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Helix derivative and low-cut filters' spectral feature and application

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ABSTRACT

A helix derivative filter can be used to roughen an image and thus enhance its details. Unlike the conventional derivative operator, the helix derivative filter has no direction orientation. I present the enhanced helix/low-cut derivative filters, in which the zero frequency response is adjustable. I analyze the quantitative effects of the adjustable parameters on the filter spectrum and propose guidelines for choosing parameters. I also show some roughened images created by the enhanced filters.

INTRODUCTION

The derivative operator is a useful tool in removing the low frequency components to enhance the details of an image (Claerbout, 1998a). However, the conventional derivative operator has a particular derivative direction. This is not desirable in some cases where the contour map has circular features, so what needed is a derivative operator without a specific direction. One isotropic alternative, the Laplacian operator, often cuts low frequencies too strongly.

The introduction of helical coordinate system (Claerbout, 1997) gives us the ability to solve the problem. The Kolmogoroff algorithm of spectral factorization (Claerbout, 1976) enables us to derive a helix derivative filter from its autocorrelation, the negative Laplacian operator (Claerbout, 1998b). The Wilson-Burg method of spectral factorization (Sava et al., 1998) makes it more convenient to compute the coefficients of a finite-length helix derivative filter.

The helix low-cut filter is another operator useful in enhancing the digital images and also has the circularly symmetric spectrum. Unlike the helix derivative filter, the helix low-cut filter removes the frequency components below an adjustable cut-off frequency from the image, and has a flat response at high frequencies.

In this paper, I present the enhanced helix derivative and low-cut filters with more adjustable parameters. Then I analyze the influence on the filter spectrum for these parameters, and present guidelines for choosing the parameters for image processing, using the examples

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from GEE.

ENHANCED HELIX FILTERS

The autocorrelation of the helix derivative filter $\mathbf{H}$ is the negative of the finite-difference representation of the Laplacian operator ∇^2,

$$\mathbf{H}^{'}\mathbf{H} = \mathbf{R} = -\nabla^2 \tag{1}$$

The coefficients of the causal helix filter $\mathbf{H}$ can be found by spectral factorization.

The helix low-cut filter H/D is designed by doing two spectral factorizations, one for the numerator of H, and another for the denominator of D. It is expressed by

$$\frac{k^2}{k^2+k_0^2} \approx \frac{-\nabla^2}{-\nabla^2+k_0^2} \approx \frac{\overline{H}H}{\overline{D}D} \tag{2}$$

where k is the frequency [2], k_0 affects the cut-off frequency, $\overline{H}$ and $\overline{D}$ are the conjugate anti-casual filters of H and D.

Both the filters do not remove the zero frequency completely, degrading the contrast and details of the roughened image. A way to solve this problem as suggested by Claerbout, is to rescale all the coefficients of H with nonzero lags by a. If s is the sum of all the coefficients with nonzero lag (which are all negative), a is expressed by

$$a = 1 + \left(\frac{1}{|s|} - 1\right)\rho \tag{3}$$

$\rho = 0$ denotes the original unscaled filter, while $\rho = 1$ guarantees the filter really removes the zero frequency component.

Now I have the enhanced helix derivative with adjustable parameters n_a and ρ, the enhanced helix low-cut filter with n_a, k_0 and ρ. Here n_a is the half length of the helix filter.

Compared with the conventional helix filters, the enhanced filters have a new adjustable parameter, ρ.

HELIX DERIVATIVE FILTER

First I apply the enhanced helix derivative filter to some familiar images and check the effects of adjustable parameters n_a and ρ.

Figure 1 shows the views of the Sea of Galilee created by helix derivative filters with different n_a and ρ. The top two plots (with $\rho = 0$) indicate that as the half filter length n_a increases, the contrast of the image increases. This means the longer filter keeps less

[2]The frequencies in this paper are all scaled frequencies, π is the Nyquist frequency.

low frequency components, indicating a lower zero-frequency response. The vertical plots (with same n_a) indicate that as ρ increases, the contrast increases also, as expected from the definition of the enhanced helix filter equation.

To compare the properties of the filters with different n_a other than the zero-frequency response, I set $\rho = 1$ in the bottom two plots, so that they have the same zero-frequency response. The plots are very similar, and the difference between them is very weak.

As the examples above show, the zero-frequency response of the enhanced helix derivative filter decreases as n_a or ρ increases. Thus filters with different n_a may create the same results by adjusting ρ, which provides the possibility of using a short filter instead of a long filter in image processing and reduce the computational cost greatly. By adjusting ρ, the enhanced short filter can reach a low level of zero-frequency response, which is done conventionally by using a long filter.

Since the short filter is equivalent to the long filter when adjusting ρ, now ρ plays the key role in image processing for the enhanced helix derivative filter, and n_a is not an important parameter. When the high frequency component is too weak compared with the low frequency, a large ρ (≈ 1) should be chosen in order resolve details in the image. Otherwise, a small ρ would be suitable. Based on the balance of the computation cost and response symmetry, I recommend the use of $n_a \approx 8$.

From the quantitative analysis of spectra of the enhanced helix derivative filter (see the appendix A), I can find ρ according to empirical formula

$$\rho = 1 - \frac{I_0 n_a}{0.44} \tag{4}$$

where I_0 is the factor of the zero-frequency component to be preserved.

For the view of the Sea of Galilee, the high frequency component is very weak, so I chose $I_0 = 0.002$, $n_a = 8$, and $\rho = 0.96$. Figure 2 shows the preferred and gradient roughened results.

HELIX LOW-CUT FILTER

I check the effects of adjustable parameters k_0, n_a and ρ on roughened images. The quantitative analysis of the effects on the filter spectrum is provided in appendix A.

When comparing the effects of k_0 and n_a, I set $\rho = 1$ to make sure the filters have the same zero-frequency response. Figure 3 shows the Bay Area map created with different k_0. As k_0 increases, more low-frequency components were removed and the detailed structure turns out to be the main focus of the map. This indicates the cut-off frequency increases as k_0 increases. In other words, k_0 governs the cut-off frequency. When k_0 remains the same, filters with different n_a can create very similar results if the zero-frequency response is the same by adjusting ρ, as shown in the middle and bottom plots in Figure 4. As expected, ρ controls the zero-frequency response. The larger ρ leads to higher contrast, as shown in the top and middle plots in Figure 4.

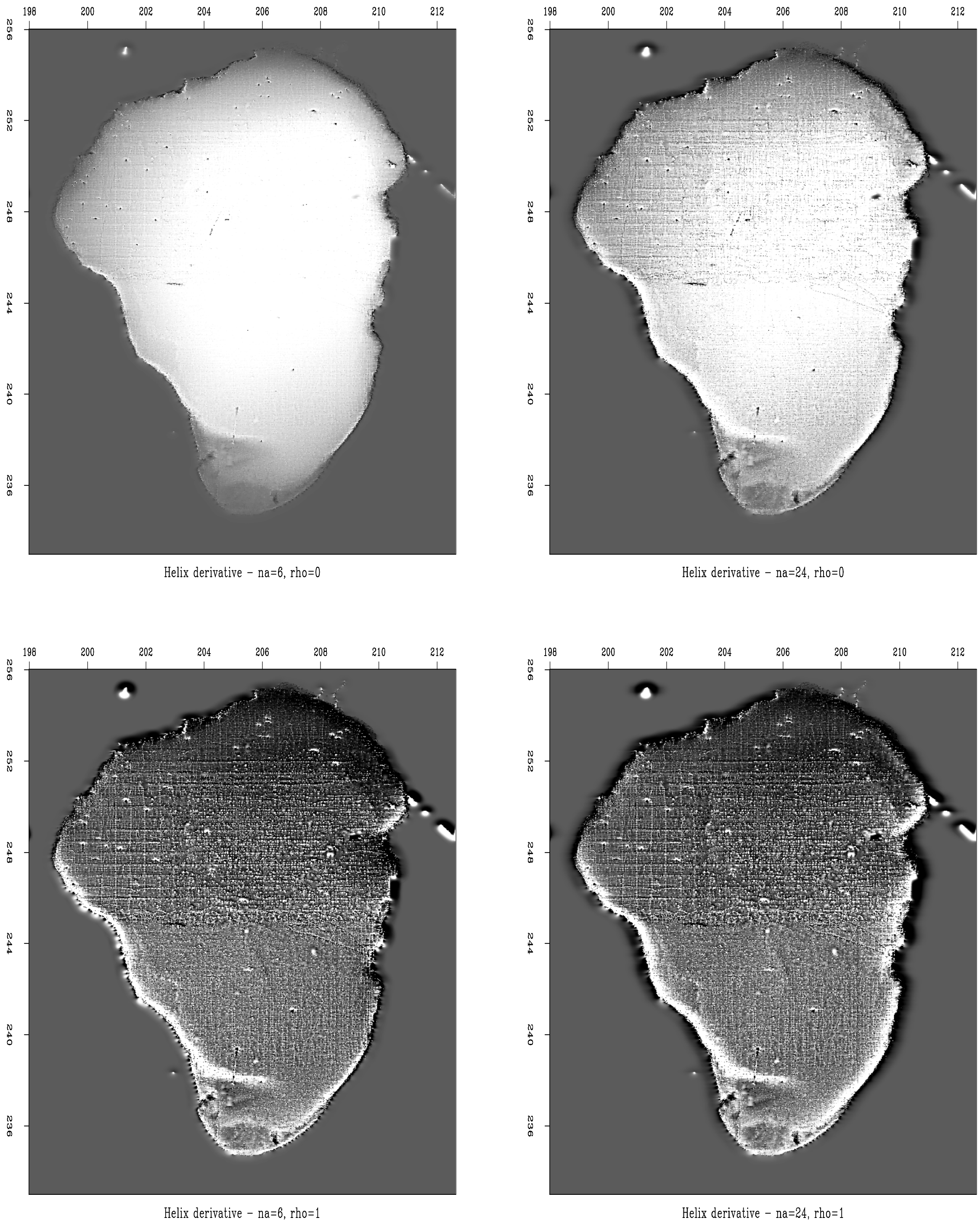

Figure 1: The Views of the Sea of Galilee roughened with helix derivative filters. Each two horizontal plots have the same ρ; each two vertical plots have the same n_a. As n_a or ρ increases, the zero-frequency response decreases. yi1-gal-drv-ar [ER]

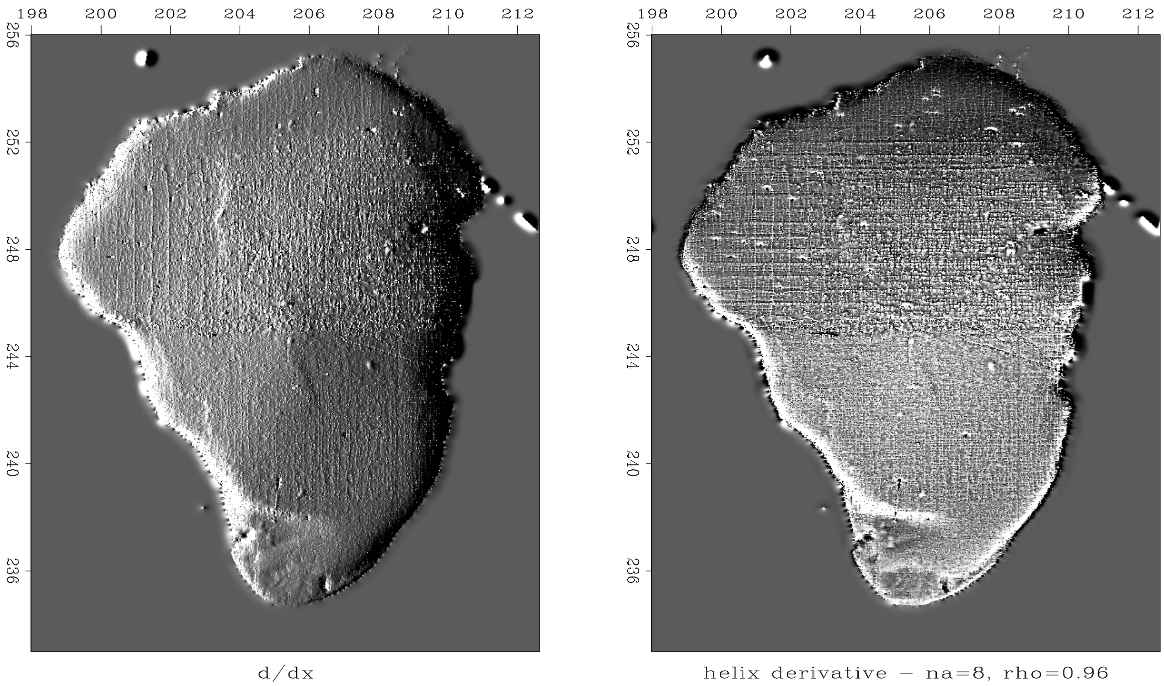

Figure 2: The views of the Sea of Galilee. The left is with the gradient operator d/dx; the right is with the helix derivative filter. The helix filter enhances the details both along the vertical and horizontal direction. yi1-gal-drv-res [ER]

Among the three adjustable parameters, n_a is the least important one because the long filter can be replaced with a short one by adjusting ρ. k_0 controls the cut-off frequency and ρ controls the zero-frequency response. It is hard to tell which one is more important if I use only this information. From the quantitative analysis, I know k_0 affects zero-frequency response significantly, but ρ does not have such an influence on cut-off frequency. So k_0 is the most important parameter.

For the enhanced helix low-cut filter, it is very reasonable to choose parameter k_0 first, then ρ and n_a.

When roughening the image with the helix low-cut filter, the key point is to choose the cut-off frequency f_0 or the adjustable parameter k_0. My suggestion is that if the lowest frequency component to be preserved is f_L, k_0 should be

$$k_0 = \frac{2}{3} f_L \tag{5}$$

Therefore, the frequency far below f_L would be cut off completely, and the component near f_L would not be affected too much.

Since the short filter is made equivalent to the long filter by adjusting ρ, I suggest the use of $n_a \approx 16$ based on the balance of computation costs and symmetric features.

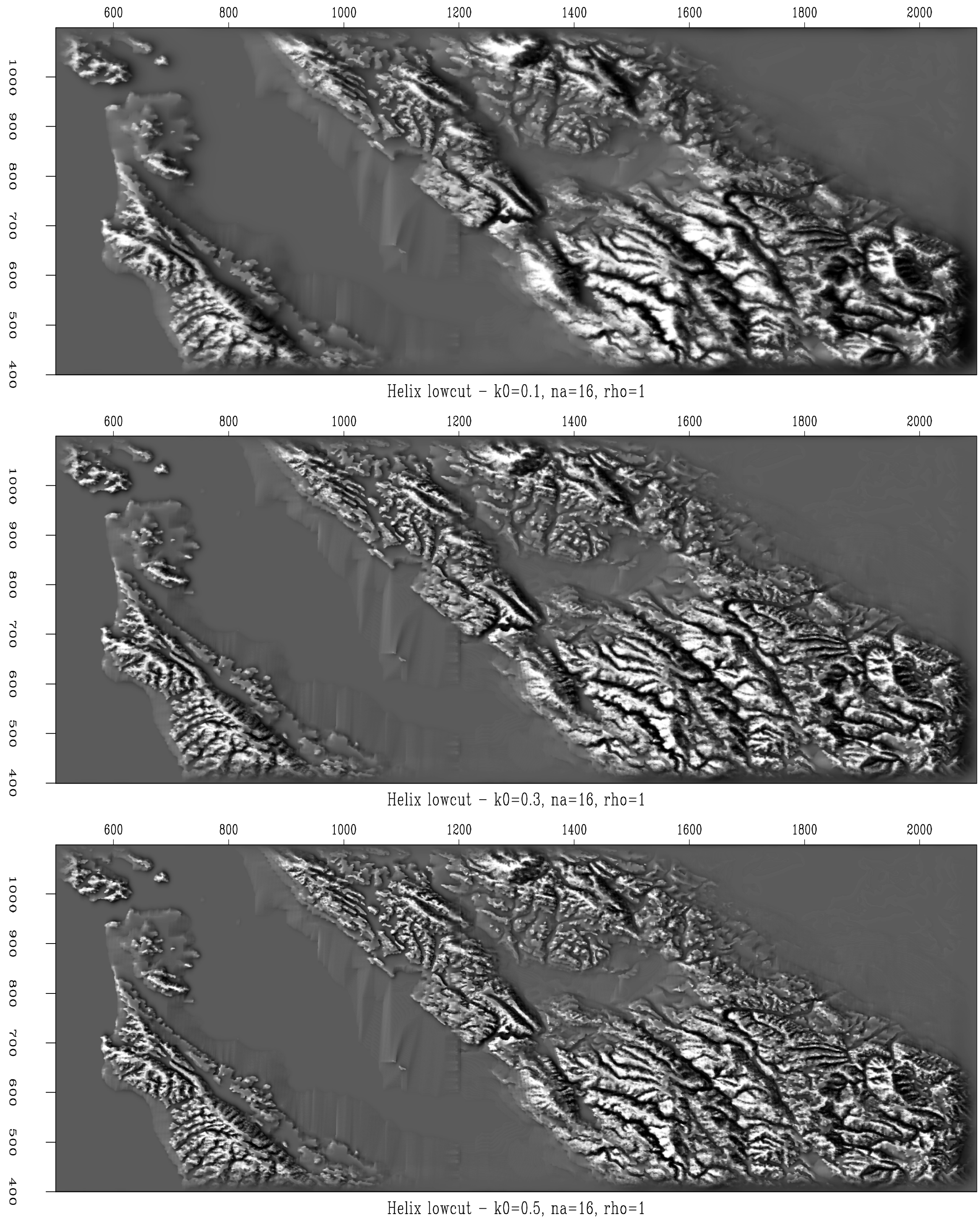

Figure 3: Bay Area maps roughened by helix low-cut filters with different k_0. The top is $k_0 = 0.1$, the middle is $k_0 = 0.3$, the bottom is $k_0 = 0.5$. $n_a = 16$, $\rho = 1$. As k_0 increases, the cut-off frequency increases. yi1-bay-lct-k-a16r1 [ER]

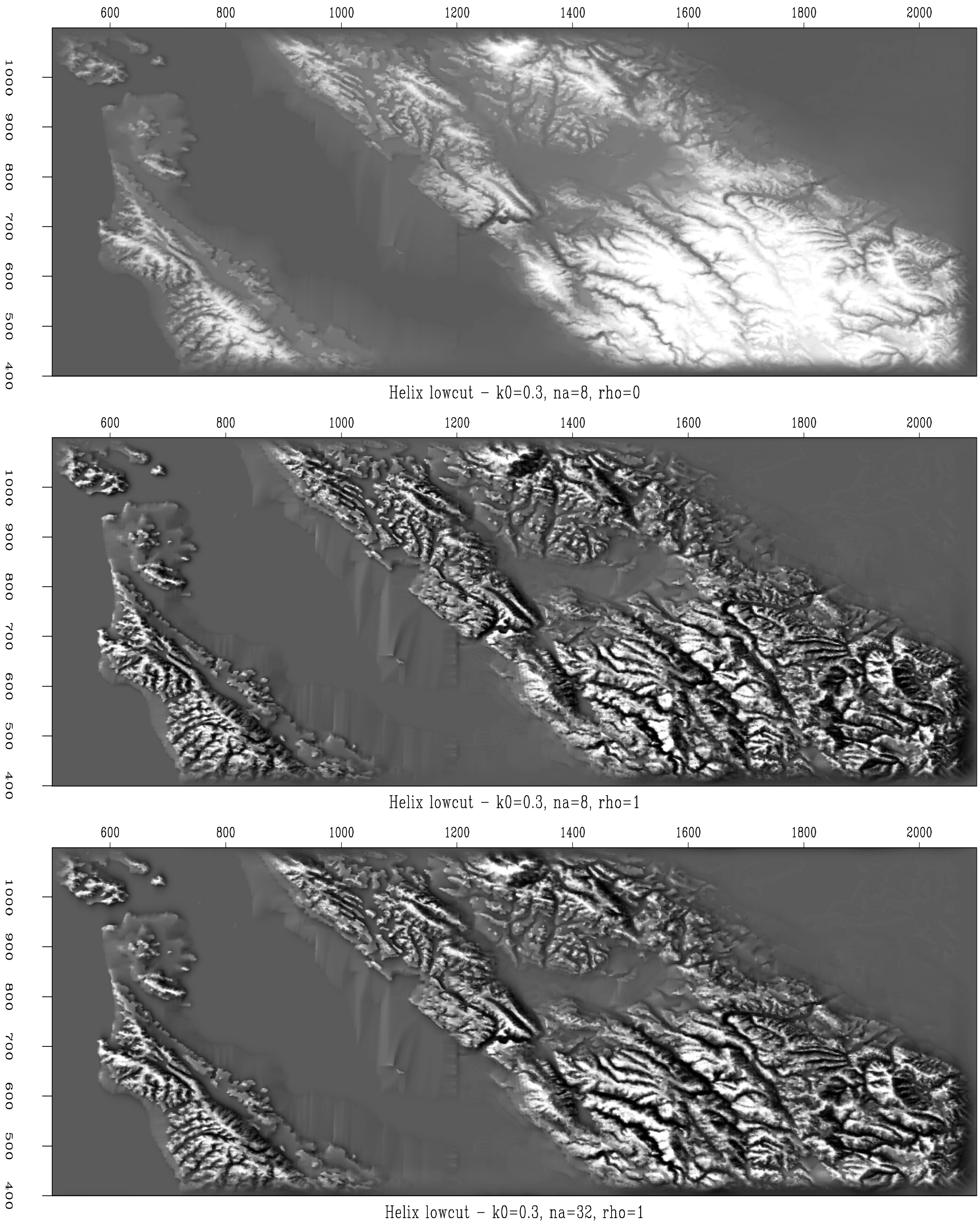

Figure 4: Bay Area maps roughened by helix low-cut filters with different filter length. The top is $n_a = 8$ and $\rho = 0$, the middle is $n_a = 16$ and $\rho = 1$, the bottom is $n_a = 32$. $k_0 = 0.3$, $\rho = 1$. $k_0 = 0.3$. As ρ increases, the zero-frequency response decreases. When the zero-frequency response remains the same, the difference of n_a does not affect results. yi1-bay-lct-ar-k3 [ER]

With k_0 and n_a determined, I can find ρ according to Equation (A-7)

$$\rho = 1 - \frac{I_0 n_a k_0}{0.82} \tag{6}$$

If $\rho = 1$, the zero-frequency is removed completely and leads to the highest contrast in the roughened image.

Figure 5 consists of three maps of the Bay Area. The top portion is a topographic map of the Bay Area. The bottom plot is the preferred result. From one slice of the Bay Area topographic map, I know one main low frequency component is about 0.4. So I chose $k_0 = 0.3$ to remove the lower frequency. I chose $n_a = 16$. In order to obtain the highest contrast, I chose $\rho = 1$. The middle one is the reference plot with $k_0 = 0.1$, $n_a = 16$ and $\rho = 1$. I notice that the bottom plot removes more low frequency components than the middle one and has clearer details, as predicted by the theory.

Figure 6 consists of a normal mammogram and the roughened images. The main low frequency component of the mammogram slice is about 0.3, so I choose $k_0 = 0.2$ in the right plot as the preferred result, and use $k_0 = 0.1$ in the middle as reference. I use $\rho = 1$ to achieve the highest contrast.

DERIVATIVE VERSUS LOW-CUT

Both helix derivative and low-cut filters can be used to enhance the details of images. But which one is better? It depends on the situation.

- Focus of the structure

  Both helix derivative and low-cut filters cut off the zero-frequency component, but the functions of the frequency are different. The derivative filter's response is approximately a linear function of frequency, while the low-cut filter's response has a hole below the cut-off frequency and is flat above it. This leads to the main difference between derivative and low-cut filters: the derivative filter enhances the small-scale structures more, while the low-cut filter makes medium-scale structure much clearer.

  In Figure 2, a long line structure in the middle of the sea is very clear in the plot created by $\frac{d}{dx}$ operator; in the plot created by helix derivative filter, it is too weak to be seen. The helix low-cut filter preserves this structure quite well, as shown in Figure 7.

- Computational cost

  Another difference is the cost of image processing:

  1. The helix low-cut filter needs to do the deconvolution with **D** besides the convolution with **H**;
  2. The worse spectral symmetry leads helix low-cut filter to the use of a long filter.

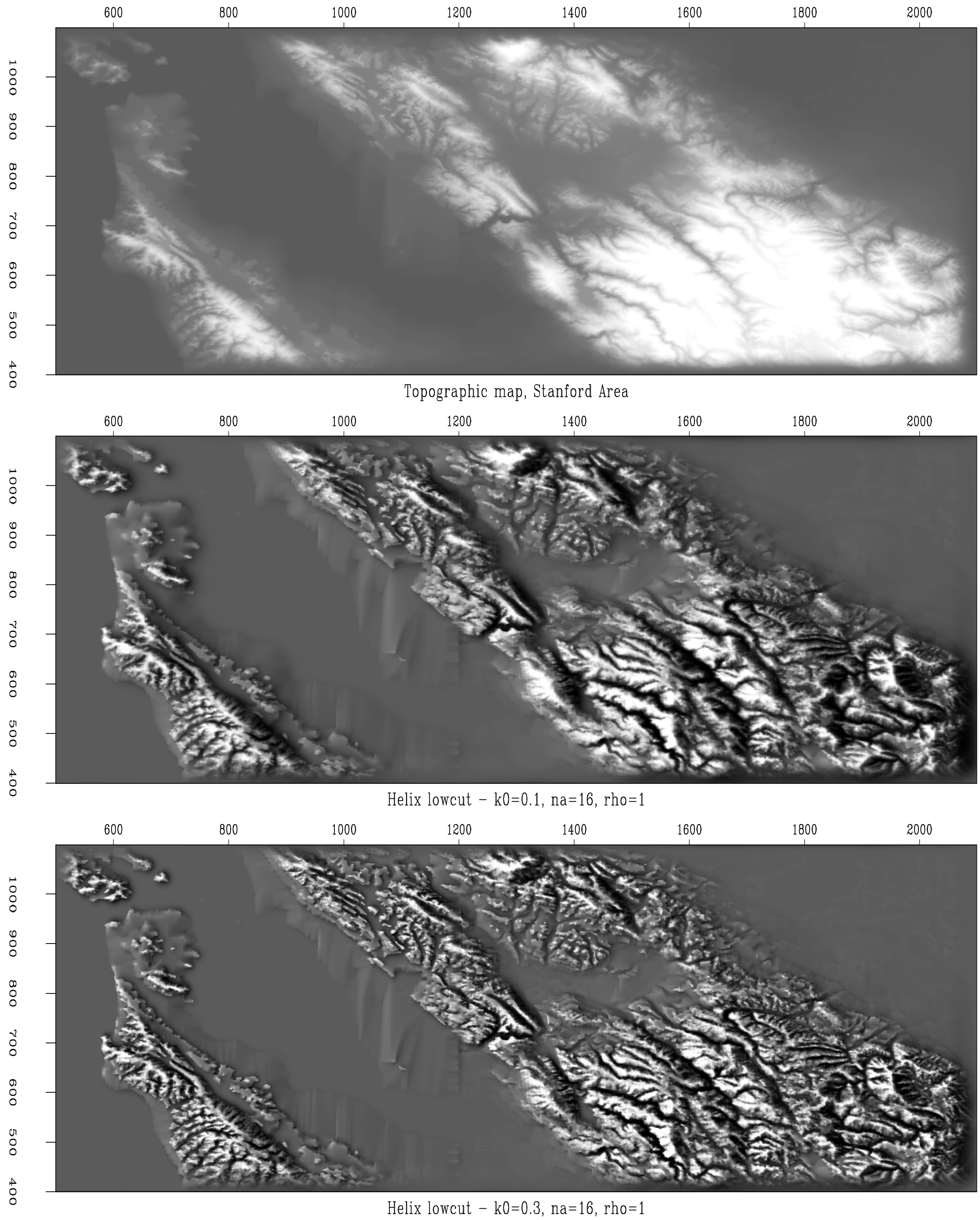

Figure 5: Bay Area maps. The top is the topographic map; the other two are roughened with helix low-cut filter. The middle is $k_0 = 0.1$; and the bottom is $k_0 = 0.3$. $n_a = 16$, $\rho = 1$. yi1-bay-lct-res [ER]

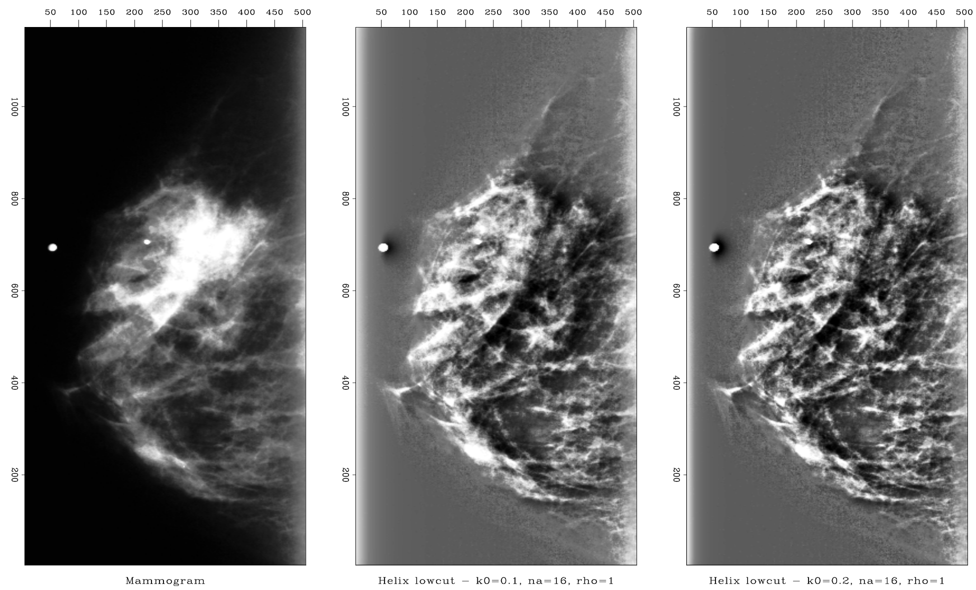

Figure 6: Mammogram (medical X-ray). The left figure is the origin map; the right two are filtered with helix low-cut filter. $n_a = 16$, $\rho = 1$, the middle is $k_0 = 0.1$, the right is $k_0 = 0.2$. yi1-mam-lct-res [ER]

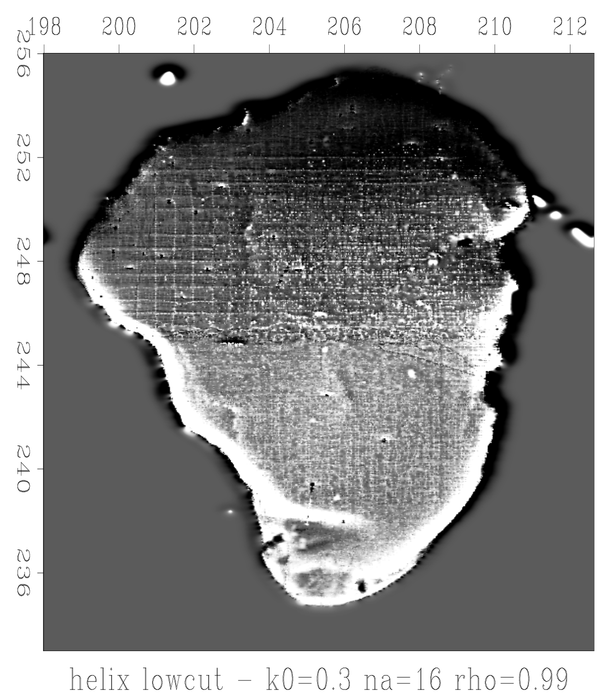

Figure 7: View of the Sea of Galilee roughened with a helix low-cut filter. $k_0 = 0.3$, $n_a = 16$ and $\rho = 0.99$. The edge of the sea fades away, but in the middle of the sea, the long line structure is persevered quite well. yi1-gal-lct-res [ER]

These two points add the burden of computation for the helix low-cut filter.

So if the main interests are the small-scale structures or the high costs of computation are not affordable, I should choose the helix derivative filter. Otherwise, the helix low-cut filter is a better choice.

CONCLUSIONS

The enhanced derivative and low-cut filters with adjustable parameters improve our ability to roughen an image for details. The introduction of adjustable parameter ρ makes it possible to gain computational savings by using a short filter instead of a long one.

For the enhanced helix derivative filter, the filter length determines its spectral symmetry and controls the zero-frequency response with ρ. For the low-cut filter, the cut-off frequency is mainly determined by k_0, and the zero-frequency response is affected by all three parameters. n_a is an unimportant parameter for helix filters.

The enhanced helix derivative filter emphasizes the small-scale structures in the image, while the low-cut filter emphasizes the medium-scale structures and costs more than derivative filter.

Based on our analysis of the effects of the parameters on the roughened image and filter spectrum, I present guidelines for choosing the adjustable parameters for image processing application.

ACKNOWLEDGMENTS

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APPENDIX A

QUANTITATIVE EFFECT OF THE ADJUSTABLE PARAMETERS

Helix derivative filter

For the helix derivative filter, the response is nearly a linear function of $|\mathbf{k}| = \sqrt{k_x^2 + k_y^2}$ and does not have the "cut-off frequency". The main feature here is the zero-frequency response. Figure A-1 shows the zero-frequency response R_0 as the function of filter length when $\rho = 0$. n_a is the half size of the filter. I find that R_0 decreases as n_a increases and $R_0 \propto n_a^{-1}$. The empirical relationship between R_0 and n_a is

$$R_0 \approx \frac{0.44}{n_a} \tag{A-1}$$

R_0 is the sum of the helix filter's coefficients, so when $\rho \neq 0$, according to Equation (3), the zero-frequency response is

$$R_0 \approx \frac{0.44}{n_a}(1 - \rho) \tag{A-2}$$

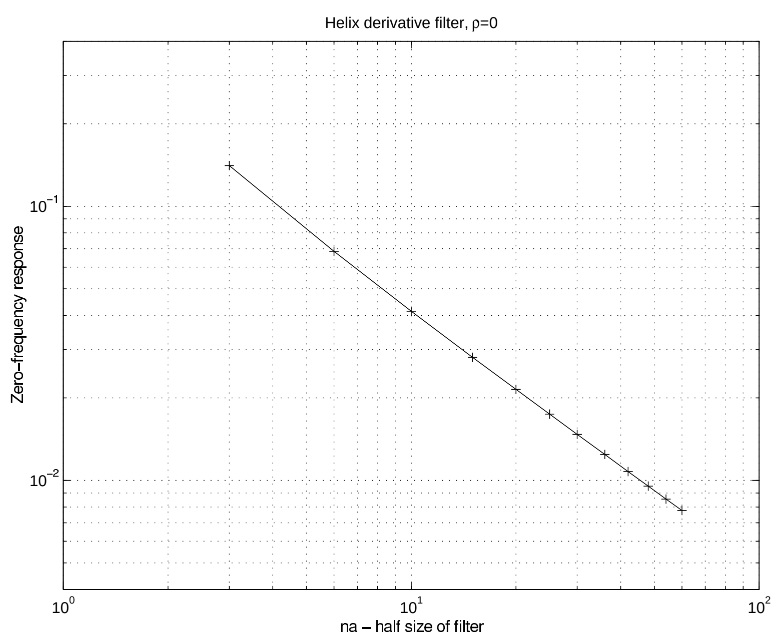

Figure A-1: The zero-frequency response of helix derivative filter when $\rho = 0$. The approximate zero-frequency response is $\frac{0.44}{n_a}$. yi1-drv-r0f-ar0 [CR]

The zero-frequency response of the enhanced helix derivative filter is controlled by both n_a and ρ, I can compute the approximate value of R_0 using Equation (A-2).

Helix low-cut filter

For the helix low-cut filter, the main features are the cut-off frequency and zero-frequency response. All the frequencies in this paper are scaled values, taking π as the Nyquist frequency.

- Effect of k_0

  Figure A-2 is the zero-frequency response and cut-off frequency of a 100-point low-cut filter for different k_0 when $\rho = 0$. I find that cut-off frequency f_0 is almost the same as k_0.

  $$f_0 \approx k_0 \tag{A-3}$$

  This satisfies the Equation (**??**) very well. In Equation (2), if I set $k = k_0$, the expression of $\frac{\overline{H}H}{\overline{D}D}$ is 0.5. According to the definition of cut-off frequency, the energy spectrum of the filter should be 0.5 at the cut-off frequency.

  The difference of the cut-off frequencies at various azimuths is very small when $k_0 > 0.1$ and can be ignored. For $k_0 < 0.1$, the difference is obvious.

  In Figure A-2, the zero-frequency response decreases as k_0 increases. For $\rho = 0$ and $k_0 > 0.03$, $R_0 \propto k_0^{-1}$, the empirical relationship between the zero-frequency response of 100-point and k_0 is

  $$R_0 \approx \frac{0.017}{k_0} \tag{A-4}$$

  For $k_0 < 0.02$, the zero frequency response curve becomes flat, and reaches the limit of 1. This is easily derived from Equation (2). When k_0 turns to zero, the difference between numerator H and denominator and D becomes smaller.

- Effect of n_a

  Figure A-3 shows the effect of n_a on helix low-cut filter when $k_0 = 0.3$ and $\rho = 0$. For small n_a, the numerical anisotropy is very strong. Although the mean value of the cut-off frequency remains the same, the azimuthal difference becomes larger when n_a becomes smaller. The zero-frequency response increases as n_a decreases, and when $k_0 = 0.3$ and $\rho = 0$, the empirical expression is

  $$R_0 \approx \frac{2.7}{n_a} \tag{A-5}$$

- Effect of ρ

  ρ directly controls the zero-frequency response and affects the cut-off frequency as well. Figure A-4 shows the cut-off frequency of the 100-point helix low-cut filter as the function of k_0 when $\rho = 0.5$ and $\rho = 1$. For larger ρ, the numerical anisotropy is stronger, especially when k_0 is small. Compared with Figure A-2, the cut-off frequency at small k_0 increases slightly with ρ.

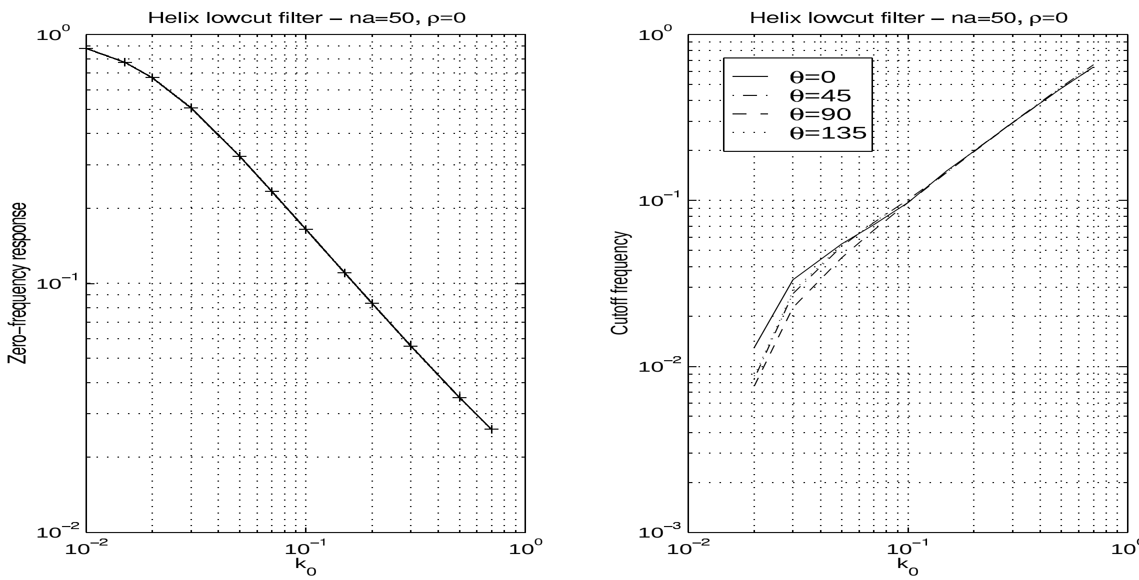

Figure A-2: The zero-frequency response and cut-off frequency of the helix low-cut filter with $n_a = 50$, $\rho = 0$. The zero-frequency response here is about $\frac{0.017}{k_0}$; the cut-off frequency is about k_0. yi1-lcut-r00-ka50r0 [CR]

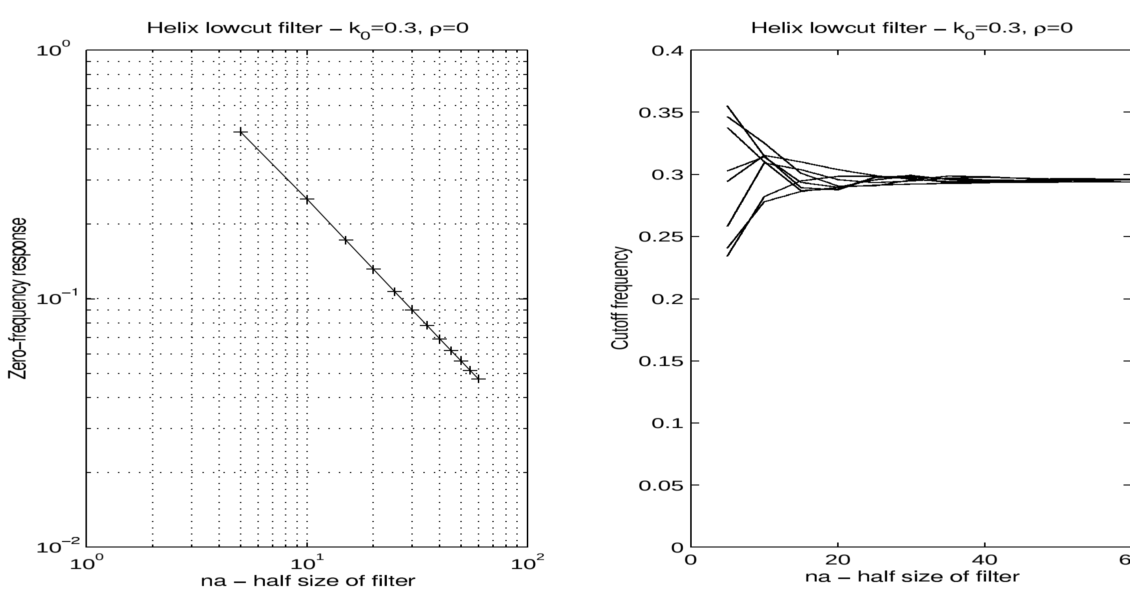

Figure A-3: The zero-frequency response and cut-off frequency of the helix low-cut filter with $k_0 = 0.3$, $\rho = 0$. The zero-frequency response here is about $\frac{2.7}{n_a}$. yi1-lcut-r00-ak3r0 [CR]

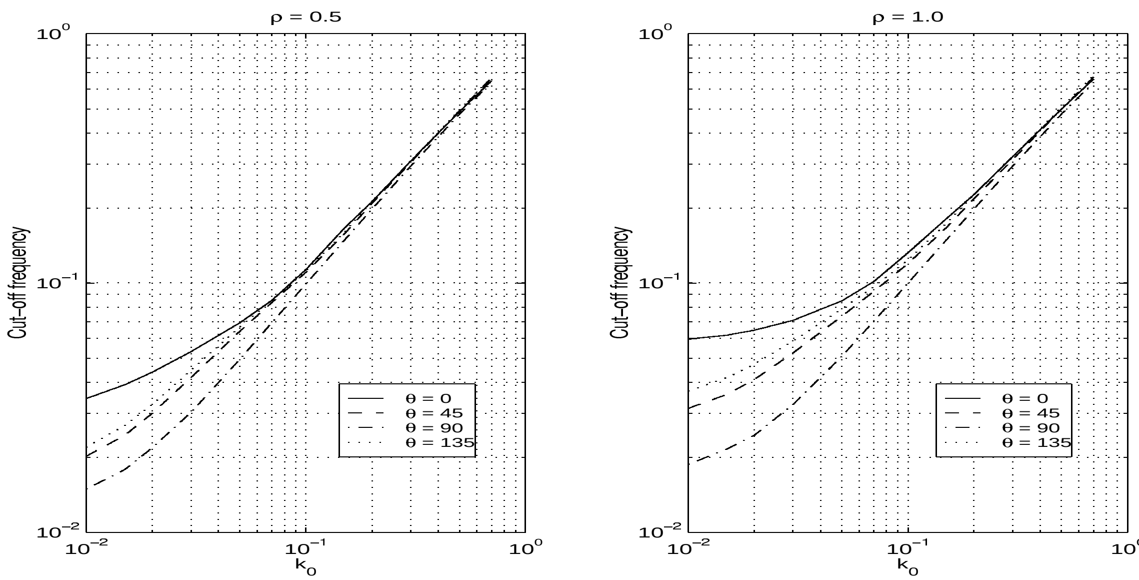

Figure A-4: The zero-frequency response and cut-off frequency of helix low-cut filter. yi1-lcut-rf0-kra50 [CR]

- Composed effects

  Based on the proceeding analysis, I can derive the composed effects of the adjustable parameters on the helix low-cut filter.

  The cut-off frequency is mainly governed by k_0.

$$f_0 \approx k_0 \tag{A-6}$$

  For large n_a, f_0 is almost the same as k_0. Both nonzero ρ and small n_a leads to the anisotropy of cut-off frequency. However, there is a difference between them: ρ causes the average cut-off frequency to increase slightly; small n_a intends to keep it.

  The zero-frequency response R_0 is under the direct control of ρ and influenced by n_a and k_0. If I assume that the influences of n_a and k_0 are independent, the empirical expression of R_0 would be

$$R_0 \approx \frac{0.82}{n_a k_0}(1-\rho) \tag{A-7}$$

Equations (A-2), (A-6) and (A-7) describe the quantitative effects of the helix derivative / low-cut filter's adjustable parameters. These empirical formulas make it quantitative for us to choose the adjustable parameters of the helix filter in practice.