \(x_n = \log |c - c_n|, y_n = |c - c_{n+1}|\). Plotting the points \(\{x_n, y_n\}\) against the theoretical line verifies experimentally the order of convergence.

In the previous homework, we found that the equation

\[
x + e^x = 0
\]

(8)
has the root at \(c \approx -0.567143\) (accurate to six significant digits).

The figure shows the logarithmic plot of bisection iterations \(\{x_n, y_n\}\) plotted against the line \(y = \log (1/2) + x\). We can see that the iterations oscillate chaotically around the line. You will investigate whether the convergence behavior of other methods is more predictable.

Implement and apply the following methods:

(a) Fixed-point iteration. Apply it to \(g(x) = -e^x\) starting with \(c_0 = -1\).
(b) Newton’s method. Apply it to \(f(x) = x + e^x\) starting with \(c_0 = -1\).
(c) Secant method. Apply it to \(f(x) = x + e^x\) starting with \(c_0 = 0\) and \(c_1 = -1\).

In each case, find the root with the accuracy of six significant digits and plot the points \(x_n, y_n\) and the theoretical convergence line. Since some methods converge faster than others, you will need to use different number of points. Use at least 19 points for (a), 2 points for (b), and 3 points for (c).
5. (Programming) In this assignment, you will compute the motion of a planet according to Kepler’s equation — one of the most famous nonlinear equations in the history of science. Kepler’s equation has the form

$$\omega t = \psi - \epsilon \sin \psi, \quad (9)$$

where $t$ is time, $\omega$ is angular frequency, $\epsilon$ is the orbit eccentricity, and $\psi$ is the angle coordinate. To find the planet location at time $t$, we need to solve equation (9) for $\psi$. The planet coordinates $x$ and $y$ are then given by

$$x = a (\cos \psi - \epsilon); \quad (10)$$
$$y = a \sqrt{1 - \epsilon^2 \sin \psi}, \quad (11)$$

where $a$ is the major semi-axis of the elliptical orbit. For our planet, we will take $a = 1$ AU (astronomical unit), and the eccentricity $\epsilon = 0.6$ (which is much larger than the orbit eccentricity of the Earth and other big planets in the Solar system). The picture shows the orbit and the planet positions in January ($\psi = \pi$) and July ($\psi = 0$).

Your task is to find the planet location in the other ten months, assuming that each month takes 1/12 of the rotation period. Solve Kepler’s equation (9) for $\omega t = 0, \pi/6, 2 \cdot \pi/6, \ldots, 11 \cdot \pi/6$. You can use any numerical method to do that (either your own program or a library program). The result should be computed with the precision of 1 second ($1/3600$ of $1^\circ$). Output a table of the form

| $\omega t$ | $\psi$ | $x$ | $y$ |

and then use a graphics program to plot the planet locations.