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Short Note

Traveltime sensitivity kernels: Banana-doughnuts or just plain bananas?

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INTRODUCTION

Estimating an accurate velocity function is one of the most critical steps in building an accurate seismic depth image of the subsurface. In areas with significant structural complexity, one-dimensional updating schemes become unstable, and more robust algorithms are needed. Reflection tomography both in the premigrated (Bishop et al., 1985) and postmigrated domains (Stork, 1992; Kosloff et al., 1996) bring the powerful technologies of geophysical inversion theory to bear on the problem.

Unfortunately, however, inversion methods can be limited by the accuracy of their forward modeling operators, and most practical implementations of traveltime tomography are based on ray-theory, which assumes a high frequency wave, propagating through a smoothly varying velocity field, perhaps interrupted with a few discrete interfaces. Real world wave-propagation is much more complicated than this, and the failure of ray-based methods to adequately model wave propagation through complex media is fueling interest in "wave-equation" migration algorithms that both accurately model finite-frequency effects, and are practical for large 3-D datasets. As a direct consequence, finite-frequency velocity analysis and tomography algorithms are also becoming an important area of research (Woodward, 1992; Biondi and Sava, 1999).

Recent work in the global seismology community (Marquering et al., 1998, 1999) is drawing attention to a non-intuitive observation first made by Woodward (1992), that in the weak-scattering limit, finite-frequency traveltimes have zero-sensitivity to velocity perturbations along the geometric ray-path. This short-note aims to explore and explain this non-intuitive observation.

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THEORY

A generic discrete linear inverse problem may be written as

$$\mathbf{d} = \mathbf{A}\,\mathbf{m} \tag{1}$$

where $\mathbf{d} = (d_1\ d_2\ ...)^T$ is the known data vector, $\mathbf{m} = (m_1\ m_2\ ...)^T$ is the unknown model vector, and **A** represents the linear relationship between them. A natural question to ask is: which parts of the model influence a given observed data-point? The answer is that the row of matrix, **A**, corresponding to the data-point of interest will be non-zero where that point in model space influences the data-value. Rows of **A** may therefore be thought of as sensitivity kernels, describing which points in model space are sensed by a given data-point.

For a generic linearized traveltime tomography problem, traveltime perturbations, $\delta\mathbf{T}$, are related to slowness perturbations, $\delta\mathbf{S}$, through a linear system,

$$\delta\mathbf{T} = \mathbf{A}\,\delta\mathbf{S}. \tag{2}$$

The form of the sensitivity kernels depend on the the modeling operator, **A**.

Under the ray-approximation, traveltime for a given ray, T, is calculated by integrating slowness along the ray-path,

$$T = \int_{\text{ray}} s(\mathbf{x})\,dl. \tag{3}$$

Assuming that the ray-path is insensitive to a small slowness perturbation, the perturbation in traveltime is given by the path integral of the slowness perturbation along the ray,

$$\delta T = \int_{\text{ray}} \delta s(\mathbf{x})\,dl. \tag{4}$$

Since traveltime perturbations given by equation (4) are insensitive to slowness perturbations anywhere off the geometric ray-path, the sensitivity kernel is identically zero everywhere in space, except along the ray-path where it is constant. The implication for ray-based traveltime tomography is that traveltime perturbations should be back-projected purely along the ray-path.

We are interested in more accurate tomographic systems of the form of equation (3), that model the effects of finite-frequency wave-propagation more accurately than simple ray-theory. Once we have such an operator, the first question to ask is: what do the rows look like?

Born traveltime sensitivity

One approach to building a linear finite-frequency traveltime operator is to apply the first-order Born approximation, to obtain a linear relationship between slowness perturbation, δS, and wavefield perturbation, δU,

$$\delta\mathbf{U} = \mathbf{B}\,\delta\mathbf{S}. \tag{5}$$

The Born operator, **B**, is a discrete implementation of equation (A-7), which is described in the Appendix.

Traveltime perturbations may then be calculated from the wavefield perturbation through a (linear) picking operator, **C**, such that

$$\delta\mathbf{T} = \mathbf{C}\,\delta\mathbf{U} = \mathbf{CB}\,\delta\mathbf{S} \tag{6}$$

where **C** is a (linearized) picking operator, and a function of the background wavefield, U_0.

Cross-correlating the total wavefield, $U(t)$, with $U_0(t)$, provides a way of measuring their relative time-shift, δT. Marquering et al. (1999) uses this to provide the following explicit linear relationship between δT and $\delta U(t)$,

$$\delta T = \frac{\int_{t_1}^{t_2} \dot{U}(t)\,\delta U(t)\,dt}{\int_{t_1}^{t_2} \ddot{U}(t)\,U(t)\,dt}, \tag{7}$$

where dots denote differentiation with respect to t, and t_1 and t_2 define a temporal window around the event of interest. Equation (7) is only valid for small time-shifts, $\delta T \ll \lambda s_0$.

Rytov traveltime sensitivity

The first Rytov approximation (or the phase-field linearization method, as it is also known) provides a linear relationship between the slowness and complex phase perturbations.

$$\delta\Psi = \mathbf{R}\,\delta\mathbf{S}, \tag{8}$$

where $\Psi = \exp(\mathbf{U})$, and the Rytov operator, **R**, is a discrete implementation of equation (A-10), which is also described in Appendix A.

Traveltime is related to the complex phase by the equation, $\Im(\delta\psi) = \omega\,\delta t$. For a band-limited arrival with amplitude spectrum, $F(\omega)$, traveltime perturbation can be calculated simply by summing over frequency (Woodward, 1992),

$$\delta\mathbf{T} = \sum_{\omega} \frac{F(\omega)}{\omega}\,\Im(\delta\Psi) = \sum_{\omega} \frac{F(\omega)}{\omega}\,\Im(\mathbf{R}\,\delta\mathbf{S}). \tag{9}$$

Of the two approximations, several authors (Beydoun and Tarantola, 1988; Woodward, 1989) note that the Born approximation is the better choice for modeling reflected waves, while the Rytov approximation is better for transmitted waves. Differences tend to zero, however, as the scattering becomes small.

KERNELS COMPARED

This section contains images of traveltime kernels computed numerically for a simple model that may be encountered in a reflection tomography problem. The source is situated at the surface, and the receiver (known reflection point) is located at a depth of 1.8 km in the subsurface. The background velocity model, $v_0(z) = 1/s_0(z)$, is a linear function of depth with $v_0(0) = 1.5\ \mathrm{km\,s^{-1}}$, and $\frac{dv_0}{dz} = 0.8\ s^{-1}$. I chose a linear velocity function since Green's functions can be computed on-the-fly with rapid two-point ray-tracing.

Figure 1 shows the ray-theoretical traveltime sensitivity kernel: zero except along the geometric ray-path.

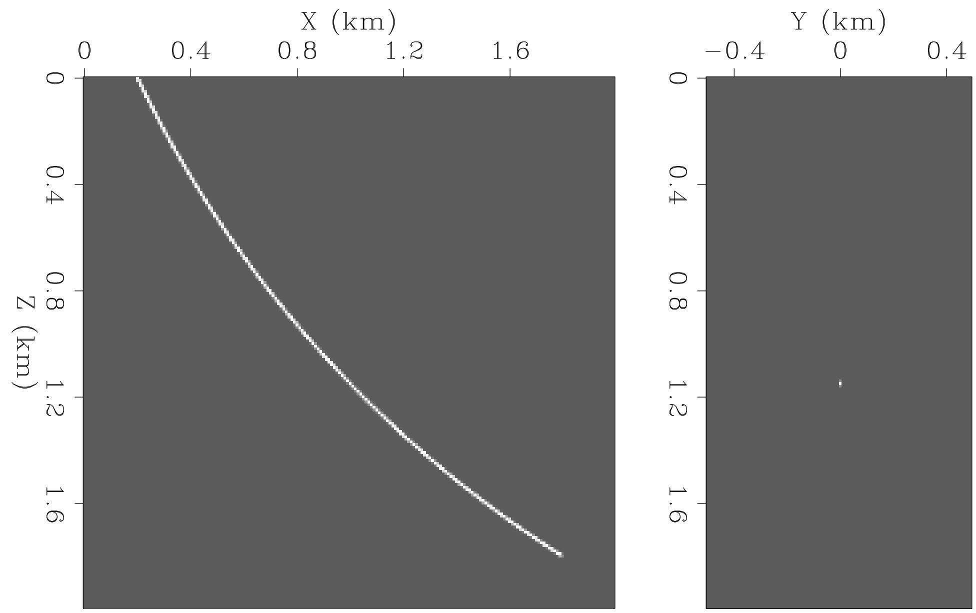

Figure 1: Traveltime sensitivity kernel for ray-based tomography in a linear $v(z)$ model. The kernel is zero everywhere *except* along geometric ray-path. Right panel shows a cross-section at $X = 1$ km. james3-RayKernel [ER]

Figures 2 and 3 show first Rytov traveltime sensitivity kernels for 30 Hz and 120 Hz wavelets respectively. The important features of these kernels are identical to the features of kernels that Marquering et al. (1999) obtained for teleseismic $S-H$ wave scattering, and to Woodward's (1992) band-limited wave-paths. They have the appearance of a hollow banana: that is appearing as a banana if visualized in the plane of propagation, but as a doughnut on a cross-section perpendicular to the ray. Somewhat counter-intuitively, this suggests that traveltimes have zero sensitivity to small velocity perturbations along the geometric raypath. Fortunately, however, as the frequency of the seismic wavelet increases, the bananas become thinner, and approach the ray-theoretical kernels in the high-frequency limit. Parenthetically, it is also worth noticing that the width of the bananas increases with depth as the velocity (and seismic wavelength) increases.

I do not show the first-Born kernels here, since, in appearance, they are identical to the Rytov kernels shown in Figures 2 and 3.

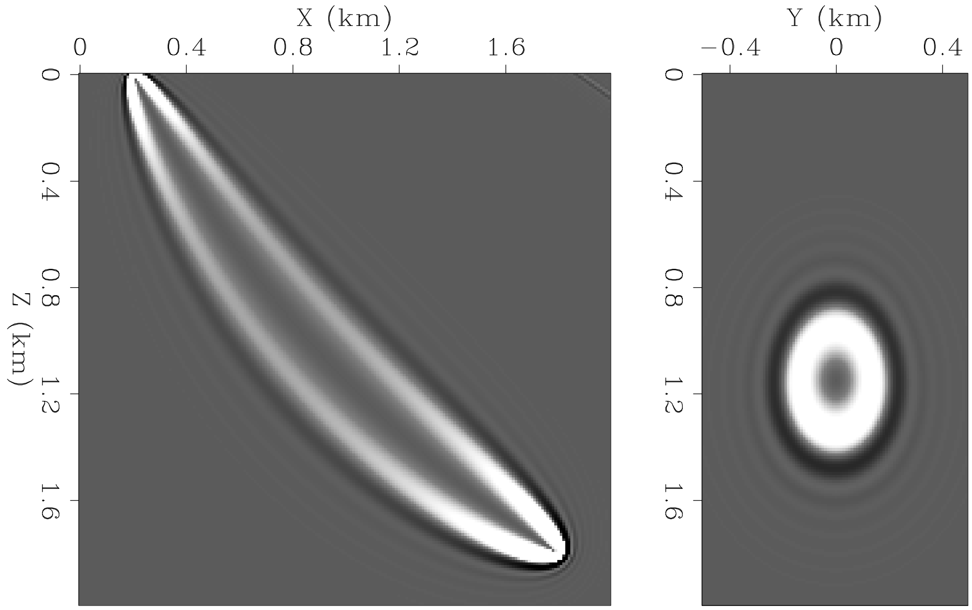

Figure 2: Rytov traveltime sensitivity kernel for 30 Hz wavelet in a linear $v(z)$ model. The kernel is zero along geometric ray-path. Right panel shows a cross-section at $X = 1$ km. james3-BananaPancake8 [ER]

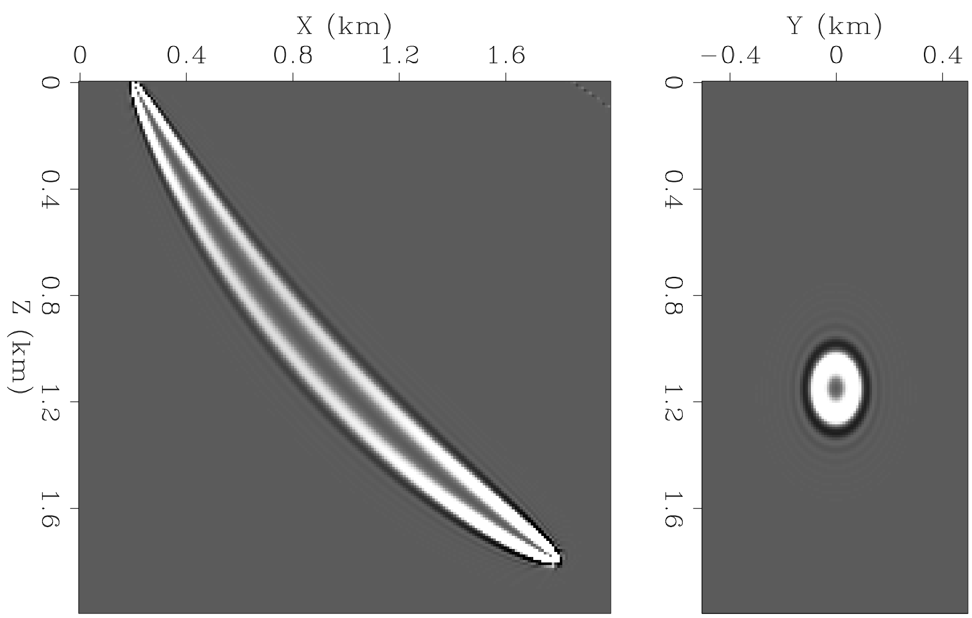

Figure 3: Rytov traveltime sensitivity kernel for 120 Hz wavelet in a linear $v(z)$ model. The kernel is zero along geometric ray-path. Right panel shows a cross-section at $X = 1$ km. james3-BananaPancake2 [ER]

THE BANANA-DOUGHNUT PARADOX

The important paradox is not the apparent contradiction between ray-theoretical and finite-frequency sensitivity kernels, since they are compatible in the high-frequency limit. Instead, the paradox is how do you reconcile the zero-sensitivity along the ray-path with your intuitive understanding of wave propagation?

A first potential resolution to the paradox is that the wavefront healing removes any effects of a slowness perturbation. This alone is a somewhat unsatisfactory explanation since it does not explain why traveltimes are sensitive to slowness perturbations just off the geometric ray-path.

A second potential resolution is that the hollowness of the banana is simply an artifact of modeling procedure. This is partially true. Both Born and Rytov are single scattering approximations, and a single scatterer located on the geometric ray-path may only contribute energy in-phase with the direct arrival. In contrast, if there are two scatterers on the geometric ray-path traveltimes may be affected. However, just because the paradox may appear to be an artifact of the modeling procedure does not mean it is not a real phenomenom. In the weak scattering limit, traveltimes will indeed be insensitive to a slowness perturbation situated on the geometric ray-path.

CONCLUSIONS: DOES IT MATTER?

Practitioners of traveltime tomography typically understand the shortcomings of ray-theory; although they realize using “fat-rays” would be better, they smooth the slowness model both explicitly and by regularizing during the inversion procedure. In practice, any shortcomings of traveltime tomography are unlikely to be caused by whether or not the fat-rays are hollow.

However, the null space of seismic tomography problems is typically huge. Smoothing and regularization are often done with very ad hoc procedures. Understanding the effects of finite-frequency through sensitivity kernels may lead to incorporating more physics during the regularization and improve tomography results.

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APPENDIX A

BORN/RYTOV REVIEW

Modeling with the first-order Born (and Rytov) approximations [e.g. Beydoun and Tarantola (1988)] can be justified by the assumption that slowness heterogeneity in the earth exists on two separate scales: a smoothly-varying background, s_0, within which the ray-approximation is valid, and weak higher-frequency perturbations, δs, that act to scatter the wavefield. The total slowness is given by the sum,

$$s(\mathbf{x}) = s_0(\mathbf{x}) + \delta s(\mathbf{x}). \tag{A-1}$$

Similarly, the total wavefield, U, can be considered as the sum of a background wavefield, U_0, and a scattered field, δU, so that

$$U(\mathbf{x},\omega) = U_0(\mathbf{x},\omega) + \delta U(\mathbf{x},\omega), \tag{A-2}$$

where U_0 satisfies the Helmholtz equation in the background medium,

$$\left[\nabla^2 + \omega^2 s_0^2(\mathbf{x})\right] U_0(\mathbf{x},\omega) = 0, \tag{A-3}$$

and the scattered wavefield is given by the (exact) non-linear integral equation (Morse and Feshbach, 1953),

$$\delta U(\mathbf{x},\omega) = \frac{\omega^2}{4\pi} \int_V G_0(\mathbf{x},\omega;\mathbf{x}')\, U(\mathbf{x},\omega;\mathbf{x}')\, \delta s(\mathbf{x}')\, dV(\mathbf{x}'). \tag{A-4}$$

In equation (A-4), G_0 is the Green's function for the Helmholtz equation in the background medium: i.e. it is a solution of the equation

$$\left[\nabla^2 + \omega^2 s_0^2(\mathbf{x})\right] G_0(\mathbf{x},\omega;\mathbf{x}_s) = -4\pi\,\delta(\mathbf{x}-\mathbf{x}_s). \tag{A-5}$$

Since the background medium is smooth, in this paper I use Green's functions of the form,

$$G_0(\mathbf{x},\omega;\mathbf{x}_s) = A_0(\mathbf{x},\mathbf{x}_s) e^{i\omega T_0(\mathbf{x},\mathbf{x}_s)}. \tag{A-6}$$

where A_0 and T_0 are ray-traced traveltimes and amplitudes respectively.

A Taylor series expansion of U about U_0 for small δs, results in the infinite Born series, which is a Neumann series solution (Arfken, 1985) to equation (A-4). The first term in the expansion is given below: it corresponds to the component of wavefield that interacts with scatters only once.

$$\delta U_{\mathrm{Born}}(\mathbf{x},\omega) = \frac{\omega^2}{4\pi} \int_V G_0(\mathbf{x},\omega;\mathbf{x}')\, U_0(\mathbf{x},\omega;\mathbf{x}') \delta s(\mathbf{x}') dV(\mathbf{x}'). \tag{A-7}$$

The approximation implied by equation (A-7) is known as the first-order Born approximation. It provides a linear relationship between δU and δs, and it can be computed more easily than the full solution to equation (A-4).

The Rytov formalism starts by assuming the heterogeneity perturbs the phase of the scattered wavefield. The total field, $U = \exp(\psi)$, is therefore given by

$$U(\mathbf{x},\omega) = U_0(\mathbf{x},\omega) \exp(\delta\psi) = \exp(\psi_0 + \delta\psi). \tag{A-8}$$

The linearization based on small $\delta\psi/\psi$ leads to the infinite Rytov series, on which the first term is given by

$$\delta\psi_{\mathrm{Rytov}}(\mathbf{x},\omega) = \frac{\delta U_{\mathrm{Born}}(\mathbf{x},\omega)}{U_0(\mathbf{x},\omega)} \tag{A-9}$$

$$= \frac{\omega^2}{4\pi\, U_0(\mathbf{x},\omega)} \int_V G_0(\mathbf{x},\omega;\mathbf{x}')\, U_0(\mathbf{x},\omega;\mathbf{x}') \delta s(\mathbf{x}') dV(\mathbf{x}'). \tag{A-10}$$

The approximation implied by equation (A-10) is known as the first-order Rytov approximation. It provides a linear relationship between $\delta\psi$ and δs.