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Spectral factorization revisited

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ABSTRACT

In this paper, we review some of the iterative methods for the square root, showing that all these methods belong to the same family, for which we find a general formula. We then explain how those iterative methods for real numbers can be extended to spectral factorization of auto-correlations. The iteration based on the Newton-Raphson method is optimal from the convergence stand point, though it is not optimal as far as stability is concerned. Finally, we show that other members of the iteration family are more stable, though slightly more expensive and slower to converge.

INTRODUCTION

Spectral factorization has been recently revived by the advent of the helical coordinate system. Several methods are reported in the literature, ranging from Fourier domain methods, such as Kolmogoroff's (Claerbout, 1992; Kolmogoroff, 1939), to iterative methods, such as the Wilson-Burg method (Claerbout, 1998; Wilson, 1969; Sava et al., 1998).

In this paper, after reviewing the general theory of root estimation by iterative methods, we derive a general square root relationship applicable to both real numbers and to auto-correlation functions. We introduce a new spectral factorization relation and show its relation to the Wilson-Burg method.

THE SQUARE ROOT OF REAL NUMBERS

This section briefly reviews some well known square root iterative algorithms, and derives the Newton-Raphson and Secant methods. It also shows that Muir's iteration for the square root (Claerbout, 1995) belongs to the same family of iterative methods, if we make an appropriate choice of the generating function.

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Root-finding recursions

Given a function $f(x)$ and an approximation for one of its roots x_n, we can find a better approximation for the root by linearizing the function around x_n

$$f(x) \approx f(x_n) + (x_{n+1} - x_n) f'(x_n)$$

and by setting $f(x)$ to be zero for $x = x_{n+1}$. We find that

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \tag{1}$$

1. **Newton-Raphson's method for the square root**

   A common choice of the function f is $f(x) = x^2 - s$. This function has the advantage that it is easily differentiable, with $f'(x) = 2x$. The recursion relation thus becomes

   $$x_{n+1} = x_n - \frac{x_n^2 - s}{2x_n} = \frac{x_n}{2} + \frac{s}{2x_n}$$

   or

   $$x_{n+1} = \frac{1}{2}\left(x_n + \frac{s}{x_n}\right)$$

   or, after rearrangement,

   $$\boxed{x_{n+1} = \frac{s + x_n^2}{2x_n}} \tag{2}$$

   The recursion (2) converges to $\pm\sqrt{s}$ depending on the sign of the starting guess $x_0 \neq 0$.

2. **Secant method for the square root**

   A variation of the Newton-Raphson method is to use a finite approximation of the derivative instead of the differential form. In this case, the approximate value of the derivative at step n is

   $$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

   For the same choice of the function f, $f(x) = x^2 - s$, we obtain

   $$x_{n+1} = x_n - \frac{x_n^2 - s}{x_n + x_{n-1}}$$

   and

   $$\boxed{x_{n+1} = \frac{s + x_n x_{n-1}}{x_n + x_{n-1}}} \tag{3}$$

   In this case, recursion (3) also converges to $\pm\sqrt{s}$ depending on the sign of the starting guesses x_0 and x_1.

3. **Muir's method for the square root**

   Another possible iterative relation for the square root is Francis Muir's, described by Jon Claerbout (1995):

$$\boxed{x_{n+1} = \frac{s + x_n}{x_n + 1}} \tag{4}$$

   This relation belongs to the same family of iterative schemes as Newton and Secant, if we make the following special choice of the function $f(x)$ in (1):

$$f(x) = |x + \sqrt{s}|^{\frac{\sqrt{s}-1}{2\sqrt{s}}} |x - \sqrt{s}|^{\frac{\sqrt{s}+1}{2\sqrt{s}}} \tag{5}$$

   Figure 1 is a graphical representation of the function f(x).

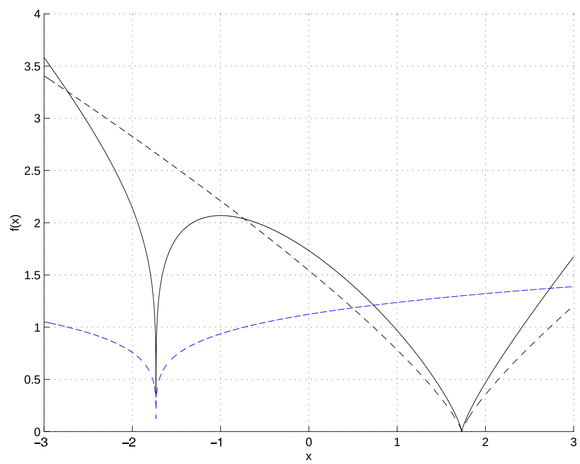

Figure 1: The graph of the function defined in Equation (1) used to generate Muir's iteration for the square root (solid line). The dashed lines are the plot of the two factors in the equation . paul3-muf [CR]

4. **A general formula for the square root**

   From the analysis of equations (2), (3), and (4), we can derive the following general form for the square root iteration:

$$\boxed{x_{n+1} = \frac{s + x_n \gamma}{x_n + \gamma}} \tag{6}$$

   where γ can be either a fixed parameter, or the value of the iteration at the preceding step, as shown in Table 1. The parameter γ is the estimate of the square root at the given step (Newton), the estimate of the square root at the preceding step (Secant), or a constant value (Muir). Ideally, this value should be as close as possible to $\sqrt{s}$.

The convergence rate

We can now analyze which of the particular choices of γ is more appropriate as far as the convergence rate is concerned.

Table 1: Recursions for the square root

	γ	Recursion
Muir	1	$x_{n+1} = \frac{s+x_n}{x_n+1}$
Secant	x_{n-1}	$x_{n+1} = \frac{s+x_n x_{n-1}}{x_n+x_{n-1}}$
Newton	x_n	$x_{n+1} = \frac{s+x_n^2}{2x_n}$
Ideal	$\sqrt{s}$	$x_{n+1} = \frac{s+x_n\sqrt{s}}{x_n+\sqrt{s}}$

If we consider the general form of the square root iteration

$$x_{n+1} = \frac{s + x_n \gamma}{x_n + \gamma}$$

we can estimate the convergence rate by the difference between the actual estimation at step $(n+1)$ and the analytical value $\sqrt{s}$. For the general case, we obtain

$$x_{n+1} - \sqrt{s} = \frac{s + \gamma x_n - x_n\sqrt{s} - \gamma\sqrt{s}}{x_n + \gamma}$$

or

$$\boxed{x_{n+1} - \sqrt{s} = \frac{(x_n - \sqrt{s})(\gamma - \sqrt{s})}{x_n + \gamma}} \qquad (7)$$

The possible selections for γ from Table 1 clearly show that the recursions described in

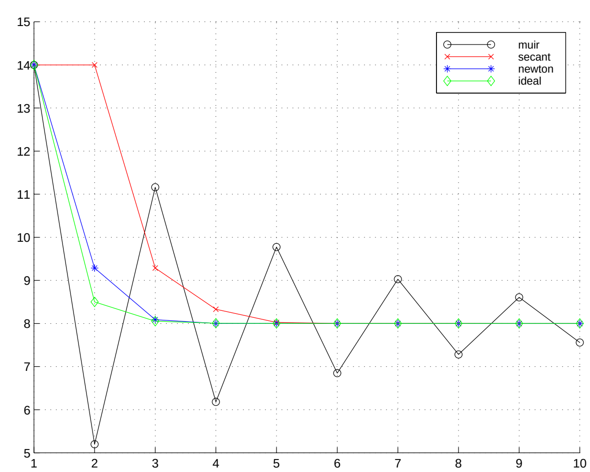

Figure 2: Convergence plots for different recursive algorithms, shown in Table 1. paul3-sqroot [CR]

the preceding subsection generally have a linear convergence rate (that is, the error at step $n+1$ is proportional to the error at step n), but can converge quadratically for an appropriate selection of the parameter γ, as shown in Table 2. Furthermore, the convergence is faster when γ is closer to $\sqrt{s}$.

Table 2: Convergence rate

	γ	Convergence
Muir	1	linear
Secant	x_{n-1}	quasi-quadratic
Newton	x_n	quadratic

We therefore conclude that Newton's iteration has the potential to achieve the fastest convergence rate. Ideally, however, we could use a fixed γ which is a good approximation to the square root. The convergence would then be slightly faster than for the Newton-Raphson method, as shown in Figure 2.

SPECTRAL FACTORIZATION

We can now extend the equations derived for real numbers to polynomials of Z, with $Z = e^{i\omega t}$, and obtain spectral factorization algorithms similar to the Wilson-Burg method (Sava et al., 1998), as follows:

$$\boxed{X_{n+1} = \frac{S + X_n \bar{G}}{\bar{X}_n + \bar{G}}} \tag{8}$$

If L represents the limit of the series in (8),

$$L\bar{L} + L\bar{G} = S + L\bar{G}$$

and so

$$L\bar{L} = S$$

Therefore, L represents the causal or anticausal part of the given spectrum $S = X\bar{X}$.

Table 3 summarizes the spectral factorization relationships equivalent to those established for real numbers in Table 1.

The convergence properties are similar to those derived for real numbers. As shown above, the Newton-Raphson method should have the fastest convergence.

A COMPARISON WITH THE WILSON-BURG METHOD

For reasons of symmetry, we can take Newton's relation from Table 3

$$X_{n+1} = \frac{S + X_n \bar{X}_n}{2\bar{X}_n}$$

Table 3: Spectral factorization

General	$X_{n+1} = \frac{S+X_n\bar{G}}{\bar{X}_n+\bar{G}}$
Muir	$X_{n+1} = \frac{S+X_n}{\bar{X}_n+1}$
Secant	$X_{n+1} = \frac{S+X_n\bar{X}_{n-1}}{\bar{X}_n+\bar{X}_{n-1}}$
Newton	$X_{n+1} = \frac{S+X_n\bar{X}_n}{2\bar{X}_n}$
Ideal	$X_{n+1} = \frac{S+X_n\sqrt{S}}{\bar{X}_n+\sqrt{S}}$

and convert it to

$$\frac{X_{n+1}}{2X_n} = \frac{S + X_n\bar{X}_n}{(2X_n)(2\bar{X}_n)}.$$

We can then consider a symmetrical relation where on the left side we insert the anticausal part of the spectrum, and obtain

$$\frac{\bar{X}_{n+1}}{2\bar{X}_n} = \frac{S + X_n\bar{X}_n}{(2X_n)(2\bar{X}_n)}.$$

Finally, we can sum the preceding two equations and get

$$\boxed{\frac{X_{n+1}}{2X_n} + \frac{\bar{X}_{n+1}}{2\bar{X}_n} = \frac{2S + X_n\bar{X}_n + \bar{X}_nX_n}{(2X_n)(2\bar{X}_n)}} \tag{9}$$

which can easily be shown to be equivalent to the Wilson-Burg relation

$$\frac{X_{n+1}}{X_n} + \frac{\bar{X}_{n+1}}{\bar{X}_n} = 1 + \frac{S}{X_n\bar{X}_n} \tag{10}$$

In an analogous way, we can take the general relation from Table 3

$$X_{n+1} = \frac{S + X_n\bar{G}}{\bar{X}_n + \bar{G}}$$

and convert it to

$$\frac{X_{n+1}}{X_n + G} = \frac{S + X_n\bar{G}}{(X_n + G)(\bar{X}_n + \bar{G})}$$

We can then consider a symmetrical relation where on the left side we insert the anticausal part of the spectrum, and obtain

$$\frac{\bar{X}_{n+1}}{\bar{X}_n + \bar{G}} = \frac{S + \bar{X}_nG}{(X_n + G)(\bar{X}_n + \bar{G})}$$

Finally, we can sum the preceding two equations and get

$$\boxed{\frac{X_{n+1}}{X_n + G} + \frac{\bar{X}_{n+1}}{\bar{X}_n + \bar{G}} = \frac{2S + X_n\bar{G} + \bar{X}_n G}{(X_n + G)(\bar{X}_n + \bar{G})}} \tag{11}$$

Equation (11) represents our general formula for spectral factorization. If we consider the particular case when G is X_n, we obtain equation (10), which we have shown to be equivalent to the Wilson-Burg formula.

From the computational standpoint, our equation is more expensive than the Wilson-Burg because it requires two more convolutions on the numerator of the right-hand side. However, our equation offers more flexibility in the convergence rate. If we try to achieve a quick convergence, we can take G to be X_n and get the Wilson-Burg equation. On the other hand, if we worry about the stability, especially when some of the roots of the auto-correlation function are close to the unit circle, and we fear losing the minimum-phase property of the factors, we can take G to be some damping function, more tolerant of numerical errors.

Moreover, by using the Equation (11), we can achieve fast convergence in cases when the auto-correlations we are factorizing have a very similar form, for example, in nonstationary filtering. In such cases, the solution at the preceding step can be used as the G function in the new factorization. Since G is already very close to the solution, the convergence is likely to occur quite fast.

CONCLUSIONS

The general iterative formula for the square root that we derived can be extended to the factorization of the auto-correlation functions. The Wilson-Burg algorithm is a special case of our more general formula. Using such a general formula provides flexibility in choosing between fast convergence and stability. We can achieve fast convergence when factorizing auto-spectra that have a very similar form. This improvement in convergence rate can have a useful application, for instance, in nonstationary preconditioning.

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