Stanford Exploration Project, Report 100, April 20, 1999, pages 251–256

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Helical meshes on spheres and cones

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keywords: *math, sphere, cone, helix, autoregression*

ABSTRACT

We embed a helix in the two-dimensional surface of a sphere; likewise, in the two-dimensional surface of a cone. This provides a one-dimensional coordinate system on a two-dimensional surface. Although mesh points are exactly evenly spaced along the helix and approximately evenly spaced in the crossline dimension, unfortunately, the angles between neighboring points are continuously changing. We seem to lose the concepts of two-dimensional autoregression that we have in cartesian space.

INTRODUCTION

Geophysical practice is filled with spheres and cones. Wave fronts are spheres. The Kirchhoff impulse response has a conical asymptote. Surface wave noises fill a conical surface. Likewise linear moveout in 3-D can be taken to be a conical surface. Because of the many applications that I found for a helix mapping of a cartesian space, I chose to examine spheres and cones. (To a small boy with a new hammer, everything looks like a nail.)

HELICAL COORDINATE ON A SPHERE

I set out to find the equations describing a rope that begins from the north pole and spirals its way around a sphere neatly covering it and ending at the south pole. Taking uniform samples along this rope gives a fairly uniform covering of the sphere. The surface of a sphere is two dimensional, but the rope gives a one dimensional covering that is uniformly sampled in that dimension. I wondered about the sampling in the other dimension so I set out to plot it. I found the "generalized spiral set" of Saff and Kuijlaars(1997). In spherical coordinates (θ, ϕ), for $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$, they set

$$\theta_k \quad = \quad \arccos(h_k), \quad h_k \; = \; -1 + \frac{2(k-1)}{(N-1)}, \quad 1 \le k \le N \tag{1}$$

$$\phi_k \quad = \quad \phi_{k-1} + \frac{3.6}{\sqrt{N}} \frac{1}{\sqrt{1-h_k^2}}, \quad 2 \le k \le N-1, \quad \phi_1 = \phi_N = 0 \tag{2}$$

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My plot of these equations is shown in Figure 1. There is no interesting pattern in the crossline direction. Although my plot looks reasonable, Saff and Kuijlaars(1997) show a curious pattern in the crossline direction that my plots do not show. A few tests with various values of N and various rotations failed to show any curious pattern.

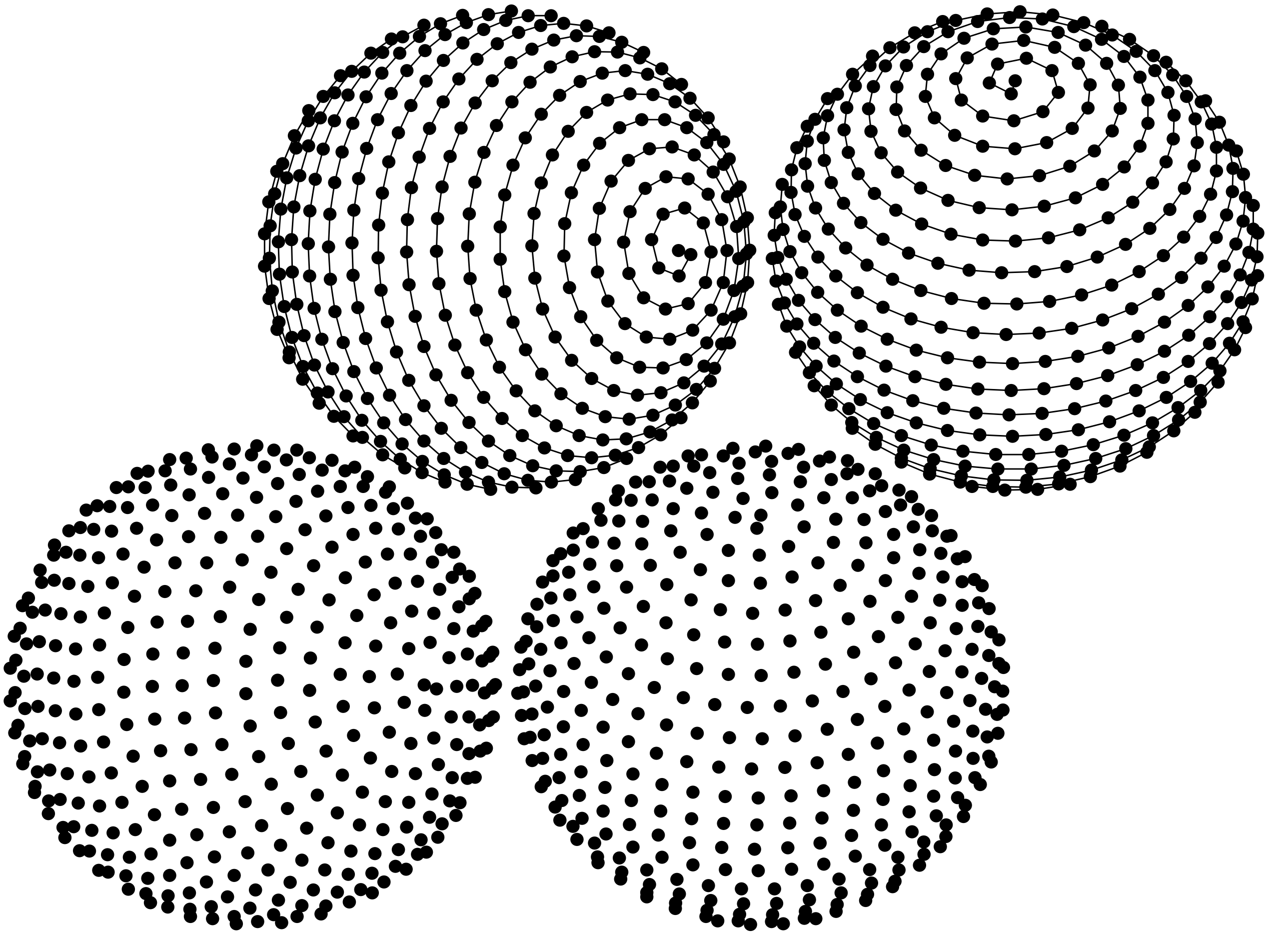

Figure 1: Helix on a sphere. Top shows the embedded helix. Bottom hides it. An interesting pattern of points that appears in the article in the Mathematical Intelligencer is inexplicably absent here (even though I tested several rotations and several values of N). jon2-sphere [ER]

HELICAL COORDINATE ON A CONE

I could not find the equations for a helix on a cone, so I derive them below. An example of results is Figure 2.

Define ϕ to be the angle from the axis of the cone to its surface. I call this the apex angle. I discovered by accident that certain apex angles give interesting patterns in the crossline directions while most do not. I found these patterns to be insensitive to the choice of N and show them here for about $N = 1400$. Another value of apex angle with an interesting

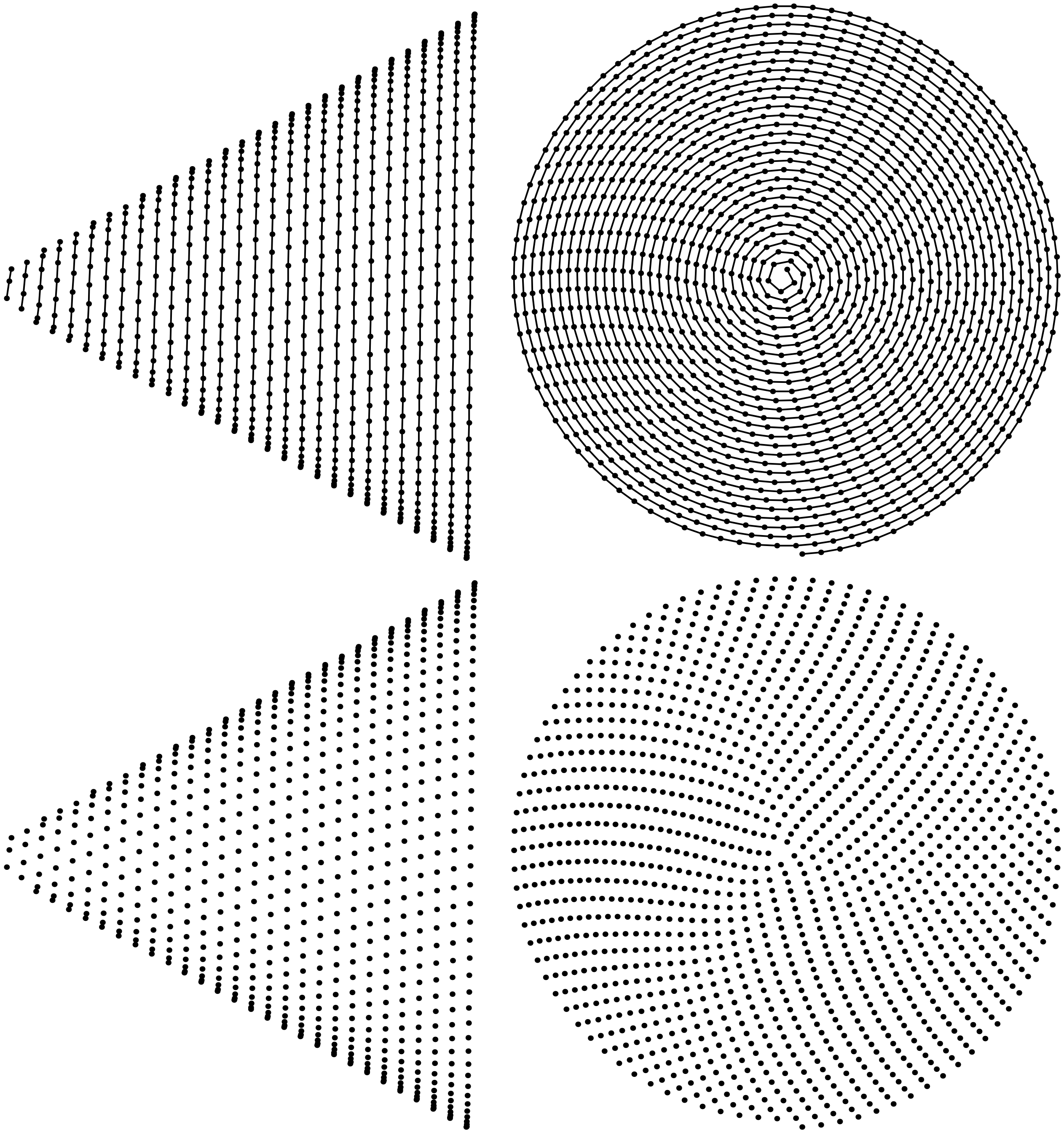

Figure 2: Helix on a cone. Top shows the embedded helix. Bottom hides it. Apex angle $\phi = 1/2$. jon2-trycart [ER]

pattern is $\phi = 1/3$. It gives the charming pattern in Figure 3. (I've seen this pattern before on a party hat. I attributed it to an ingenious artist. Now I realize that like all mathematics, this art existed before the big bang.)

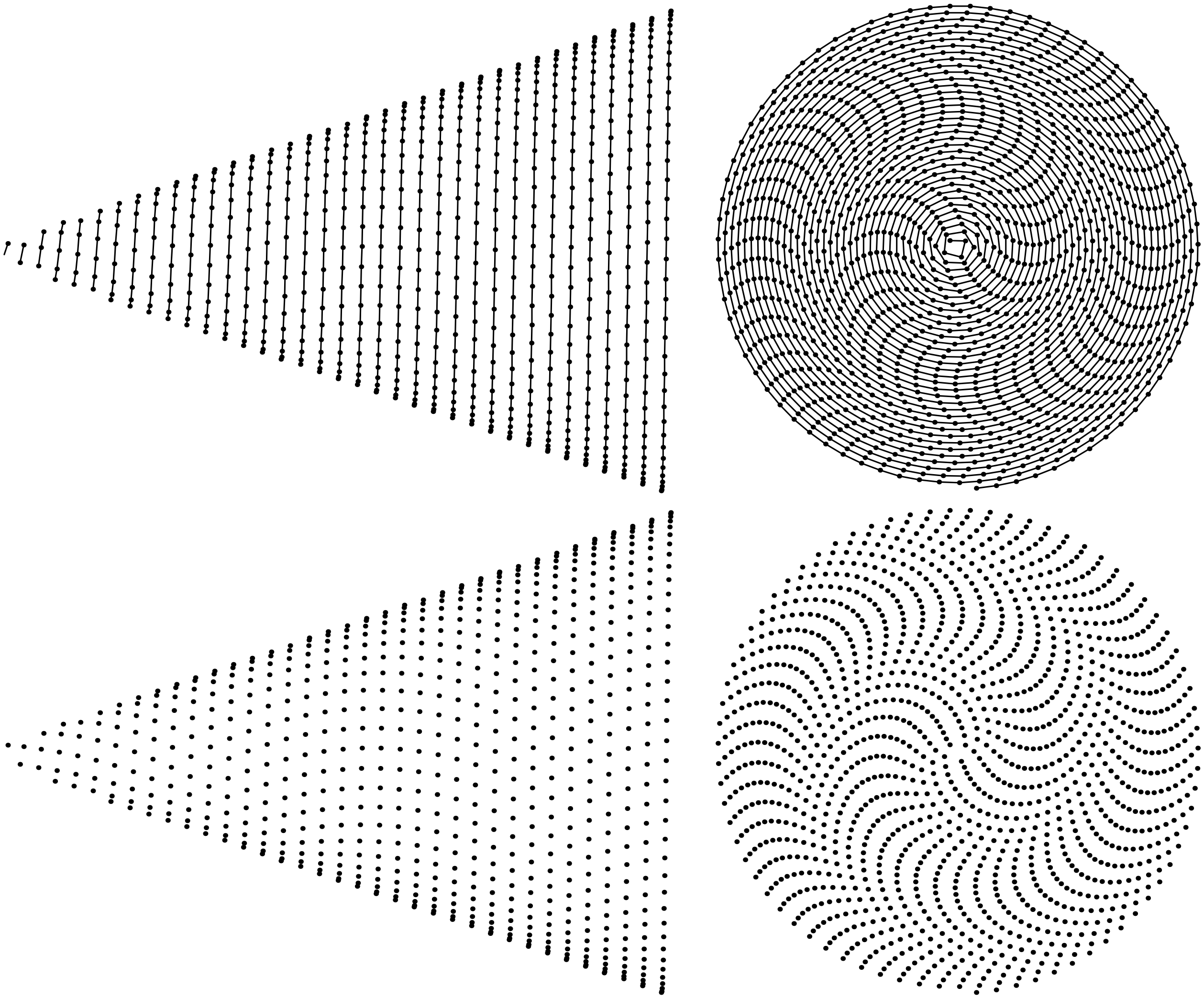

Figure 3: Helix on a cone. Top shows the embedded helix. Bottom hides it. Apex angle equals one third radian. jon2-YinYang [ER]

The radius r of the cone divided by its altitude z has a ratio $r/z = \tan\phi$ where ϕ is the angle from the axis of the cone to its surface. The area of a circle is πR^2. The surface area of a cone is $\pi R^2/\sin\phi$. Dividing the area into N cells, the surface area per cell is $\pi R^2/(N\sin\phi)$. Taking any cell area to be square, the length of a side is

$$\Delta s \quad = \quad R\sqrt{\frac{\pi}{N\sin\phi}} \tag{3}$$

The number of points running one cycle around a rim is

$$N_{\text{rim}} \quad = \quad \frac{2\pi r}{\Delta s} \quad = \quad \frac{2r}{R}\sqrt{N\pi\sin\phi} \tag{4}$$

On one cycle around the rim, the radius must change by $\Delta r = \Delta h \sin\phi$ where the hypotenuse Δh of the triangle lies on the surface of the cone. Thus, for each mesh point going around the rim

$$\Delta r = \frac{\Delta h \ \sin\phi}{N_{\rm rim}} \tag{5}$$

Since we want "square" mesh points, Δh must equal Δs, hence

$$\Delta r = \frac{R^2}{2r\,N} \tag{6}$$

The algorithm starts on the rim at $\theta = 0$ and $r = R$. At each step, update r and θ with $-\Delta r$ and $\Delta\theta = r\,\Delta s$. Stop before $r - \Delta r$ becomes negative.

FILLING VOLUMES

Nested spherical shells can fill a volume. A question is whether the "string" that winds around each shell should wind from north to south on every layer suffering a discontinuity at the poles, or whether layers of string should alternate between winding south and winding north.

Likewise cones can nest with cones, as a stack of ice-cream cones.

CONCLUSIONS

The useful property of all these fillings of space is that the spacing between the grid points is basically constant. Two-dimensional "pixels" have about equal area, likewise three-dimensional voxels have about equal volume.

Although mesh points are exactly evenly spaced along the helix and approximately evenly spaced in the crossline dimension, unfortunately, the angles between neighboring points are continuously changing.

I hope I am wrong, but this appears to defeat the many useful tricks we play in cartesian space such as using autoregression for multidimensional prediction and using prediction-error filters to characterize multidimensional spectra. Thus we cannot interpolate by expressing physical stationarity as stationarity of filter coefficients.

REFERENCES

Saff, E., and Kuijlaars, A., 1997, Distributing many points on a sphere: Mathematical Intelligencer, **19**, no. 1, 5–11.