Nonstationary Signal Tutorial
(release alpha 0.3)

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0.1 PREFACE

Putting this out right at my 80th birthday, I am not planning another book; but since 2014, I stumbled on an approach to a large amount of geophysical data modeling and fitting that is much simpler than traditional approaches. Even better, it avoids the often unreasonable academic presumption of stationarity (i.e. time and space invariant statistics). Therefore, I could not resist embarking on this booklet. Any participant is welcome to contribute illustrations (and ideas)—perhaps becoming a coauthor, even eventually taking it over. The first need now is more examples. Ultimately, all the examples should be presented in reader rebuildable form.

My previous 2014 book Geophysical Image Estimation by Example (GIEE) is widely referenced herein. It is freely available at http://sep.stanford.edu/sep/prof/ or in paper for a small price at many booksellers, or at the printer, Lulu.com.

Alfa and early beta versions of this booklet will fail to provide rebuildable illustrations. I am no longer coding myself, so if there are ever to be rebuildable illustrations, I need coauthors. I set for myself the goal to take this booklet out out of beta when 50% of the illustrations can be destroyed and rebuilt by readers.

The most recent version of this manuscript should be at Jon Claerbout’s classroom website. Check here: http://sep.stanford.edu/sep/prof/. This manuscript version was formed March 15, 2018.
0.2 INTRODUCTION

Nonstationary data is data with statistics that vary with time. Concepts here are illuminated with pseudo code, with modeled data, and with real data.

0.2.1 What can you do with this stuff?

1. Model and fit data with nonstationary statistics.
2. Perform blind deconvolution (estimate and remove a source wavelet).
3. Bring randomly positioned data to a regular Cartesian mesh.
5. Transform residuals to IID (Independent, Identically Distributed) before fitting.
6. Swap easily among $\ell_1$, $\ell_2$, hyperbolic, and inequality penalties.
7. Stretch signals unevenly to match others
8. Predict price based on diverse aspects.
9. Remove crosstalk in multichannel signals (vector data).
10. Model robustly (i.e. multivariate median versus the mean).
11. Use Occam's razor to shave models (beats the $\ell_1$ norm).
12. Join the world of BIG DATA by grasping multiple aspects of back projection.

This booklet is novel by attacking data what is nonstationary, meaning that its statistical characterization is not constant in time or space. This approach works by adding a new data value to a previously solved imaging problem. The newly arrived data value requires us to make a small adjustment to the solution. Then we continue with all the other data values.

Although we begin here narrowly with a single 1-D scalar signal $y_t$, we soon expand broadly with $y_t(x, y, z)$ representing multidimensional data (images and voxels) and then multicomponent (vector-valued) signals $\mathbf{y}_t$.

Many researchers dealing with physical continua use “inverse theory” with little grasp of how to supply the “inverse covariance matrix.” For Cartesian gridded data, the needed code is here.

Most of the time-series literature presumes data spectra are time invariant (stationary), while most applications involve spectra that change in time and space. Therefore there is a real need for nonstationary tools. These capabilities are particularly important in higher dimensional problems, slightly in time, and more strongly in space. A basic problem in geophysics is the lack of adequately dense data sampling in space, and wide-spread nonstationarity (also in space). Such nonstationarity arises both for statistical reasons, as well as physical reasons such as location versus angle.
0.3 PREDICTION ERROR FILTER == PEF

Knowledge of an autocorrelation is equivalent to knowledge of a spectrum. Less well known is that both are equivalent to knowledge of a Prediction Error Filter (PEF). PEFs share many properties with differential equations. A PEF is like a differential equation that when driven by a random noise source produces data with the spectrum of interest. The PEF is commonly estimated from data by least squares procedures that we soon see.

0.3.1 PEF history

The name “Prediction Error Filter” appears first in the petroleum exploration industry although the idea emerges initially in the British market forecasting industry in the 1920s as the Yule-Walker equations. The same equations next appear in 1949 in a book by Norbert Wiener in an appendix by Norman Levinson. Soon after, Enders Robinson extended the PEF idea to multichannel (vector-valued) signals. Meanwhile, as the petroleum exploration industry became computerized it found a physical model for scalar-valued PEFs. They found a lot of oil with it; and they pursued PEFs vigorously until about 1970 when their main focus shifted (to where it remains today) to image estimation. My friends John Burg and John Sherwood understood a 2-D extension to the PEF idea but it went unused until I discovered the helix interpretation of it (in about 1998) and used it extensively in my 2014 book *Geophysical Image Estimation by Example* (GIEE). Beyond 2-D, the PEF idea naturally extends to any number of dimensions. (Exploration industry data exists in a 5-D space, time plus two Earth surface geographical coordinates for each energy source plus another two for each sound receiver.) I expected the study of submarine warfare to result in conceptual advances with acoustic antennas; etc, but, I am not aware of whatever fundamentals it may have uncovered.

0.3.2 PEFs present and future

From an application perspective the weakness of autocorrelation, spectrum, and classic PEF is the lack of a natural extension to nonstationarity. Like autocorrelation and spectrum, the PEF theory became clumsy when applied to real-world data in which the statistics varied with time and space. Since my 2014 book, Sergey Fomel and I have found a way to extend the PEF concept to nonstationary data. Not only have we discovered an extension, it is also easier to understand and to code! This ease promises quick results and, it looks like fun! Although I recently turned 80, I cannot stop thinking about it.

Besides, all the old-time activities that are beginning to get easier and better, we can speculate it is going to be fun for even more reasons. The nonstationary PEFs in this document bring us many flavors of back projection. They share this aspect with the emerging field of machine learning. Who is going to have more fun, them or us? We are quite model-based; therefore, we should have the advantage of needing less data to find robust methods and more reliable results. Never-the-less, they bring a young, rapidly-growing, energetic community to the table, and that’s another reason we’re going to have fun.

Beyond introduction to nonstationary PEFs, find in this booklet a surprising new concept promising a new direction. Quantum theory associates momentum with Fourier space
leading to the uncertainty principle that says they cannot simultaneously know position and velocity. Although we are not doing quantum mechanics, we too often feel such mathematical constraints in our work. PEFs like differential equations, such as finite-difference equations, often have coefficients that change abruptly. Therein the character of a signal changes abruptly. With an image of a person—even a black and white image—the spatial spectrum changes abruptly at the transition from skin to hair. Also in this document is a strategy for recognition of such a sharp boundary along with the PEFs that pertain to each side of the boundary.

0.4 CREDITS AND THANKS

Sergey Fomel triggered this direction of research when he solved the nonstationarity problem that I had posed but could not solve. Bob Clapp ran an inspiring summer research group. Stewart Levin generously welcomes my incomplete thoughts on many topics. He page edited and provided a vastly cleaner 1-D whiteness proof. John Burg set me on the track for understanding the 2-D PEF. Kaiwen Wang worked with me and made all the illustrations in the multichannel chapter. Joseph Jennings provided the field-data debubble example and commented on early versions of the multichannel chapter. Jason Chang assisted me with LaTeX. Many thanks to all.
Chapter 1

Nonstationary scalar signals

The field of Statistics has a principle that residuals in model building should be Independent, Identically Distributed (IID). In practice with gridded data, the “ID” means gain should be chosen to give residuals a generally uniform variance over physical space; it is simply a matter of scaling. The remaining “I” in IID means “independent,” our topic here. Autocorrelations should tend to impulses, meaning spectra should tend to whiteness (uniform variance) which is accomplished herein with Prediction-Error Filters (PEFs) subsequently to be defined.

1.0.1 Why should residuals be IID?

With least squares fitting modeled data to observed data, small residuals are squared; so they tend to be ignored. Thus, the entire physical space and Fourier space of residuals should be scaled up to easily visible levels to ensure all aspects of the data have been probed. The statistical principle that model fitting residuals should be IID says almost every image estimation situation calls for using PEFs. They are introduced next.

1.0.2 Prediction-error filtering (deconvolution)

A widespread generic model for signal and image data is that the data originated from independent causes that were somehow smoothed, filtered, or convolved before reaching us as data. Because the PEF returns us to uncorrelated and apparently independent sources, it is said to be “deconvolved.” Those uncorrelated sources will tend to contain all frequencies in a generally uniform amount.

Start with a channel of data (a signal or signals of many thousands of values). We denote these data numbers by \( y = (y_0, y_1, y_2, \ldots) \). A little patch of numbers that we call a “filter” is denoted by \( a = (a_0, a_1, a_2, \ldots, a_n) \). In pseudo code these filter numbers are denoted by \( a(0), a(1), \ldots, a(\text{ntau}) \). Likewise code for the data. The filter numbers slide across the data numbers with the leader being \( a(0) \). An equation for sliding the filter numbers across the data numbers obtaining the output \( r_t \) is \( r_t = \sum_{\tau=0}^{n_\tau} a_\tau y_{t-\tau} \). In a stationary world, the filter values are constants. In our nonstationary world, the filter values change a tiny bit after the arrival of each new residual value.
CHAPTER 1. NONSTATIONARY SCALAR SIGNALS

Several computer languages allow the calculation $x \leftarrow x + y$ to be represented by $x+=y$. We use this notation herein, likewise $x-=y$ for subtraction. Pseudo code for finding $r(t)$ is:

```python
# CODE = CONVOLUTION
r(....) = 0.
for all t {
  do tau = 0, ntau
  r(t) += a(tau) * y(t-tau)
}
```

With each step in time we prepare to change the filter $a(tau)$ a tiny bit. To specify the change, we need a goal for the filter outputs $r(t)$ to have minimum energy. To prevent the filter $a$ from becoming all zeros, we constrain the first filter coefficient to be unity.

$$a = [1, a_1, a_2, a_3, \cdots] \quad (1.1)$$

To contend with the initial unit “1.0” outputting an input data value, the remaining filter coefficients try to destroy that data value. They must attempt to predict the value’s negative. The filter output $r_t$ is the residual of the attempted prediction. The name of the filter itself is the Prediction-Error Filter (PEF). PEFs are slightly misnamed because their prediction portion predicts the data negative.

Intuitively, PEF output has sucked all the predictability from its input. Appendix 5.1.1 Why 1-D PEFs have white output shows the PEF output tends to be spectrally white—to be a uniform function of frequency. The longer the filter, the whiter the output. The name deconvolution came about from a hypothetical model that the original sources were random impulses, but the received signal became spectrally colored (convolved) by reasons of wave propagation. Thus, a PEF should return the data to its original state. It cannot restore any delays—because the PEF is causal meaning it has only knowledge of the past; therefore $[\cdots, a_{-2}, a_{-1}] = 0$. P.E. filtering is sometimes called blind deconvolution—stressing that $a$ is estimated as well as applied.

For now $y_t$ and $a_{\tau}$ are scalar time functions at a point in space. Before we finish, they will become time functions in an $N$-dimensional physical space like the four component array $y_t(x, y, z)$. Beyond that we will revert to a point in space, but extend to a vector-valued signal. Theory proceeds somewhat like with scalar signals, but data and prediction error will be vector functions like $\vec{r}_t = (u_t, v_t, w_t)$, where the PEF begins not with a $1.0$ but with a three component identity matrix $I$.

1.1 THE HEART OF NONSTATIONARY PEF WITH NO CALCULUS

At any one moment in time, we may think of the PEF output $\sum_{\tau} a_{\tau} y_{t-\tau}$ as a dot product of the filter $a$ onto some backwards piece of input data. Denote that backwards piece by $d$. (Other moments in time have other values in the $d$ vector.) At that moment, the PEF output is $a \cdot d$. Consider the exploratory filter $a - \epsilon d$ where $\epsilon$ is a tiny positive number. Its output $r_t$ is $(a - \epsilon d) \cdot d = (a \cdot d) - \epsilon(d \cdot d)$. To reduce the size $|r_t|$ of the new output residual, these two terms must have opposite polarity; but $r_t = (a \cdot d)$ may have either polarity. Try instead the filter update $(a - \epsilon r_t d)$. Its output is $(a \cdot d) - \epsilon r_t(d \cdot d)$, which
1.1. THE HEART OF NONSTATIONARY PEF WITH NO CALCULUS

on rearrangement is \((a \cdot d)(1 - \epsilon (d \cdot d))\), is easily assured smaller than \(r_t = (a \cdot d)\). Thus,

\[
\Delta a = -\epsilon r_t d = -\epsilon r_t y_{t-\tau}
\]

(1.2)

In summary:

<table>
<thead>
<tr>
<th>Filter</th>
<th>Definition</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a = a_\tau)</td>
<td>(d = y_{t-\tau})</td>
<td>(r_t = a \cdot d)</td>
</tr>
<tr>
<td>(a - \epsilon d)</td>
<td>First trial</td>
<td>(a \cdot d - \epsilon (d \cdot d))</td>
</tr>
<tr>
<td>(a - \epsilon r_t d)</td>
<td>Revision</td>
<td>(a \cdot d - \epsilon r_t (d \cdot d))</td>
</tr>
<tr>
<td>(a + \Delta a)</td>
<td>Success!</td>
<td>((a \cdot d)(1 - \epsilon (d \cdot d)))</td>
</tr>
</tbody>
</table>

The last line says the filter change reduces the size of the residual \(r_t = (a \cdot d)\). Our choice of \(\epsilon\) balances fitting the present versus fitting the past.

The polarity of components of \(d\) are all that matters. Their magnitude does not. We can relate this conclusion to basic mathematics. Take anyone’s favorite penalty function. Take its gradient \(g\) of the newest data value’s residual with respect to the PEF coefficients \(a\). One characteristic of the gradient of any penalty function is that moving along any gradient component increases the penalty. (Its negative decreases penalty). Thus, the result of your mathematical exercise must conclude with components of \(g\) that are are sign-preserving functions of components of \(d\), such as \(d\) itself. For example we could update with \(\Delta a = \epsilon r_t / |r_t| d\) which has an \(\ell_1\)-norm flavor. Currently new (interpreted later) is that \(\epsilon d\) can also be stretched analogously.

Appendix 5.2 The heart of nonstationary PEF using calculus has a two-page calculation showing that the gradient of the \(\ell_2\)-norm residual induced by including a new data value \(y_t\) is \(\Delta a_\tau = -\epsilon r_t y_{t-\tau}\), namely, our Equation (1.2). Penalty functions need not be convex(!), but they should have a single unique minimum. Convex functions do not have banana-shaped contours, a problem for many methodologies, but not a problem herein.

1.1.1 Code for prediction error = deconvolution = autoregression

The following code does “deconvolution” or more loosely, “decon,” also known as “autoregression.” In 1-D filtering applications, the universal need for IID residuals gives it value.

```c
r(...) = 0. # CODE = NONSTATIONARY PREDICTION ERROR
a(...) = 0.
a( 0 ) = 1.0
do over time t {
    # r(t) = nonstationary prediction error.
    do tau= 0, ntau
        da(tau) = 0
        r(t) += a(tau) * y(t-tau) # forward
    do tau= 0, ntau
        da(tau) += r(t) * y(t-tau) # adjoint
    da(0) = 0. # constraint
    do tau= 0, ntau
        a(tau) -= da(tau) * epsilon
}
```

The #forward line in the code above applies the filter to get the residual. The #adjoint line in the code is building Equation (1.2). The adjoint operation is also called back-projection.
The code above, based on little more than the definition of dot product, is a demonstration of a deeper principle in classroom mathematics. The line `#forward` is a matrix times a vector. The line `#adjoint` is also a matrix times a vector. It is the same matrix $y(t-\tau)$, but one $\tau$ loop does the matrix transpose of the other because one carries $\tau$ space to $t$ space, while the other carries $t$ space to $\tau$ space. The transpose of any matrix is $M_{ij}^T = M_{ji}$. The line $\text{da}(0)=0$ is a constraint to prevent changing the $a(0)=1$ maintaining the definition of $r(t)$ as a residual. Common sense has given us the above example of classroom fundamentals: Put the residual into the adjoint (transpose) to get the gradient; then go down. We got the gradient without ever calculating a derivative! If coding adjoints is new to you, I recommend Chapter 1 in GIEE (Claerbout, 2014). It’s free on the internet.

Suppose while running along time $t$, we find the line in the code above computing $\Delta a$ saying for practical purposes $\text{da}(\tau)=r(t)\ast y(t-\tau)$ vanishes for all $\tau>0$. This statement would delight any stationary theorist, because the gradient vanishing $\Delta a = 0$ says we are finished. It is saying the residual $r$ is orthogonal to all the fitting functions $y_\tau = y_{t-\tau}$. (A particular fitting function is $y_9 = y_{t-9}$.) With our nonstationary technology, we do not expect $\Delta a$ ever to be exactly zero, but we do expect it to get small and then bounce around. The fluctuation in size of $|\Delta a|$ is not simply epsilon, but the fluctuations diminish as the residual becomes more and more orthogonal to all the fitting functions. We are too new at this game to know precisely how to choose $\epsilon$, how much bouncing around to expect, or how to characterize nonstationarity; but, we will come up with a good starting guess for $\epsilon$. While theorizing, there is much we will learn by experience.

1.1.2 Why does the residual into the adjoint give the gradient?

Basic geophysical model $m$ estimation is summarized by the residual minimization $0 \approx r(m) = Fm-d$. In the special case in which $F$ is a convolution matrix (downshifted columns of data $d$), this formulation fits the estimation of a prediction filter $m$. But, for most applications we want a prediction-error filter $a$. Thus we think of PEF estimation as the constraint $a_0 = 1$ along with the augmented matrix $Y = [d|F]$ where $Y$ is also a convolution matrix. PEF estimation coding is repetitions of $r = Ya$, followed by $\Delta a = -Y^* r$, followed by the constraint $\Delta a_0 = 0$, followed by $a = a + \epsilon \Delta a$.

This tutorial document shows six codes for diverse applications of PEFs; therefore let us be sure that everyone is on board with the idea that the gradient is a residual dumped into a transposed modeling operator. Chapter 2 of my textbook GIEE (Claerbout, 2014) guides you through every step with a $2 \times 3$ matrix. In summary, the quadratic form you are minimizing is $r \cdot r = (m^*F^* - d^*)(Fm - d)$ with derivative by $m^*$ being $\Delta m = F^* r$. Likewise, the derivative of $a^*Y^*Ya$ by $a^*$ is $\Delta a = Y^* r$.

1.1.3 The outside world

The regression updating approach introduced here is not limited to convolutional matrices. It applies to all regression equations. For each new regression row, subtract a tiny suitably scaled copy of the new row into the solution. Move along; keep doing it. When you run out of equations, you can recycle the old ones. By cycling around a vast number of times with an epsilon tending to zero, you converge to the stationary solution. This updating
procedure should be some long-known principle in mathematics. I have stumbled upon something called the Widrow-Hoff learning rule, which feels just like this updating.

For example, imagine a stack of records of home sales. The \( i \)-th member of the stack is like the \( t \)-th time of a signal. The first column contains the recorded sales prices. The next column contains the square footages, the third column contains the number of bathrooms, etc. Because many of these variables have all positive elements, we should allow for removing their collective means by including a column of all “ones.” In the signal application, the \( i \)-th column contains the signal at the \( i \)-th lag. Columns containing all positive numbers might be replaced by their logarithms. The previously shown code finds \( a_i \) coefficients to predict (negatively) the signal. Associating lags with real-estate aspects, the code would predict (the negative and possibly the logarithm of) the sales price. You have made the first step towards a learning machine.

1.1.4 Hyperbolic penalty function

Most people do data fitting by minimizing the sum of the squared residuals—called “least squares” or the \( \ell_2 \)-norm approach. Computations are generally easy, but a single outlandish residual ruins everything. The \( \ell_1 \)-norm approach minimizes the sum of absolute values of residuals. The median is a child of \( \ell_1 \). Occasional humongous residuals detract little from the solutions. Regressions solved with \( \ell_1 \) are described as “robust.”

GIEE has many examples of practical use of the hyperbolic penalty function. Loosely, we call it \( \ell_h \). For small residuals it is like \( \ell_2 \), and for large ones it is like \( \ell_1 \). Results with \( \ell_h \) are critically dependent on scaling the residual, say \( q = r/\bar{r} \). Our choice of \( \bar{r} \) specifies the location of the transition between \( \ell_1 \) and \( \ell_2 \) behavior. I have often taken \( \bar{r} \) to be at the 75\(^{th} \) percentile of the residuals.

A marvelous feature of \( \ell_1 \) and \( \ell_h \) emerges on model space regularizations. They encourage models to contain many zero or small values, thereby leaving the essence of the model in a small number of locations. Thus we build sparse models, the goal of Occam’s razor.

Happily, the nonstationary approach allows easy mixing and switching among norms. In summary:

<table>
<thead>
<tr>
<th>Name</th>
<th>Scalar Residual</th>
<th>Scalar Penalty</th>
<th>Scalar Gradient</th>
<th>Vector Gradient</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \ell_2 )</td>
<td>( q = r )</td>
<td>( q^2/2 )</td>
<td>( q )</td>
<td>( q )</td>
</tr>
<tr>
<td>( \ell_1 )</td>
<td>( q = r )</td>
<td>(</td>
<td>q</td>
<td>)</td>
</tr>
<tr>
<td>( \ell_h )</td>
<td>( q = r/\bar{r} )</td>
<td>( (1 + q^2)^{1/2} - 1 )</td>
<td>( q/(1 + q^2)^{1/2} )</td>
<td>( \text{softclip}(q) )</td>
</tr>
</tbody>
</table>

From the table, observe at \( q \) large, \( \ell_h \) tends to \( \ell_1 \). At \( q \) small, \( \ell_h \) tends to \( q^2/2 \) which matches \( \ell_2 \). To see a hyperbola \( h(q) \), set \( h = 1 \) equal to the Scalar Penalty in the table, getting \( h^2 = 1 + q^2 \). The \( \text{softclip}() \) function of a signal applies the \( \ell_h \) Scalar Gradient \( q/(1 + q^2)^{1/2} \) to each value in the residual.

Coding requires a model gradient \( \Delta m \) or \( \Delta a \) that you form by putting the Vector Gradient into the adjoint of the modeling operator, then taking the negative. If you want \( \ell_2 \), \( \ell_1 \), or \( \ell_h \), then your gradient is either \( \Delta a = -Y^*q, -Y^*\text{sgn}(q) \), or \( -Y^*\text{softclip}(q) \). You may also tilt the \( \ell_1 \) penalty making it into a "soft" inequality.

(Quick derivation: People choose \( \ell_2 \) because its line search is analytic. Instead, we chose...
epsilon instead. For the search direction, let \( P(q(a)) \) be the Scalar Penalty function. The step direction is \( -\Delta a = \partial P / \partial q = \partial P / \partial a^* = \partial q^* / \partial a^* \partial P / \partial q^* = Y^* \partial P / \partial q^* \) where for \( \partial P / \partial q^* \) you get to choose a Vector Gradient from the table above.)

An attribute of \( \ell_1 \) and \( \ell_2 \) fitting is that \( ||\alpha r|| = \alpha ||r|| \). This attribute is not shared by \( \ell_h \). Technically \( \ell_h \) is not a norm; it should be called a “measure.”

### 1.1.5 Other coordinates

If all we need to do is keep \( d \cdot d = d^*d \) positive, we immediately envision more general linear changes of variables in which we keep \( d^*B^*Bd \) positive. This suggests the update \( \Delta a = -\epsilon r_t \cdot d^*B^*B \). I’ve done no example of that yet.

### 1.1.6 Nonlinear norms and penalties

Besides the “no calculus” update Equation (1.2) which amounts to a least-squares (\( \ell_2 \)-norm) approach (see Appendix: The heart of nonstationary PEF using calculus), an \( \ell_1 \)-norm style update is as follows:

\[
\Delta a = -\epsilon \text{sgn}(r_t) \text{sgn}(d) = -\epsilon \text{sgn}(r_t) \text{sgn}(y_{t-\tau}) \tag{1.3}
\]

With this update there is the possibility of deconvolving data containing a sprinkling of huge spikes. More likely, a soft-clip function would work better than the hard-clip \( \text{sgn}() \) function in Equation (1.3).

We need not begin our thoughts with penalty functions and their gradients. Any polarity preserving amplitude stretching functions may be applied to \( r_t \) and \( y_{t-\tau} \) which is clear from our original non-mathematical derivation of Equation (1.2). Corresponding to this idea is the general principle that scaling axes on a penalty function leads to scaled components of gradients, not polarity changes. The main property of gradients is that each component is pulling the penalty in the desired direction. We need know nothing about a penalty function or its derivative.

The difference between \( \text{sgn}(r_t) \) and \( \text{sgn}(y_{t-\tau}) \) is interesting. Deconvolution in the presence of large spike noise is improved using \( \text{sgn}(r_t) \) to downplay predicting corrupted data. It is also improved by downplaying—with \( \text{sgn}(y_{t-\tau}) \)—regression equations that use corrupted data to try predicting good data.

Notice that \( d \) contains a lag axis; therefore the stretching function (norm choice) can vary with lag. In practice, how should we play the statistics of \( r \) and \( d \)? I have never undertaken to play with these options so I have little feel for opportunities they may present. To fuel our creative imaginations, perhaps we need interactive programs to play with diverse data sets. I also need an energetic reader to test my explanations—to verify, or possibly reject some of my claims.

Because a humongous data value is easy to recognize, perhaps we should simply mark such a location as missing data and switch to the method of the next program.

What is the most general formulation? When we get to vector-valued signals, we may...
find unexpected opportunities, such as vorticity in the ocean or Alfven waves in the ionosphere. In time, diverse applications will crop up.

1.1.7 Application aspects of these fundamentals

Consider diagonal weighting functions of the form \(d^*W_d\). Take a diagonal \(W\) with some components vanishing. The vanishing corresponds constraints that some filter coefficients remain constant (0 or 1). An example of a “gapped PEF,” is the debubble process described in Chapter 2.

In reflection seismology, \(t^2\) gain and debubble do not commute. Do the physics right by applying debubble first; then get a bad answer (because late data has been ignored). Do the statistics right; apply gain first; then violate the physics. How do we make a proper nonstationary inverse problem? I think the way is to merge the \(t^2\) gain with the \(\epsilon\).

1.2 ESTIMATING TOGETHER MISSING DATA WITH ITS PEF

One of the smartest guys I have known came up with a new general-purpose nonlinear solver for our lab. He asked us all for some simple test cases. I suggested, “How about simultaneous estimation of PEF and missing data?”

“That’s too tough,” he replied.

We do it easily now by appending three lines to code above. The #forward line is the usual computation of the prediction error. To understand that the lines labeled #adjoint are adjoints (transposes), compare each to the forward line, and observe that input and output spaces have been swapped. At the bottom are the three lines for missing-data estimation. The code “looks canonical” (by sticking a residual into an adjoint), but what is it doing?

```python
# CODE = ESTIMATING TOGETHER MISSING DATA WITH IT'S PEF
# y(t) is data.
# miss(t) = "true" where y(miss(t)) is missing (but zero)
r(...) = 0; # prediction error
a(...) = 0; a(0) = 1. # PEF
do t = ntau, infinity {
    do tau= 0,ntau
        r(t) += y(t-tau) * a(tau) # forward
        if( tau > 0)
            a(tau) -= epsilonA * r(t) * y(t-tau) # adjointA
        do tau= 0,ntau
            if( miss(t-tau))
                y(t-tau) -= epsilonY * r(t) * a(tau) # adjointY
    }
}
```

It is a simple program, but not easy to explain that it is doing something logical. It is easier to assert that it must work because a residual \(r(t)\) is going into the transpose of forward modeling, the spaces swapped being \(r(t)\) and \(y(t-tau)\).

---

1 See the youtube for “Perpetual ocean.”

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It would be fun to view the data, the PEF, and the inverse PEF as the data streams through the code. It would be even more fun to have an interactive code with sliders to choose $\epsilon_\text{A}$, $\epsilon_\text{Y}$, and our $\Delta t$ viewing rate.

It would be still more fun to have this happening on images (Chapter 2). Playing with your constructions cultivates creative thinking, asserts the author of the MIT Scratch computer language in his book *Lifelong Kindergarten* (Resnick 2017). Sharing your rebuildable projects with peers also cultivates the same.

PEF estimation proceeds quickly on early parts of the data. Filling missing data is not so easy. You may need to run the above code over all the data many times. To maintain continuity on both sides of large gaps you could run the time loop backward on alternate passes. (Simply time reverse $y$ and $r$ after each pass.) To speed the code, one might capture the $t$ values that are affected by missing data, and iterate only on those.

The above code is quite easily extended to 2-D and 3-D spaces. The only complication (explained later) is the shape of PEFs in higher dimensional spaces.

I wondered if our missing data code would work in the wider world of applications—the world beyond mere signals. Most likely not. A single missing data value affects $\tau_n$ regression equations while a missing home square footage affects only one regression equation.

### 1.2.1 Old 1-D examples I have done in the stationary world

Figure 1.1 shows a nice test case. The result shown solved for missing data given the PEF of $(1, 0, -1)$. When I solved for the optimal PEF (many years ago), both it and the interpolated data was much the same as shown herein. The conclusion to draw is that PEF interpolation preserves the character of the given data, unlike linear or cubic interpolation.

Figure 1.2: Another problem of missing data with unknown PEF that I once solved is copied in Figure 1.2. It clearly shows interpolation beyond aliasing. I took it from page 197 of the 2012 version of *GIEE*. A sinusoid was sampled densely on the left and sparsely on the right. Toward the right, the interpolated function is well sampled despite widely separated data values, i.e., data sampled beyond aliasing.
Figure 1.2: Simultaneous estimation of PEF and stationary missing data (taken from the 2012 version of GIEE, on page 197) (Claerbout)

1.2.2 Epsilon

An application parameter like \( \epsilon \) requires some practitioner to choose its numerical value. This choice is best rationalized by making sure \( \epsilon \) is free from physical units. Let us now attend to that. From the past of \( y \), the filter \( a \) predicts the future of \( y \), so \( a \) itself must be without physical units. The data \( y_t \) might have units of voltage. Its prediction error \( r_t \) has the same units. To repair the units in \( \epsilon \) we need something with units of voltage squared for the denominator. Let us call it the variance \( \sigma_y^2 \). You can compute it globally for your whole data set \( y \), or you can compute it by leaky integration (such as \( \sigma_y^2 \leftarrow 0.99 \sigma_y^2 - 0.01 y_t^2 \)) to adjust itself with the nonstationary changes in data \( y_t \). The filter update \( \Delta a \) with a unit-free \( \epsilon \) is

\[
\Delta a = -\frac{\epsilon r_t}{\sigma_y^2} d
\]

That is the story for \( \epsilon_A \) in the code above. For the missing data adaptation rate, \( \epsilon_Y \), no normalization is required because \( r(t) \) and \( y(t) \) have the same physical units; therefore the missing data \( y_{t-\tau} \) updates scale from the residual \( r_t \) by the unit-free \( \epsilon_Y \).

Epsilon \( \epsilon \) is the fractional change to the filter at each time step. In a process called “leaky integration,” any long-range average of the filter at time \( t \) is reduced by the \((1 - \epsilon)\) factor; then it is augmented by \( \epsilon \) times a current estimate of it. After \( \lambda \) steps, the influence of any original time is reduced by the factor \((1 - \epsilon)^\lambda\). Setting that to \( 1/e = 1/2.718 \) says \((1 - \epsilon)^\lambda = 1/e \). Taking the natural logarithm, \( 1 = -\lambda \ln(1 - \epsilon) \approx \lambda \epsilon \), so to good approximation

\[
\epsilon = 1/\lambda
\]

By the well known property of exponentials, half the area in the decaying signal appears before the distance \( \lambda \)—the other half after.

I often think of the memory function \((1 - \epsilon)^t\) as a rectangle function of length \( \lambda \). Least squares analysis begins with the idea that there should be more regression equations than unknowns. Therefore \( \lambda \) should roughly exceed the number of filter coefficients \( n_{\tau} \). To avoid overfitting, I suggest beginning with \( \lambda = 100 \times n_{\tau} \).

There is a pitfall in the paragraph above. With synthetic data, you may have runs of zero values. These do not count as data. Then, you need a bigger \( \lambda \) because the zeros do not provide the needed information.

Mathematicians are skilled at dealing with the stationary case. They are inclined to consider all residuals \( r_t \) to carry equal information. They may keep a running average \( m_t \)

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of a residual $r_t$ by the identity (proof by induction)

$$m_t = \frac{t-1}{t} m_{t-1} + \frac{1}{t} r_t = \frac{1}{t} \sum_{k=1}^{t} r_k$$  \hspace{1cm} (1.6)$$

This equation suggests an $\epsilon$ decreasing proportional to $1/t$ (which is like $\lambda$ proportional to $t$) may in some instances be a guide to practice although it offers little guidance for nonstationarity other than that $\epsilon$ should drop off less rapidly than does $1/t$.

Given an immense amount of data, a “learning machine” should be able to come up with a way of choosing the adaptivity rate $\epsilon$. But, besides needing an immense amount of data, learning machines are notoriously fragile. We should try conjuring up some physical/geometric concepts for dealing with the kind of nonstationarity that our data exhibits. With such concepts we should require far less data to achieve more robust results. We need examples to fire up our imaginations.

### 1.2.3 Opportunities in a time-variable epsilon

Examine the logic leading up to Equation [1.2]. Recall that $d$ is the chunk of data lying under the filter; therefore $d$ is actually a function of time $d_t$. The logic of equation [1.2] says we should assure $|1 - \epsilon (d_t \cdot d_t)| < 1$. But, for any fixed $\epsilon$ there will surely be moments in time when $|d_t|$ is large enough to violate this inequality. Instead of taking an $\epsilon$ constant in time we might consider this time-variable epsilon $\epsilon_t$.

$$\epsilon_t = 2 \eta / (d_t \cdot d_t) \quad \text{for} \quad 0 < \eta < 1$$  \hspace{1cm} (1.7)$$

However, since $\epsilon_t$ may now change rapidly with time, the overall process becomes a dynamical system in which behavior is difficult to anticipate, though easy enough to try out. The possibilities are even greater. Eta $\eta$ could be a function of lag $\tau$. Is there an opportunity here? We don’t know. If I were coding it today, I would stick with the constant epsilon defined previously, but I would have a running cap on epsilon that $\epsilon (d_t \cdot d_t)$ not be allowed to exceed $1/2$.

### 1.2.4 How can the nonstationary PEF operator be linear?

Formally, finding the PEF is $a = \text{argmin}_a (Ya)$ subject to $a_0 = 1$, while using it is $r = Ay$. The combination is a nonlinear function of the data $y$. But it is nearly linear. Notice that $A$ could have been built entirely from spatially nearby data, not at all from $y$. Then $A$ would be nonstationary, yet a perfectly linear operator on $y$.

I am no longer focused on conjugate-direction solutions to stationary linear problems, but if I were, I could at any stage make two copies of all data and models. The solution copy would evolve with iteration while the other copy would be fixed and would be used solely as the basis for PEFs. Thus, the PEFs would be changing with time while not changing with iteration, which makes the optimization problem a linear one, fully amenable to linear methods. In the spirit of conjugate gradients (as it is commonly practiced), on occasion we might restart with an updated copy. People with inaccurate adjoints often need to restart. (Ha ha.)
1.2.5 My favorite wavelet for modelers

I digress to view current industrial marine wavelet deconvolution. Because acoustic pressure vanishes on the ocean surface, upcoming waves reflect back down with opposite polarity. This reflection happens twice, once at the air gun (about 10 meters deep) and once again at the hydrophones yielding roughly a second finite-difference response called a “ghost.”

The Ricker wavelet, a second derivative of a Gaussian, is often chosen for modeling. Unfortunately, the Gaussian function is not causal (not vanishing before \( t = 0 \)). A more natural choice derives from the Futterman wavelet \((GIEE)\) which is a causal representation of the spectrum \(\exp(-|\omega|t/Q)\) where \(Q\) is the quality constant of rock. Figure 1.3 shows the Futterman wavelet and also its second finite difference. I advocate this latter wavelet for modelers because it is solidly backed by theory; and I often see it on data.

![Futterman wavelet and its second finite difference](image)

Figure 1.3: The causal constant \(Q\) response and its second finite difference. It explains why the water bottom could seem a Ricker wavelet (second derivative of a Gaussian) while the top of salt would seem a doublet. That third lobe is really small. The first two lobes are approximately the same height, but the middle lobe has more area. (Claerbout)

1.2.6 Sparse decon

Antoine Guitton \((Guitton and Claerbout, 2015)\) analyzed five data sets getting amazing results on all five. Two are shown in Figures 1.4 and 1.5. The clarity of polarity in every case is wonderful for geologic interpretation. (UnFortunately, Antoine and I did not appreciate the Futterman effect when we made the sparse deconvolutions.) Guitton’s examples were done with a stationary theory that allows the inverse shot wavelet being slightly noncausal. Unfortunately for us (and many others before us) PEFs lose their whiteness when extended non causally. I think we should try to spike not at the wavelet onset (which is what PEFs do), but somewhere more like the center lobe of the Gaussian or the second lobe of the second derivative of the Futterman.

We have the ability to seek sparseness as Guitton did with the hyperbolic penalty function. His work did not extend readily to wider shot-hydrophone separations, a feature we have with our current nonstationary approach. If we could get this all together, we could reasonably hope to make millions. Let us begin guessing!
For each travel time depth $\tau = z/v$, we need the phase correction of $\exp(-|\omega|\tau/Q)$. Is that easy (like Stolt migration) or harder like downward continuation? Hmm.

REFERENCES

Figure 1.4: Gulf of Mexico. Top is before sparse decon, bottom after. Between 2.25s to 2.70s, the right side is salt (no reflectors). Notice salt top reflection is white, bottom black. Notice that sparse decon has eliminated bubble reverberation in the reflection-free salt zone (as well as elsewhere). (Antoine Guitton)
Figure 1.5: Offshore west Australia. Notice how the sparse decon creates many events that are pure white or pure black. White denotes a hard reflector, black a soft one.
Chapter 2

Spatial deconvolution

2.1 TIME AND SPACE

A streaming 1-D prediction filter is a decaying average of earlier prediction filters, however these earlier filters need not be all saved in memory. Since they vary smoothly we may simply use the most recent one. Call it $\bar{a}$. In two dimensions $\bar{a}$ becomes some average of its previous value on each of its two axes. For example instead of updating from the old filter $a(t - \Delta t, x)$, we could update from the old filter located at $a(t, x - \Delta x)$. That would be learning over $x$ while filtering over $t$. More generally, an update could leap from a base that is a weighted average over time and space. We would update

$$a \leftarrow \bar{a} + \Delta a$$

with

$$\bar{a} = a(t - \Delta t, x) \frac{\lambda_t^2}{\lambda_t^2 + \lambda_x^2} + a(t, x - \Delta x) \frac{\lambda_x^2}{\lambda_t^2 + \lambda_x^2}$$

(2.1)

Notice that the weights sum to unity. The averaging region is an area roughly $\lambda_x\lambda_t$ pixels squared in size. The coding requires not only saving $a$ at the previous time, it requires a saved $a$ for every time at the previous $x$, namely at $x - \Delta x$.

In 3-D it looks like we will need a plane of saved PEFs. In higher dimensional spaces we need store PEFs only in the zone of the transition from the filtered to the unfiltered. Thus in 5-D we need to store a 4-D volume of PEFs. Don’t let that trouble you though. Since the PEFs are generally smoothly variable they can be linearly interpolated from a sparse mesh. PEFs on the previous trace $a(t, x - \Delta x)$ can be smoothed symmetrically on the time axis so the region of averaging expands from the quadrant behind $(t, x)$ to the halfspace behind $x$.

Stationary decon should remove a shot waveform. Nonstationary decon starts from there but has the added possibility of removing the waveform of the propagating wave. It evolves with traveltime ($Q$ and forward scattered multiples). It also evolves with space, especially shot to receiver offset. We could build such nonstationary filters on either marine data or synthetic data, then apply them to marine data. The relations among pressure, velocity, upcoming, and downgoing waves vary systematically with offset. You could work out theoretical expressions for these relations, but instead you could see how this data fitting code would handle it.

\footnotesize
\begin{footnotesize}
\[\text{1Drawn from Fomel et al. (2016).}\]
\end{footnotesize}
2.1.1 Bubble removal

The internet easily yields slow-motion video of gun shots under water. Perhaps unexpectedly, the rapidly expanding exhaust gas bubble soon slows, then collapses to a point, where it behaves like a second shot, repeating again and again. This reverberation period (the interval between collapses) for exploration air guns (“guns” shooting bubbles of compressed air) is here about 120 milliseconds. Imagers hate it. Interpreters hate it. Figure 2.1 shows marine data and a gapped PEF applied to it. It is a large gap, 80 ms (milliseconds), or 80/4=20 samples on data sampled at 4 ms, say $\Delta a = (1, 0, 0, \text{more zeros}, 0, a_{20}, a_{21}, \cdots, a_{80})$.

Figure 2.1: Debubble done by the nonstationary method. Original (top), debubbled (bottom). On the right third of the top plot, prominent bubbles appear as three quasi-horizontal black bands between times 2.4 and 2.7 seconds. Blink overlay display makes it more evident that there is bubble removal in many locations. (Joseph Jennings)
2.1.2 Two dimensional PEF

We have seen 1-D PEFs applied to 2-D data. Now for 2-D PEFs. Signal analysis extends to image analysis quite easily except for the fact that the spike on the PEF is not in the middle or on a corner of the 2-D filter array but on its side. This old knowledge is summarized in the appendix: Why 2-D PEFs have white output.

Figure 2.2: A PEF is a function of lag $a(t_l,x_l)$. It is lain backwards here so we see it crosscorrelating seismic data with $t$ down, $x$ to the right. On the filter $\tau$ runs up, $x$ runs left.

( Claerbout [chap2/. pef2-d ])

Unlike the 1-D code, here we use negative subscripts on time. As in 1-D, the PEF output is aligned with its input because $a(0,0)=1$. To avoid filters trying to use off-end inputs, no output is computed (first two loops) at the beginning of the $x$ axis nor at both ends of the time axis. At three locations below the lag loops $(t_l,x_l)$ cover the entire filter. First the residual $r(t,x)$ calculation (# Filter) is simply the usual 1-D convolution seen additionally on the 2-axis. Next the adjoint follows the usual rule of swapping input and output spaces. Next the constraint line preserves not only the 1.0, but also the zeros before it. Finally, the update line $a-=da$ is trivial.

\[
\begin{array}{ccc}
  a(2,2) & a(2,1) & a(2,0) \\
  a(1,2) & a(1,1) & a(1,0) \\
  a(0,2) & a(0,1) & 1.0 \\
  a(-1,2) & a(-1,1) & 0 \\
  a(-2,2) & a(-2,1) & 0 \\
\end{array}
\]

This code whitens (flattens) nonstationary spectra in the 2-D $(\omega,k_x)$-space. The local autocorrelation tends to a delta function in 2-D lag $(t_l,x_l)$-space. Everybody’s 2-D image estimations need code like this to achieve IID residuals.

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Figure 2.3 shows an old stationary example from GIEE. In the stationary case a global PEF is computed first, then it is used to fill missing data.

![Figure 2.3](chap2/ seapef90)

Figure 2.3: (left) Seabeam data of mid-Pacific transform fault. (right) After interpolation by stationary 2-D PEF. The purpose here is to guess what the ship would have recorded if there were more hours in a day. (GIEE)  

2.1.3 Example of a 2-D PEF applied to a seismic section

Figure 2.4 shows the world’s first “industrial scale” nonstationary 2-D prediction error. It was the principal product of a summer research group led by Bob Clapp ([Ruan et al., 2015](chap2/ wgstack)). It demonstrates a 2-D nonstationary deconvolution based on the idea of using a coarse mesh to store many 2-D PEFs. Someone should try it with the code above which is much simpler. This data set and others used by Antoine ([Guitton and Claerbout, 2015](chap2/ wgstack)) are available for testing the streaming ideas of this tutorial.

![Figure 2.4](chap2/ wgstack)

Figure 2.4: A CDP stack before and after 2-D decon. Its goal is to suppress predictable events. It does not attempt to give zero strength to the layers, but to give all slopes uniform strength. That’s the meaning of whiteness in $(\omega, k_x)$-space. The residual hyperbolas are called diffractions. ([Ruan et al., 2015](chap2/ wgstack))
2.1.4 Stretch matching

Sometimes we have two signals that are nearly the same but for some reason, one is stretched a little from place to place. Tree rings seem an obvious example. I mostly encounter seismograms where a survey was done both before and after oil and gas production, so there are stretches along the seismogram that have shrunken or grown. A decade or two back, navigation was not what it is now, especially for seismograms recorded at sea. Navigation was one reason, tidal currents are another. Towed cables might not be where intended. So, signals might shift in both time and space. A first thought is to make a running crosscorrelation. The trouble is, crosscorrelation tends to square spectra which diminishes the high frequencies, those being just the ones most needed to resolve small shifts. Let us take a look at a time-variable filter that best converts one signal to the other.

Take the filter \( a \) to predict signal \( x \) from signal \( y \). Either signal might lag the other. Take the filter to be two-sided, \( [a(-9), a(-8), \ldots, a(0), a(1), \ldots, a(9)] \). Let us begin from \( a(0)=1 \), but not hold that as a constraint.

\[
\begin{align*}
    r(...) &= 0. \quad \# \text{ CODE } = \text{NONSTATIONARY EXTRAPOLATION FILTER} \\
    a(...) &= 0. \\
    a(0) &= 1. \\
    \text{do over time } t \{ \quad \# \text{ r(t) = nonstationary extrapolation error} \\
        \text{do } i= -ni, ni \\
        r(t) &+= a(i) \ast y(t-i) - x(t) \quad \# \text{ forward} \\
        \text{do } i= -ni, ni \\
        a(i) &-= r(t) \ast y(t-i) \ast \epsilon \quad \# \text{ adjoint} \\
        \text{do } i= -ni, ni \\
        \text{shift}(t) &= i \ast a(i) \\
    \}
\end{align*}
\]

The last step is to convert the filters to a time shift. Here I have simply computed the moment. That would be correct if signals \( x \) and \( y \) had the same variance. If not, I leave it to you calculate their standard deviations \( \sigma_x \) and \( \sigma_y \) and scale the shift in the code above by \( \sigma_x/\sigma_y \) thus yielding the shift in pixels.

Don’t forget, if you have only one signal, or if it is short, you likely should loop over this code multiple times while decreasing epsilon.

Besides time shifting, the filtering operator has the power of gaining and of changing color. Suppose, for example that brother \( y \) and sister \( x \) each recited a message. This filtering could not only bring them into synchronization, it would raise his pitch. Likewise in 2-D starting from their photos, he might come out resembling her too much!

2.1.5 Geostatistics

Figure 2.5 illustrates using PEF technology refilling an artificial hole in an image of the Gulf of Mexico. This illustration (taken from GIEE) uses mature stationary technology. The center panel illustrates filling in missing data from knowledge of a PEF gained outside the hole. The statistics at the hole in the center panel appears and is different from the statistics of the surrounding data. Long wavelengths have entered the hole and diminish slowly in strength as they propagate away from the edges of known data. Shorter wavelengths are
less predictable and diminish rapidly to zero as we enter the unknown. Actually, it is not low frequency but narrow bandedness that enables projection far into the hole from its boundaries.

![Figure 2.5: 2-D stationary example from GIEE. CDP stack with a hole punched in it. The center panel attempts to fill the hole by methodology like here. The rightmost panel uses random numbers inverse to the PEF to create panel fill with the global spectrum while assuring continuity at the hole boundary.](chap2/WGstack-hole-fillr)

The right panel illustrates a concept we have not covered. It has the same spectrum inside the hole as outside. Nice. And, it does not decay in strength going inward from the boundaries of the hole. Nice. Before I ask you which you prefer, the central panel or the right panel, I should tell you that the right panel is one of millions of panels that might have been shown. Each of the millions uses a different set of random numbers. A statistician (like Albert Tarantola) would say the solution to a geophysical inverse problem is a random variable. The center panel is the mean of the random variable. The right panel is one realization of the many possible realizations. The average of all the realizations is the center panel.

Geophysicists tend to like the center panel, Geostatisticians tend to prefer an ensemble of solutions like the right panel. In stationary theory the center panel would be a solution to a problem like $0 \approx Fm - d$ with regularization $0 \approx Am$. The solution to the right panel uses a different regularization, $0 \approx Am - r$ where $r$ is random numbers inside the hole and zeros outside. The variance of the prediction error outside would match the variance of the random numbers inside. Got it? Good. Now it’s your turn to write a nonstationary program.

Start from my missing data program. Debug your code. Clean it up a little and give it to me. Maybe I can use it. Better yet, make me a nice illustration and your name will go in the caption. Still better, make sure your illustration and code is presented in reader rebuildable form.

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2.1.6 Gap filling

When filling a 1-D gap, I wonder if we’d get the same fill if we scanned time backwards. Stationary theory finds a PEF from the autocorrelation function. In that world the PEF of forward going data must be identical with that of backward going data. But, when it comes to filling a gap in data shouldn’t we be using that PEF going in both directions? We should experiment with this by comparing one direction to two. Would convergence run faster if we ran alternating directions? After each time scan we would simply time reverse both the input and the output, $y_t$ and $r_t$, for the next scan. In 2-D, reversal would run over both axes.

2.1.7 Rapid recognition of a spectral change

This booklet begins with the goal of escaping the strait jacket of stationarity, intending merely to allow for slowly variable spectral change. Real life, of course has many important examples where a spectral change is so rapid that our methods cannot adapt to it. Imagine you are tracking a sandstone. Suddenly you encounter a fault with shale on the other side. Permeability is blocked. This could be bad fortune or very good fortune.

Warming up to an unexpectedly precise measurement of location of spectral change consider this one-dimensional example: Let $T = 1$ and $o = -1$. The time function

$$(..., T, T, T, o, o, o, T, T, o, o, o, T, T, o, o, o, T, T, o, o, T, T, o, o, T, T, o, o,...)$$

begins with period 6 and abruptly switches to period 4. The magnitude of the prediction error running to the right is quite different from that running to the left. Running right the prediction error is about zero, but it suddenly thunders at the moment of spectral change, thunder gradually dying away again as the PEF adapts. Running left, again there is another thunder of prediction error, but this thunder is on the opposite side of the abrupt spectral change. Having both directions is the key to defining a sharp boundary between the two spectra. Let the prediction variance going right be $\sigma_{\text{right}}$ and going left be $\sigma_{\text{left}}$. The local PEF will be defined by a weighted average of the two PEFs.

$$a = \frac{\sigma_{\text{right}}}{\sigma_{\text{right}} + \sigma_{\text{left}}} a_{\text{left}} + \frac{\sigma_{\text{left}}}{\sigma_{\text{right}} + \sigma_{\text{left}}} a_{\text{right}} \quad (2.2)$$

A weight is big where the other side has big error variance. The width of the zone of transition is comparable to the duration of the PEFs, much shorter than the distance of adaptation. This is an amazing result. We have sharply defined the location for the spectral change even though the PEF estimation cannot be expected to adapt rapidly to spectral changes. Amazing! This completes your preparation for the image of Lenna, figure 2.7.

2.1.8 Boundaries between regions of constant spectrum

There is no direct application to predicting financial markets. But, with recorded data one could experiment with predictions forwards and backwards in time. Including space with time makes it more intriguing. In space there is not only forwards and backwards but sideways and at other angles. The PEF idea in 3-D (Figure 2.6) shows that sweeping a plane...
(the top surface) upward through a volume transforms an unfiltered upper half space to a filtered lower one. Whatever trajectory the sweep takes, it may also be done backwards, even at other angles.

You are trying to remove noise from the test photo of Lenna (figure 2.7). Your sweep abruptly transitions from her smooth cheek to her straight hair, to the curly fabric of her hat. To win this competition, you surely want sweeps in opposite directions or even more directions. Fear not that mathematics limits us to slow spectral transitions. The location of a sharp spectral transition can be defined by having colliding sweeps, each sweep abruptly losing its predictability along the same edge. But we haven’t got Lenna yet.

How should we composite the additional sweeps that are available in higher dimensional spaces? Obviously we get two sweep directions for each spatial dimension, but more might be possible at 45° angles or with hexagonal coordinates. I would start with an unnormalized weight from each \( i \)th pass of \( N \) passes by the inverse of a smoothing of the PEF output variance.

\[
\begin{align*}
    w_i &= 1/ \langle \sigma_i \rangle \\
    a &= \frac{\sum_i^N w_i a_i}{\sum_i^N w_i} 
\end{align*}
\]

I confess, all of this is pretty speculative. I have my doubts about averaging PEFs in different directions. In 1-D we we have some feeling for it because stationary theory tells us forward and backward PEFs are the same. Something similar might happen in 2-D, but I have doubts about merging a north-south PEF with an east-west PEF.

### 2.1.9 What physical phenomena gives the spectra of a 3-D PEF?

Although it’s clear how to fit a single 3-D PEF to data, it might not be relevant to seismic data. Waves fill a volume with pancakes, not with noodles. When I see 3-D data, \( y(t, x, y) \),
Figure 2.7: Lenna, a widely known photo used for testing engineering objectives in photometry. (Wikipedia)
I visualize it containing planes. A plane in 3-D looks like a line in both \((t, x)\) and \((t, y)\) space. It's more efficient to fit two planes each with a 2-D PEF \([a(t, x), a(t, y)]\) than with a single 3-D PEF \(a(t, x, y)\). If you have been thinking about a regularization, it now becomes two regularizations. What physical 3-D fields call for 3-D PEFs? I could guess, but this is not the time and place.

REFERENCES


Chapter 3

Fitting while whitening residuals

PEFs require a regular mesh. Data space frequently is not a regular mesh. We can invent a model space that is a regular mesh. From that regular mesh we should be able to use linear interpolation to best fit the data. That’s a central idea in this chapter.

A typical geophysical data fitting problem may be expressed as \( 0 \approx r = Fm - d \), where \( d \) is data, \( m \) is a model, \( r \) is a residual, and \( F \) is an operator representing physics. For statistical reasons (to achieve IID) we also need a PEF applied to residuals, say \( A \). You find here the code for finding the nonstationary PEF \( A \) while finding the model \( m \) in

\[
0 \approx q(m) = A(Fm - d)
\]

Later a regularization term is included. The approach here is called “streaming,” meaning that including \( A \) does not prevent streaming, that the entire data volume need not be kept in memory—it all flows through the box defined by codes here. Whether your entire process does allow streaming naturally depends on whether your \( F \) operator in \( 0 \approx Fm - d \) allows it.

It is shown in an appendix that PEFs whiten their output signals. Starting out on a model fitting project we’d like to know the PEF that whitens the ultimate fitting residuals that we don’t have yet. So, let us change the PEF at every iteration. By whitening our current residuals we whiten our final residuals. This is more proper than guessing that the final residual spectrum is the same as the data spectrum! (Theory states that iteration-variable weighting harms conjugate gradient solvers while not bothering conjugate direction solvers.) What happens to people who use suboptimal weights and PEFs? They do not use their data “efficiently” (a statistical term). We are ready to go, wondering if we might consistently outperform our forebears in iteration count as well as solution quality. Even if not, we remain enthusiastic. The tool here handles both nonstationarity and spatial aliasing, massive practical issues in seismic imaging.

3.1 Applying the adjoint of a streaming filter

Those of us with a long history of filtering think of a filter adjoint as running the filter backwards. That view arises with recursive filters whose adjoint must indeed run backwards.

---

1This chapter is largely drawn from [Claerbout] (2017).
With nonrecursive filters, such as prediction error, there is a more basic view. In a (nonrecursive) linear operator program, the inputs and outputs can be exchanged to produce the adjoint. For example the pseudocode below applies a filter $a$ to the physical residual $y$ to get a statistical (whitened) residual $r$. We get the adjoint by swapping spaces.

```plaintext
# CODE = CONVOLUTION AND ITS ADJOINT
do it= ntau, nt {
do tau = 0, na {
  if( operator itself)
    r(it) += y(it-tau) * a(tau)  # one output r(t) pulls many
  if( adjoint )
    y(it-tau) += r(it) * a(tau)  # one input r(t) pushes many
}
do tau = 0, ntau {
  q(t-tau+1) += r(t) * a(tau)  # Apply adjoint
}
do tau = 1, ntau {
  a(tau) -= y(t-tau) * r(t) * epsilon  # Update filter
}
}
```

Observe the time axis runs forward for both the operator and its adjoint $A^\ast$. It could as well run backwards.

### 3.1.1 Applying the PEF with its adjoint

The residual after PEF whitening in a simple data fitting problem is $r = A(Fm - d)$. The energy in it is $E = (m^*F^* - d^*)A^*A(Fm - d)$. The gradient of $\partial E/\partial m^*$ is $\Delta m = F^*A^*A(Fm - d)$. Here are the steps to compute it.

\[
\begin{align*}
y &= (Fm - d) \\
r &= A(Fm - d) = Ay \\
q &= A^*A(Fm - d) = A^*r \\
\Delta m &= F^*A^*A(Fm - d) = F^*q
\end{align*}
\]

Equations above are in code below for $q = A^*Ay$. Your job is to make $y$ and $F^*q$.

```plaintext
# CODE = DATA FITTING WITH PEFed RESIDUALS.
a(*) = 0; a(0) = 1.  # You compute y=Fm-d.
r(*) = 0
q(*) = 0
do t= ntau, nt {
do tau = 0, ntau {
  r(t) += y(t-tau) * a(tau)  # Make statistical residual
}
do tau = 0, ntau {
  q(t-tau+1) += r(t) * a(tau)  # Apply adjoint
}
do tau = 1, ntau {
  a(tau) -= y(t-tau) * r(t) * epsilon  # Update filter
}
}  # You apply F' to q.
```

Only the middle tau loop is new. This code is untested. Notice the code assures that $A$ and $A^*$ apply the same filter. Notice that the program also works when the time axis is run backwards. In two dimensions, either or both the axes may be run backwards. Flipping axes flips the region in which statistics are gathered.

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3.1. APPLYING THE ADJOINT OF A STREAMING FILTER

3.1.2 PEFs for both fitting and styling

Having PEF $A$ on the regularization and PEF $B$ on the fitting, the gradient is

$$\Delta m = F^\ast B^\ast Br + \epsilon^2 A^\ast Am$$

(3.6)

which may be coded as we did equations (3.2)-(3.5). Naturally, the PEF $B$ is applicable only on those data-space axes that are regularly sampled such as on a marine cable.

We have ignored preconditioning. It’s important for covering large gaps such as at cable ends. But in most applications we have more modest goals such as data sampling irregularities and gaps the size of streamer separations. Moreover, the speed of this method might render preconditioning irrelevant even on larger gaps.

3.1.3 Signal regridding

Let data $d$ lie wherever it likes sprinkled on a surface. Let $m$ be a regular mesh to which we will move the data, and let $L$ be a linear interpolation operator. Like equations (3.2)–(3.5) and (3.6) we may deduce PEFs $B$ and $A$ and the gradient $\Delta m$

$$\Delta m = L^\ast B^\ast B(Lm - d) + \epsilon^2 A^\ast Am$$

(3.7)

I like to daydream about equation (3.7) and its relationship to the land surface of the USA. Many kinds of geophysical recorders lay sparse and irregular on the ground, so the $Lm - d$ seems central to our efforts. Of course we need to flatten the sphere giving us to wonder whether PEF concepts are limited to Cartesian spaces. The land surface $m$ is somewhat smooth in the plains while rough in the mountains. Where $m$ in the plains is very smooth, $A$ must turn out to be a powerful roughener. There can be the occasional sharply defined texture in the plains, so we will want softclip($Am$) in the plains as much as in the mountains.

Have a look with Google Earth or satellite maps. In the Appalachians there is a pattern to the mountains not found in the Rockies. Follow the track from Harrisburg, Pennsylvania to Birmingham Alabama. Occasionally these rolling mountains are broken through by rivers. After the land, look at the bottom of the oceans.

Ocean bottoms are tough places to get data. Many kinds of data (and data gaps!) affect what you see on the ocean floor. Everywhere there are stories to be told, half geological, half data acquisition limitations. Awesome! Let your imagination run.

Equation (3.7) applies to 2-D and 3-D data regularization. I believe it should also be able to fill holes.

REFERENCES

Chapter 4

Multi channels = Vector-valued signals

We have done much with PEFs on scalar-valued signals. Vector-valued signals are for 3-component seismographs and the like. The idea of deconvolution with a PEF extends to multi-component signals. In ideal geometries different wave types arrive on different channels, but in real life, wave types get mixed. Pressure waves tend to arrive on vertical seismographs, shear waves on horizontals, but dipping waves corrupt each channel with the other. The main goal here is to disentangle this channel crosstalk.

Scalar blind deconvolution is widely used in the seismic survey industry. The simple information flow in the upper quarter of Figure 4.1 is pretty much what we’ve done in Chapter 1 with the addition of the Bandpass filter at the end. Oversimplifying, the idea is that earth layers have random densities (impedances) hence random echo polarities at a fine scale. This layering $z_t$ gets smeared by the source wavelet which is not an ideal impulse, instead being a mixture of air bubbles, ghosts, and weathered-layer reverberations leading to the observed output $y_t$. Those corrupting processes amount to causal filters, best undone with a PEF producing the output $r_t$. The bandpass filter at the end is there for subjective reasons, mainly we don’t want to clutter our view with the highest possible frequency that a mesh can hold because we know it is just noise. A popular alternative to the bandpass filter is gapping the PEF. Instead of limiting high frequencies, it does much the same by broadening the autocorrelation spike of the “white” output.

Widespread adaptation of multicomponent recorders leads to new opportunities indicated by the lower bulk of Figure 4.1. Hypothetical statistically independent channels $z_1$ and $z_2$ become colored making our ideal unpolluted channels $x_1$ and $x_2$ which unfortunately “crosstalk” before giving us our observations $y_1$ and $y_2$. Learning here the theory of matrix valued PEFs, we design a matrix of filters, say $A = a_{ij}$ attempting to achieve the original purity of $z$. Normally we do not wish to achieve the pure whiteness of $z$. Rather than apply a bandpass filter here, we use our estimates $\hat{b}_{11}$ and $\hat{b}_{22}$ to find $\hat{x}$ as our attempt to restore the original colored signals $x$.

Others may make other choices, but we are choosing to display $\hat{x}$ for a reason. We want tests of whether our method works in practice. If it does, we can expect to see the S-wave

---

1 This chapter draws from [Claerbout and Wang 2017](#).
channel coming out lower frequency than the P-wave channel. This because the earth acts as a wavelength filter. It’s generally believed the earth dissipates waves proportional to their spatial frequencies. Cutting both P and S at the same spatial frequency implies S will cut off at a lower temporal frequency than P because its velocity is lower. The scalar wave equation explains it $\omega^2 = v^2 k^2$.

The multichannel structure of Figure 4.1 arises in diverse physical settings. Not only does the earth contain pressure waves and shear waves while we measure vertical and horizontal motions. Additionally ocean bottom recordings contain pressure as well as three component velocity sensors. It is useful to extract upgoing from downgoing waves. Since pressure and velocity are sensed in different but overlapping frequency bands, the idea of $b_{11}$ and $b_{22}$ having different passbands is another valuable aspect of this model.

Fourier analysis suggests a crude approach to Figure 4.1. For scalar waves, given the spectrum $Y(\omega)^{*} Y(\omega)$ the solution to the problem is $A(\omega) = 1/\sqrt{Y(\omega)^{*} Y(\omega)}$. But this implies a symmetric function of time, not causal. Fourier space requires stationary statistics, forbids $\ell_1$-norm. The square root of a matrix of Fourier functions is easily found, but the disadvantages of Fourier space are overwhelmed by the simplicity of the time domain. Causality is easily expressed with $Z$-transforms, equivalently either as a matrix of polynomials or as a polynomial of matrix coefficients.

Figure 4.1: Top is scalar decon. Bottom is vector decon. In nature two uncorrelated white random signals $z$ get colored creating $x$ which then get mixed creating our observations $y$. Vector decon converts $y$ to uncorrelated white signals $r$ which hopefully are a reasonable approximation to $z$. If $r \approx z$, then $AB \approx I$, so recoloring $r$ without mixing gives us $\hat{x}$, which should match the original colored signals $x$. (Kaiwen Wang)
4.1 No time delays please

This mathematical model applies to one point in space where it is based on causality and simultaneity of the two channels responding to the world around. The two-component signal model here is not suitable for two scalar signals recorded at separate locations. At separate locations there naturally would be time delays between the locations. If the underlying model \( \mathbf{B} \) were to introduce delay, its hypothetical inverse \( \mathbf{A} \) would need to contain inverse delay (anti-causality!). Since \( \mathbf{A} \), a PEF, is casual by construction, it cannot function anti-causally. Whatever \( \mathbf{A} \) would come out of this process, it could not satisfy \( \mathbf{BA} = \mathbf{I} \). In other words, there are many ways \( \mathbf{B} \) could contain delays without changing its covariance \( \mathbf{BB}^* \). Our inverse operator \( \mathbf{A} \) is fundamentally based on \( \mathbf{BB}^* \) which contains no phase. We get phase by insisting on causality in \( \mathbf{A} \).

If you are processing a string of multicomponent recorders (down a well, for example) each multicomponent recorder yields statistics which may be shared and averaged with neighboring recorders, but the signals themselves do not mix. The process described here is simply a vector-valued, time-variable linear operator. The same process could be independently applied to other channels.

Delay causes the method of this paper to fail in principle. In marginal cases (tiny delay) the notion of sparsity has helped for scalar signals ([Claerbout and Guitton](2013)). There is an example in Chapter 1. Minuscule delays are a promising area beyond our present scope. Differential equations apply to a point in space. Their finite difference representations cover slightly more than a point. There may be some ticklish but promising aspects of merging such notions.

The multichannel model would seem to extend to three and more physical dimensions though we'll never know until we try. Whether it is suitable for many-channel market signals I cannot predict.

4.1.1 Vector signal scaling

When components of data or model are out of scale with one another, bad things happen: The adjoint operator will not be a good approximation to the inverse. Physical units may be contradictory. Steepest descent creeps along slowly. These dangers would arise with vector-valued signals if the observations \( y_1 \) and \( y_2 \) had different physical units such as pressure and velocity recorded from up- and down-going waves. Or such as uncalibrated vertical and horizontal seismograms.

We need to prepare ourselves for channels being out of scale with one another. Thus we scale each component of data \( \mathbf{y} \) and residual \( \mathbf{r} \) by dividing out their variances. Recall that any component of a gradient may be scaled by any positive number. Such scaling is merely a change in coordinates.

With scalar signals we updated using \( \Delta \mathbf{a} = -\left(\epsilon \sigma_y^2\right) y_{t-\tau} \). With multiple channels we will be a bit more cautious and allow for data variance to differ from prediction-error variance. More importantly, the two components of \( \mathbf{y} \) might have differing physical units. Let \( \sigma_r \) be an estimate of the standard deviation of the prediction error in each channel. The
code below resembles this update

\[ \Delta a = - \left( \frac{\epsilon \sigma_r \sigma_y}{\sigma_r \sigma_y} \right) y_{t-r} \] (4.1)

Our original code contained leaky integrations for \( \sigma_y \) and \( \sigma_r \), but we had no vision of data to test that aspect. It also gave odd behavior when we adapted too rapidly. Since we had more pressing areas to direct our attention to, the code exposition below simply replaces \( \sigma_y \) and \( \sigma_r \) by their global averages.

### 4.1.2 Pseudo code for vector signals

Compared with earlier pseudocode for scalar signals where the gradient is a scaled adjoint, the gradient here has divided out the variances \( \sigma_r \) and \( \sigma_y \). That because we may always scale gradient components by positive numbers, say \( \text{sig}_y \) and \( \text{sig}_r \). Look at the code below for the four do loops following Happy streaming. You see a matrix full of PEFs at work. The three loops next below the PEF filtering are simply its adjoint (except for the complication of the \( \sigma_r \) and \( \sigma_y \) scaling) something you easily recognize by the interchange of inputs and outputs, \( r \) and \( a \).

```python
# CODE = PREDICTION ERROR FOR VECTOR SIGNALS
#
integer it, nt=1000, tau, ntau=10, gap=0, ic, jc, nc=2
real y(nc,nt), r(nc,nt), aa(nc,nc,na), sige(nc), sigy(nc), eps
e (*,*) = 0.
aa(*,*,*) = 0.
do ic=1,nc {
    aa(ic,ic,0) = 1.  # Make a 2x2 identity matrix.
}
read input y(nc,nt) # Read multichannel data.
#
do ic=1,nc {
    # Initial variance estimates.
    sumsq=0
    do it=0,nt
        sumsq += y(ic,it)**2
    sigy(ic) = sqrt(sumsq/nt)
    sigr(ic) = sigy(ic)/2.
    }
    # Here we go! Happy streaming. Wheee!
do it= ntau, nt {
    do tau=1,ntau { # lag axis.
        do ic =1,nc { # Take a signal vector into a filter matrix.
            do jc =1,nc { #
                r(ic,it) += aa(ic,jc,tau) * y(jc, it-tau)
            }
        }
        # Optionally update sigy and sige
        do tau=gap+1, ntau { # adjoint = r * y' (outer product)
            do ic= 1, nc { #
                do jc= 1, nc { #
                    aa(ic,jc,tau) -= eps * (r(ic,it)/sigr(ic)) * ( y(jc, it-tau) /sigy(jc))
                }
            }
        }
    }
}
```

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4.1. NO TIME DELAYS PLEASE

Now it’s easy to say that the code above is really quite trivial, but I breathed a sigh of relief when Kaiwen showed me the first results. (It worked on the first try!) Before I conceived the calculation as explained above, I had quite a struggle attempting the derivative of a quadratic form by a matrix filter, even more doubts that I’d be able to explain my analysis to other people, and a debt to Mohammed Hadidi whose derivation showed that my derivative was the transpose of the correct one. Then I tried thinking carefully about Figure 4.1. But, better not to think at all, instead simply code the modeling, its adjoint, and stuff in the residual! Phew.

4.1.3 The PEF output is orthogonal to its inputs

Let us try to understand what this program has accomplished. If the program ran a long time in a stationary environment with a tiny $\epsilon$ the filter $\mathbf{A}$ namely $\mathbf{a}(\bullet,\bullet,\bullet)$ would no longer be changing. The last line of the code would then say the residual $\mathbf{r}(\mathbf{ic, it})$ is orthogonal to the fitting functions $\mathbf{y}(\mathbf{jc, it-tau+1})$. We’d have a square matrix full of such statements. The fitting functions are all channel combinations of the shifted data. That’s the main ingredient to Levin’s whiteness proof for scalar signals in Chapter 5. I believe this means we can presume Levin’s whiteness proof applies to vector signals. As we’ll see a bit later, however, the situation at zero lag does bring up something new (Cholesky, etc).

4.1.4 Restoring source spectra

White signals are not ideal for display. Before corruption from channel 2, channel 1 had the spectrum of $\mathbf{b}_{11}$. Consider restoring to the white output $\mathbf{r}_1$ the original spectrum, namely $\mathbf{b}_{11}$. Since $\mathbf{B} = \mathbf{A}^{-1}$ we can deduce $\mathbf{b}_{11}$.

$$
\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \quad (4.2)
$$

Under the assumption that the crossover filters are less significant than the pass-through filters we may simplify the result for initial trials.

$$
\begin{align*}
\mathbf{b}_{11} & = \frac{a_{22}}{(a_{11}a_{22} - a_{21}a_{21})} \approx \frac{1}{a_{11}} \\
\mathbf{b}_{22} & = \frac{a_{11}}{(a_{11}a_{22} - a_{21}a_{21})} \approx \frac{1}{a_{22}}
\end{align*} \quad (4.3) \quad (4.4)
$$

I believe we should test the stability of the simplified approximation before thinking of adding in the complicating terms. We do this by appending some code to our SEP 170 code. The result of polynomial division $\hat{x}(Z) = r(Z)/A(Z)$ is recognizable in the code by $\hat{x}_t = \hat{x}(ichan, t)$. Here is the polynomial division code fragment.

```python
# CODE = Polynomial division
xhat(1,t) = r(1,t)                      # xhat1(Z) = r1(Z)/a11(Z)
do tau=1,ntau  # xhat1(Z) = r1(Z)/a11(Z)
xhat(1,t) -= aa(1,1,tau) * xhat(1,t-tau)

xhat(2,t) = r(2,t)                      # xhat2(Z) = r2(Z)/a22(Z)
do tau=1,ntau  # xhat2(Z) = r2(Z)/a22(Z)
xhat(2,t) -= aa(2,2,tau) * xhat(2,t-tau)
}
```
We’ve been doing this for some time with no stability problems yet.

### 4.2 CHOLESKY DECORRELATING AND SCALING

The two independent channels of unit-variance random numbers in \( r \) entering filter \( B \) in Figure 4.1 have the identity matrix \( I \) as a covariance. Here we arrange to have the same identity covariance for the values \( r \) exiting from \( A \) on the right.

By construction, the multicomponent PEF output chews up nonzero lagged correlations within and among channels. By construction, it does not chew up correlations among channels at zero lag. With two components we are left at the zero lag with a nice \( 2 \times 2 \) matrix of prediction-error variances \( W \).

\[
W(\tau = 0) = \begin{bmatrix}
\sigma^2_{r_{11}} & \sigma^2_{r_{12}} \\
\sigma^2_{r_{21}} & \sigma^2_{r_{22}}
\end{bmatrix} \approx \begin{bmatrix}
(r_1 \cdot r_1) & (r_1 \cdot r_2) \\
(r_2 \cdot r_1) & (r_2 \cdot r_2)
\end{bmatrix}
\]

(4.5)

Consider the expectation (leaky sum over time) \( E[rr^*] \). Theoretically it’s a 3-D function of lag and the two channels. We’re going to assume our PEFs are perfect so that it is no longer a function of lag. Thus we presume that \( E[rr^*] \) is like the \( W \) we computed with equation (4.5) at zero lag \( \tau \).

Use the Cholesky method to factor \( W \) into a triangular matrix \( V \) times its transpose, so \( W = VV^* \). (The Cholesky method is nearly trivial: Write a triangular matrix of unknown elements. Multiply it by its transpose. Notice a sequential method that unravels the unknown elements.)

\[
W = VV^*
\]

(4.6)

\[
V^{-1}W(V^*)^{-1} = I
\]

(4.7)

\[
CWC^* = I
\]

(4.8)

where we have defined \( C = V^{-1} \). Using this new matrix operator \( C \) we get a new vector signal \( q \).

\[
q = Cr
\]

(4.9)

The expectation of this new variable \( q \) is

\[
E[qq^*] = E[Crr^*C^*] = CE[rr^*]C^* = CW C^* = I
\]

(4.10)

(4.11)

(4.12)

This proves Cholesky meets our goals: (1) it descales, and (2) it decorrelates \( r \) at zero lag.

### 4.3 ROTATING FOR SPARSITY

Intriguing is what comes last, something wholly unfamiliar. Even after solving the problem posed in Figure 4.1 the solution is unique only within an arbitrary unitary matrix. (With scalar signals the arbitrariness is in a scale factor \( e^{i\phi} \).) We get to choose the \( U \) having
4.3. ROTATING FOR SPARSITY

Minimum entropy \( r \) output. Unexpected! Intriguing! Luckily, this two channel problem, although nonlinear, is easily amenable to a one-parameter exhaustive search. That search can be done to maximize sparsity of the final signals. We humans love the simplest representation of our data. This should be it. Hooray!

The most intriguing part of the entire process arrives at this the last stage. As the universe marches on, things get mixed and entropy increases. We seek the opposite.

Rotations and reflections are called unitary operators. For now we are ignoring reflections (polarity changes). (Consider that to be an application labeling issue.) Scanning a single parameter \( \theta \) through all angles allows us to choose the one with the most sparsity (least clutter). A general form for a \( 2 \times 2 \) rotation operator is

\[
U = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \tag{4.13}
\]

We will meet our goal of finding \( A \) and \( r \) of Figure 4.1 with:

\[
r = Uq = UCr = UCEy = Ay \tag{4.14}
\]

A unitary operator \( U \) does not change the length of any vector. It satisfies \( U^*U = I \), so for any \( v \) we see \( (Uv)^*Uv = v^*U^*Uv = v^*v \). Let us check that the covariance of \( r = Uq \) is constant independent of \( \theta \). Equation (4.12) leads to \( rr^* = U E[qq^*] U^* = UIU = I \). This says the energy stays constant as we sweep through \( \theta \).

4.3.1 Finding the angle of maximum sparsity (minimum entropy)

Given any angle \( \theta \) for equation (4.13) we have \( r = Uq \). We can scan \( \theta \) over one degree increments. Defining the entropy at any particular time as \( (|r_1| + |r_2|)/\sqrt{r_1^2 + r_2^2} \) we easily choose the angle of minimum entropy for that time. We may define the entropy for the entire time range of the signal as

\[
\text{Entropy}(\theta) = \frac{\sum_{t=1}^{\infty} |r_1(t)| + |r_2(t)|}{\sqrt{\sum_{t=1}^{\infty} r_1^2(t) + r_2^2(t)}} \tag{4.15}
\]

Since the denominator should be a constant function of \( \theta \) we may as well define entropy by

\[
\text{Entropy}(\theta) = \sum_{t=1}^{\infty} |r_1(t)| + |r_2(t)| \tag{4.16}
\]

Why the scan works

Why does this \( U \) process of scanning \( \theta \) lead to sparsity? Suppose the vector signal element \( q_N \) at time at \( t = N \) has all its energy in its first component. Say the vector signal is \([-1, 0]^*\) with energy and magnitude both now equal unity. The rotated signal is now

\[
\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\cos \theta \\ \sin \theta \end{bmatrix} \tag{4.17}
\]

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Let the rotation angle be $45^\circ$ so sine and cosine are both $1/\sqrt{2}$. The sum of the magnitudes becomes $2/\sqrt{2} = \sqrt{2} > 1$. As expected the rotation took away the original sparsity.

We experimented with taking the matrix $U$ to be time variable. There are some pitfalls we are not yet prepared to explain.

### 4.3.2 3-component vector data

For 3-component vectors the scan would run over two angles so the $u(itheta)$ would be expanded to $u(itheta,iphi)$.

### 4.3.3 Channel order and polarity

Although our first synthetic data had the strongest pressure wave on the first channel, our first successful run yielded the pressure wave on the second channel. The channel flip operation is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$  \hfill (4.18)

Now we flip channels when we find the expression $|r_1 \cdot y_1| + |r_2 \cdot y_2| < |r_1 \cdot y_2| + |r_2 \cdot y_1|$. Our initial P-wave result had a flipped polarity. The operation for flipping the polarity for channel 1 is

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$  \hfill (4.19)

We change the polarity of channel 1 when $(y_1 \cdot r_1) < 0$ and likewise for channel 2.

It is easy to show for signals with an identity $I$ correlation matrix, the channel flip and polarity change operations do not change the $I$ correlation matrix.

It is easy to imagine situations where flip and polarity should change with time. For example, there may be more than two wave types present. One may die out while another grows. We have not yet synthesized such data for testing and are unclear how we might proceed. We will, no doubt, be strongly influenced by the data at hand.

### 4.4 RESULTS OF KAIWEN WANG

Figure 4.2 is our first test data, synthetic data with a vertical component and a horizontal component. Both a P wave and an S wave are emerging at a fairly steep angle so the vertical is mostly a P is corrupted by a little S, while on the horizontal it is the opposite.

On Figure 4.3 we notice that the spike estimates become sharper and sharper with time as the filter $A$ adapts with time. Oddly, there is some crosstalk on the P channel that does not seem to be diminishing with time. I don’t know why that is. Perhaps I should run the program over the panel a zillion times, at each end, capturing the filter and reinstalling it at the beginning.
4.4. RESULTS OF KAIWEN WANG

Figure 4.2: Synthetic data input is vertical and horizontal components. Model is a mix of sharp, unipolar P waves and S waves of lower frequency with alternating polarity. Stronger P waves on the vertical, and stronger S waves on the horizontal. (Kaiwen Wang)  

Figure 4.3: Output results: Deconvolved P wave on vertical component (top), S on horizontal (bottom). Spiking improves with time. (Kaiwen Wang)

Figure 4.4: V=vertical, H=horizontal. The traces P and S are overlays of the original impulsive waves and their attempted reconstruction from (V,H). The pulses get sharper with time as the PEFs adapt. (Kaiwen Wang)
On Figure 4.4 the P and S channels contain two signals, the original spikes, and their estimates. We notice that crosstalk nearly diminishes to zero on the P channel, likewise on the S channel.

Figure 4.5 is like figure 4.4 but denser spikes, a spike every 4 pixels, each spike topped by a small circle. Vertical lines mostly connect to the dots. Ideally, between the dots are vertical lines of zero height, the nonzero height exhibiting the limitations of the overall process.

Notice the vertical trace (top in upper panel) being dominated by P waves is higher frequency than the horizontal trace “H” (top in lower panel) which is dominated by S waves. Results are about the same quality as Figure 4.4 proving that having much wavelet overlap creates no real problems. Fitting on the S channel (bottom in lower panel) gets much better with time. Fitting on the P channel is so good near the beginning that we hardly notice improvement with time.

Figure 4.5: The top panel refers to the vertical motions V and the pressure waves P. The second signal in that panel is a superposition of the sparse original impulses (tiny circles) that made the data and the pulses as estimated by the entire process. These should match. They mostly do match, but small nonzero values appear between the dots. The lower panel is likewise for the horizontal H seismograph and the S wave (Kaiwen Wang)

REFERENCES


Evolving document. Save the link, not the PDF. March 15, 2018
5.1 WHY PEFs HAVE WHITE OUTPUT

It is somewhat intuitive that 1-D PEFs have a white output, but it is really amazing that 2-D PEFs tend to spectral whiteness in a 2-D space, yet this is extensively demonstrated in GIEE \cite{Claerbout2014}, while it is simply introduced and proven here now.

5.1.1 Why 1-D PEFs have white output

The basic idea of least-squares fitting is that the residual is orthogonal to each of the fitting functions. Applied to the PEF this idea means the output of the PEF is orthogonal to lagged inputs. The orthogonality applies only for lags in the past, because prediction knows only the past while it aims to the future. What we soon see here is different; namely, the output is uncorrelated with itself (as opposed to the input) for lags in both directions; hence the autocorrelation of the output is a delta function and the output spectrum is white. Knowing the PEF and having output whiteness has many applications.

Let \( \mathbf{d} \) be a vector with components containing a time function. Let \( Z^n \mathbf{d} \) represent shifting the components to delay the signal in \( \mathbf{d} \) by \( n \) samples. The definition of a PEF is that it minimizes \( ||\mathbf{r}|| \) by adjusting filter coefficients \( a_\tau \). The PEF output is:

\[
\mathbf{r} = \mathbf{d} + a_1 Z^1 \mathbf{d} + a_2 Z^2 \mathbf{d} + a_3 Z^3 \mathbf{d} + \cdots \tag{5.1}
\]

We set out to choose the best \( a_\tau \) by setting to zero the derivative of \( (\mathbf{r} \cdot \mathbf{r}) \) by \( a_\tau \). After the best \( a_\tau \) are chosen, the residual is perpendicular to each of the fitting functions:

\[
0 = \frac{d}{da_\tau} (\mathbf{r} \cdot \mathbf{r}) \tag{5.2}
\]

\[
0 = \mathbf{r} \cdot \frac{dr}{da_\tau} = \mathbf{r} \cdot Z^\tau \mathbf{d} \quad \text{for} \; \tau > 0. \tag{5.3}
\]

Given that \( 0 = \mathbf{r} \cdot Z^\tau \mathbf{d} \), we examine \( \mathbf{r} \cdot Z^\tau \mathbf{r} \) and see that it vanishes too. Using Equation \[5.1\], we have for any autocorrelation lag \( k > 0 \),

\[
\mathbf{r} \cdot Z^k \mathbf{r} = \mathbf{r} \cdot (Z^k \mathbf{d} + a_1 Z^{k+1} \mathbf{d} + a_2 Z^{k+2} \mathbf{d} + \cdots)
\]

\footnote{This subsection draws from \cite{Levin2013}, also included in \cite{Claerbout2014}.}
\[
    r \cdot Z^k d + a_1 r \cdot Z^{k+1} d + a_2 r \cdot Z^{k+2} d + \ldots = 0 + a_1 0 + a_2 0 + \ldots = 0.
\]

Because the autocorrelation is symmetric, \( r \cdot Z^{-k} r \) is also zero for \( k < 0 \); therefore, the autocorrelation of \( r \) is an impulse. In other words, the spectrum of the time function \( r_t \) is white. Thus, \( d \) and \( a \) have mutually inverse spectra. Because the output of a PEF is white, the PEF itself has a spectrum inverse to its input.

### 5.1.2 Why 2-D PEFs have white output

Chapter 4 in my GIEE book [Claerbout, 2014](#) extends 1-D signal analysis to 2-D and 3-D physical space. There are also many examples in GIEE Chapter 7. In summary, to visualize the 2-D notion of a 1-D PEF, wrap a long rope tightly spiraling around a silo inching down by covering many revolutions. The surface of the silo and rope coils are 2-D spaces for our 2-D imaging games. Let the silo hold the 2-D data and the rope hold the filter. Let the rope be slippery so it can slide over the silo in a 2-D space. Such sliding may be along the axis of the silo, or along the rope, or any direction in the 2-D surface.

Figure 5.1: The end of a 1-D rope wrapped on a silo. We consider only the filter coefficients inside the semicircle, outside coefficients theoretically negligible.

Figure 5.1 shows how you can think of the rope as either a 1-D or a 2-D filter. At the end of the rope, one filter coefficient is constrained to be a “1.” Filter coefficients in the semicircle near the “1” in the 2-D space are typically the most significant ones because being nearby the “1,” they most likely give the best predictions of what lies under the “1.” In principle all the coefficients outside the semicircle vanish. In practice, the nonvanishing coefficients lie in a box about the size of the semicircle.

Stew Levin points out that once you have mastered the 1-D whiteness proof, you don’t need the 2-D proof in GIEE if you know about the helix. Why? Because wrapping one side of a long, long 1-D autocorrelation spike many turns around the helix on the silo shows you a 2-D spike of an autocorrelation which implies 2-D spectral whiteness.

I don’t like proving theorems, especially those with negative consequences, but I may save you some trouble if I tell you a curious fact. If you put adjustable (by least squares) coefficients on both sides of the “1,” you spoil the whiteness of the output.
5.2 THE HEART OF NONSTATIONARY PEF USING CALCULUS

Suppose we have a PEF that represents all previous moments in time. Call it \( \bar{a} = (1, \bar{a}_1, \bar{a}_2, \bar{a}_3, \ldots) \). Say that \( \bar{a} \) represents the PEF (inverse spectrum) of the data values \( (d_1, d_2, d_3, \ldots, d_{98}) \). We seek to define the \( a \) that represents the PEF with an appended data value \( d_{99} \). Consider the regression:

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} \approx
\begin{bmatrix}
\frac{d_{99}}{\gamma} & \frac{d_{98}}{\gamma} & \frac{d_{97}}{\gamma} & \frac{d_{96}}{\gamma} \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{bmatrix}
\begin{bmatrix}
1 \\
\bar{a}_1 \\
\bar{a}_2 \\
\bar{a}_3
\end{bmatrix} - \gamma
\begin{bmatrix}
0 \\
\bar{a}_1 \\
\bar{a}_2 \\
\bar{a}_3
\end{bmatrix}
\tag{5.4}
\]

The top row says we are trying to fit a new data point \( d_{99} \). The bottom block says the new PEF \( a \) should be pretty similar to the PEF that fit earlier data, \( \bar{a} \). The parameter \( \gamma \) should be big enough that the new data point \( d_{99} \) does not change \( a \) very much. Rewrite equation (5.4) as

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} \approx
\begin{bmatrix}
\frac{d_n}{\gamma} & \frac{d_{n-1}}{\gamma} & \frac{d_{n-2}}{\gamma} \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix} - \gamma
\begin{bmatrix}
-d_{n+1} \\
\gamma \bar{a}_1 \\
\gamma \bar{a}_2 \\
\gamma \bar{a}_3
\end{bmatrix}
\tag{5.5}
\]

or, in a shortened block-matrix notation, we have the residual to minimize

\[
0 \approx r = \begin{bmatrix} d^* \\ \gamma \mathbf{I} \end{bmatrix} a - \begin{bmatrix} -d_{n+1} \\ \gamma \bar{a} \end{bmatrix},
\tag{5.6}
\]

where \( \mathbf{I} \) is the identity matrix and

\[
d = \begin{bmatrix} d_n \\ d_{n-1} \\ d_{n-2} \end{bmatrix}, \quad a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.
\]

For decades Bernard “Bernie” Widrow (Wikipedia) attacked problems of this nature by defining a quadratic form and finding its gradient. (Actually, he thinks in terms of circuit diagrams.) Then he repeatedly made small steps down the gradient (not up). How big are the small steps? Experience teaches. The quadratic form is \( r^* r \). We take its derivative to find the search direction.

\[
\Delta a = -(some \ constant) \left. \frac{\partial}{\partial a^*} \right|_{a=\bar{a}} r^* r \tag{5.7}
\]

Form the transpose of the residual (5.6) and then differentiate by \( a^* \). (By \( a^* \) we mean the complex conjugate transpose of \( a \).)

\[
\frac{\partial r^*}{\partial a^*} = \left. \frac{\partial}{\partial a^*} \right| \{ a^* [d \ \gamma \mathbf{I}] - [-d_{n+1} \ \gamma \bar{a}] \} = [d \ \gamma \mathbf{I}] \tag{5.8}
\]

---

\(^2\)This section drawn on Fomel et al. (2016) and Claerbout (2017).
and multiply that onto \( r \) from (5.6) keeping in mind that \( d^* \bar{a} \) is a scalar.

\[
\Delta a \propto \frac{\partial r^*}{\partial a^*} \ \bar{r} \quad = \quad [d \ \gamma I] \begin{bmatrix} d^* \bar{a} - \begin{bmatrix} -d_{n+1} \end{bmatrix} \end{bmatrix} \\
= \quad d(d^*a) + \gamma^2 a + dd_{n+1} - \gamma^2 a
\]

(5.9)

\[
\Delta a \propto \frac{\partial r^*}{\partial a^*} \bigg|_{a=\bar{a}} \quad = \quad (d^*\bar{a} + d_{n+1}) \ d
\]

(5.10)

\[
\Delta a \quad = \quad - \epsilon r_t \ d
\]

(5.11)

(5.12)

It is certainly surprising that the analytic solution to the regression (5.4) computationally amounts to a single step of the optimization strategy (5.11), a strategy so crude as to be absent from textbooks. Yet that is so (Fomel et al., 2016). Experimentalists will first notice that (5.4) demands we supply a not-given constant \( \gamma \) while (1.4) or (5.12) demands a not-given constant \( \epsilon \) (or \( \lambda \)).

REFERENCES