Example Function

An example function for studying polynomial interpolation is

\[ f(x) = \frac{1}{x}. \]

We select three nodes and find the interpolating polynomial for them.

<table>
<thead>
<tr>
<th>( x )</th>
<th>2</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f )</td>
<td>0.5</td>
<td>0.25</td>
<td>0.2</td>
</tr>
</tbody>
</table>

The second-order interpolation polynomial does not reconstruct the function exactly but fits the input data.
Lagrange Interpolation

The Lagrange form of the interpolation polynomial is

\[ P(x) = \sum_{k=1}^{n} f_k L_k(x), \]

where

\[ L_k(x) = \prod_{i \neq k} \frac{x - x_i}{x_k - x_i}. \]

In our example,

\[
\begin{align*}
L_1(x) &= \frac{(x - 4)(x - 5)}{(2 - 4)(2 - 5)} = \frac{(x - 4)(x - 5)}{6} \\
L_2(x) &= \frac{(x - 2)(x - 5)}{(4 - 2)(4 - 5)} = -\frac{(x - 2)(x - 5)}{2} \\
L_3(x) &= \frac{(x - 2)(x - 4)}{(5 - 2)(5 - 4)} = \frac{(x - 2)(x - 4)}{3}
\end{align*}
\]
Lagrange Polynomials

$L_1(x)$ is one at $x_1 = 2$ and zero at $x_2 = 4$ and $x_3 = 5$. Likewise, $L_2(x)$ is one at $x_2$ and zero at $x_1$ and $x_3$. $L_3(x)$ is one at $x_3$ and zero at $x_1$ and $x_2$.

Putting it all together,

$$P(x) = \frac{(x-4)(x-5)}{12} - \frac{(x-2)(x-5)}{8} + \frac{(x-2)(x-4)}{15} = 0.025x^2 - 0.275x + 0.95.$$  

Newton Interpolation

The Newton form of the interpolation polynomial is

$$P(x) = \sum_{k=1}^{n} f[x_1, x_2, \ldots, x_k] N_k(x),$$
where

\[ N_k(x) = \prod_{i=1}^{k-1} (x - x_i), \]

and \( f[x_1, x_2, \ldots, x_k] \) is the divided difference, evaluated with the help of the recursive relationship

\[
\begin{align*}
    f[x_k] &= f_k \\
    f[x_1, x_2, \ldots, x_k] &= \frac{f[x_2, \ldots, x_k] - f[x_1 \ldots, x_{k-1}]}{x_k - x_1}
\end{align*}
\]

In our example,

\[
\begin{align*}
    N_1(x) &= 1 \\
    N_2(x) &= x - 2 \\
    N_3(x) &= (x - 2)(x - 4)
\end{align*}
\]

The divided difference table is

<table>
<thead>
<tr>
<th></th>
<th>( f[x_1] = 0.5 )</th>
<th>( f[x_2] = 0.25 )</th>
<th>( f[x_1, x_2] = \frac{0.25 - 0.5}{4 - 2} = -0.125 )</th>
<th>( f[x_2, x_3] = \frac{0.2 - 0.25}{5 - 4} = -0.05 )</th>
<th>( f[x_1, x_2, x_3] = \frac{-0.05 + 0.125}{5 - 2} = 0.025 )</th>
</tr>
</thead>
</table>

Putting it all together,

\[
P(x) = \frac{1}{2} - \frac{x - 2}{8} + \frac{(x - 2)(x - 4)}{40} = 0.025x^2 - 0.275x + 0.95.
\]
Neville Interpolation

The Neville form of the interpolation polynomial is defined by recursion

\[
\begin{align*}
P_0(x) &= f_1 \\
P_{k-1}(x) &= P_{k-2}(x)(x_k - x) - Q_{k-2}(x)(x - x_1) / (x_k - x_1),
\end{align*}
\]

where \( P_{k-1}(x) \) interpolates at nodes \( x_1, x_2, \ldots, x_k \), and \( Q_{k-2}(x) \) interpolates at nodes \( x_2, \ldots, x_k \).

The zeroth-order Neville polynomials are constant functions.
The first-order Neville polynomials are

\[ P_1(x) = \frac{\frac{1}{3}(4 - x) + \frac{1}{4}(x - 2)}{4 - 2} = \frac{6 - x}{8} \]

\[ Q_1(x) = \frac{\frac{1}{4}(5 - x) + \frac{1}{5}(x - 4)}{5 - 4} = \frac{9 - x}{20} \]

The second-order Neville polynomial is

\[ P_2(x) = \frac{\frac{6 - x}{8}(5 - x) + \frac{9 - x}{20}(x - 2)}{5 - 2} = \frac{(6 - x)(5 - x)}{24} + \frac{(9 - x)(x - 2)}{60} = 0.025x^2 - 0.275x + 0.95 \]