Appendix A

Second-order reflection traveltime derivatives

In this appendix, I derive equations connecting second-order partial derivatives of the reflection traveltime with the geometric properties of the reflector in a constant velocity medium. These equations are used in the main text of Chapter **??** for the amplitude behavior description. Let $\tau(s,r)$ be the reflection traveltime from the source s to the receiver r. Consider a formal equality

$$\tau(s,r) = \tau_1\,(s, x(s,r)) + \tau_2\,(x(s,r), r)\ , \tag{A.1}$$

where x is the reflection point parameter, τ_1 corresponds to the incident ray, and τ_2 corresponds to the reflected ray. Differentiating (A.1) with respect to s and r yields

$$\frac{\partial \tau}{\partial s} = \frac{\partial \tau_1}{\partial s} + \frac{\partial \tau}{\partial x}\,\frac{\partial x}{\partial s}\ , \tag{A.2}$$

$$\frac{\partial \tau}{\partial r} = \frac{\partial \tau_2}{\partial r} + \frac{\partial \tau}{\partial x}\,\frac{\partial x}{\partial r}\ . \tag{A.3}$$

According to Fermat's principle, the two-point reflection ray path must correspond to the traveltime stationary point. Therefore

$$\frac{\partial \tau}{\partial x} \equiv 0 \tag{A.4}$$

for any s and r. Taking into account (A.4) while differentiating (A.2) and (A.3), we get

$$\frac{\partial^2 \tau}{\partial s^2} = \frac{\partial^2 \tau_1}{\partial s^2} + B_1\,\frac{\partial x}{\partial s}\,, \tag{A.5}$$

$$\frac{\partial^2 \tau}{\partial r^2} = \frac{\partial^2 \tau_2}{\partial r^2} + B_2\,\frac{\partial x}{\partial r}\,, \tag{A.6}$$

$$\frac{\partial^2 \tau}{\partial s \partial r} = B_1\,\frac{\partial x}{\partial r} = B_2\,\frac{\partial x}{\partial s}\,, \tag{A.7}$$

where

$$B_1 = \frac{\partial^2 \tau_1}{\partial s \partial x}\;;\; B_2 = \frac{\partial^2 \tau_2}{\partial r \partial x}\,.$$

Differentiating equation (A.4) gives us the additional pair of equations

$$C\,\frac{\partial x}{\partial s} + B_1 = 0\,, \tag{A.8}$$

$$C\,\frac{\partial x}{\partial r} + B_2 = 0\,, \tag{A.9}$$

where

$$C = \frac{\partial^2 \tau}{\partial x^2} = \frac{\partial^2 \tau_1}{\partial x^2} + \frac{\partial^2 \tau_2}{\partial x^2}\,.$$

Solving the system (A.8) - (A.9) for $\frac{\partial x}{\partial s}$ and $\frac{\partial x}{\partial r}$ and substituting the result into (A.5) - (A.7) produces the following set of expressions:

$$\frac{\partial^2 \tau}{\partial s^2} = \frac{\partial^2 \tau_1}{\partial s^2} - C^{-1}\,B_1^2\,; \tag{A.10}$$

$$\frac{\partial^2 \tau}{\partial r^2} = \frac{\partial^2 \tau_2}{\partial r^2} - C^{-1}\,B_2^2\,; \tag{A.11}$$

$$\frac{\partial^2 \tau}{\partial s \partial r} = -C^{-1}\,B_1\,B_2\,. \tag{A.12}$$

In the case of a constant velocity medium, expressions (A.10) to (A.12) can be applied directly to the explicit equation for the two-point eikonal

$$\tau_1(y,x) = \tau_2(x,y) = \frac{\sqrt{(x-y)^2+z^2(x)}}{v}\,. \tag{A.13}$$

Differentiating (A.13) and taking into account the trigonometric relationships for the incident and reflected rays (Figure **??**), one can evaluate all the quantities in (A.10) to (A.12) explicitly. After some heavy algebra, the resultant expressions for the traveltime derivatives take the form

$$\frac{\partial \tau}{\partial s} = \frac{\partial \tau_1}{\partial s} = \frac{\sin\alpha_1}{v} \quad ; \quad \frac{\partial \tau}{\partial r} = \frac{\partial \tau_2}{\partial r} = \frac{\sin\alpha_2}{v}\,; \tag{A.14}$$

$$\frac{\partial \tau_1}{\partial x} = \frac{\sin\gamma}{v\cos\alpha} \quad ; \quad \frac{\partial \tau_2}{\partial x} = -\frac{\sin\gamma}{v\cos\alpha}\,; \tag{A.15}$$

$$B_1 = \frac{\partial^2 \tau_1}{\partial s\,\partial x} = \frac{\cos\alpha_1}{v\,D\,\cos\alpha}\left(-1-\frac{\sin\gamma}{\cos\alpha}\,\sin\alpha_1\right)\,; \tag{A.16}$$

$$B_2 = \frac{\partial^2 \tau_2}{\partial r\,\partial x} = \frac{\cos\alpha_2}{v\,D\,\cos\alpha}\left(-1+\frac{\sin\gamma}{\cos\alpha}\,\sin\alpha_2\right)\,; \tag{A.17}$$

$$B_1\,B_2 = \frac{\cos^6\gamma}{v^2\,D^2\,a^4}\ ;\ B_1+B_2 = -2\,\frac{\cos^3\gamma}{v\,D\,a^2}\left(2\,a^2-1\right)\,; \tag{A.18}$$

$$\frac{\partial^2 \tau_1}{\partial x^2} = \frac{\cos^2\gamma + D\,K}{v\,D\,\cos^3\alpha}\,\cos\alpha_1\ ;\ \frac{\partial^2 \tau_2}{\partial x^2} = \frac{\cos^2\gamma + D\,K}{v\,D\,\cos^3\alpha}\,\cos\alpha_2\,; \tag{A.19}$$

$$C = \frac{\partial^2 \tau_1}{\partial x^2} + \frac{\partial^2 \tau_2}{\partial x^2} = 2\,\cos\gamma\,\frac{\cos^2\gamma + D\,K}{v\,D\,\cos^3\alpha}\,. \tag{A.20}$$

Here D is the length of the normal (central) ray, α is its dip angle ($\alpha = \frac{\alpha_1+\alpha_2}{2}$, $\tan\alpha = z'(x)$), γ is the reflection angle $\left(\gamma = \frac{\alpha_2-\alpha_1}{2}\right)$, K is the reflector curvature at the reflection point $\left(K = z''(x)\,\cos^3\alpha\right)$, and a is the dimensionless function of α and γ defined in (**??**).

The equations derived in this appendix were used to get the equation

$$\tau_n \left(\frac{\partial^2 \tau_n}{\partial y^2} - \frac{\partial^2 \tau_n}{\partial h^2} \right) = 4 \left(\tau \, \frac{\partial^2 \tau}{\partial s \, \partial r} + \frac{\cos^2 \gamma}{v^2} \right) = 4 \, \frac{\cos^2 \gamma}{v^2} \left(\frac{\sin^2 \alpha + DK}{\cos^2 \gamma + DK} \right) , \qquad \text{(A.21)}$$

which coincides with (**??**) in the main text.

Appendix B

Solving the Cauchy problem

To obtain an explicit solution of the Cauchy problem (**??**-**??**) for equation (**??**), it is convenient to apply the following simple transform of the wavefield P:

$$P(t_n,h,y) = Q(t_n,h,y)\,t_n\,H(t_n)\,. \tag{B.1}$$

Here the Heavyside function H is included to take into account the causality of the reflection seismic gathers (note that the time $t_n = 0$ corresponds to the direct wave arrival). We can extrapolate Q as an even function to negative times, writing the reverse of (B.1) as follows:

$$Q(t_n,h,y) = Q(-t_n,h,y) = P(|t_n|,h,y)/|t_n|\,. \tag{B.2}$$

With the change of function (B.1), equation (**??**) transforms to

$$h\,\frac{\partial^2 Q}{\partial y^2} = h\,\frac{\partial^2 Q}{\partial h^2} + t_n\,\frac{\partial^2 Q}{\partial t_n\,\partial h} + \frac{\partial Q}{\partial h} = \frac{\partial}{\partial h}\left(h\,\frac{\partial Q}{\partial h} + t_n\,\frac{\partial Q}{\partial t_n}\right)\,. \tag{B.3}$$

Applying the change of variables

$$\rho = \frac{t_n^2}{2}\,,\ \nu = \frac{h^2}{2\,t_n^2} \tag{B.4}$$

and Fourier transform in the midpoint coordinate y

$$\widetilde{Q}(\rho,\nu,k) = \int Q(\rho,\nu,y)\exp(-iky)\,dy\,, \tag{B.5}$$

I further transform equation (B.3) to the canonical form of a hyperbolic-type partial differential equation with two variables:

$$\frac{\partial^2 \widetilde{Q}}{\partial\rho\,\partial\nu} + k^2\,\widetilde{Q} = 0\,. \tag{B.6}$$

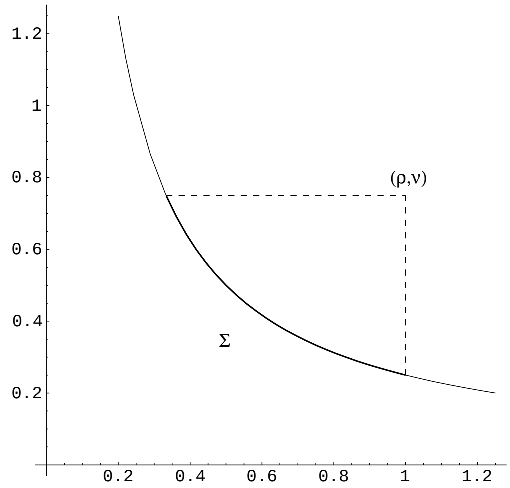

Figure B.1: Domain of dependence of a point in the transformed coordinate system. $\boxed{\text{appen-rim}}$ [NR]

The initial value conditions (**??**) and (**??**) in the $\{\rho,\nu\}$ space are defined on a hyperbola of the form $\rho\,\nu = \left(\frac{h_1}{2}\right)^2 =$ constant. Now the solution of the Cauchy problem follows directly from Riemann's method (Courant, 1962). According to this method, the domain of dependence of each point $\{\rho,\nu\}$ is a part of the hyperbola between the points $\{\rho,\frac{h_1^2}{4\rho}\}$ and $\{\frac{h_1^2}{4\nu},\nu\}$ (Figure B.1). If we let Σ denote this curve, the solution takes an explicit integral form:

$$\begin{aligned}
\widetilde{Q}(\rho,\nu) &= \frac{1}{2}\,\widetilde{Q}(\rho,\frac{h_1^2}{4\rho}) + \frac{1}{2}\,\widetilde{Q}(\frac{h_1^2}{4\nu},\nu) \\
&+ \frac{1}{2}\int_\Sigma \left(R(\rho_1,\nu_1;\rho,\nu)\,\frac{\partial\widetilde{Q}(\rho_1,\nu_1)}{\partial\rho_1} - \widetilde{Q}(\rho_1,\nu_1)\,\frac{\partial R(\rho_1,\nu_1;\rho,\nu)}{\partial\rho_1}\right)d\rho_1 \\
&- \frac{1}{2}\int_\Sigma \left(R(\rho_1,\nu_1;\rho,\nu)\,\frac{\partial\widetilde{Q}(\rho_1,\nu_1)}{\partial\nu_1} - \widetilde{Q}(\rho_1,\nu_1)\,\frac{\partial R(\rho_1,\nu_1;\rho,\nu)}{\partial\nu_1}\right)d\nu_1\,.
\end{aligned} \tag{B.7}$$

Here R is the Riemann's function of equation (B.6), which has the known explicit analytical expression

$$R(\rho_1,\nu_1;\rho,\nu) = J_0\left(2k\sqrt{(\rho_1-\rho)\,(\nu_1-\nu)}\right)\,, \tag{B.8}$$

where J_0 is the Bessel function of zeroth order. Integrating by parts and taking into account the connection of the variables on the curve Σ, we can simplify equation (B.7) to the form

$$\widetilde{Q}(\rho,\nu) = \widetilde{Q}_0(\rho,\nu) + \widetilde{Q}_1(\rho,\nu)\,, \tag{B.9}$$

where

$$\widetilde{Q}_0(\rho,\nu) = \frac{\partial}{\partial\rho}\int_\Sigma R(\rho_1,\nu_1;\rho,\nu)\,\widetilde{Q}(\rho_1,\nu_1)\,d\rho_1\,, \tag{B.10}$$

$$\widetilde{Q}_1(\rho,\nu) = -\int_\Sigma R(\rho_1,\nu_1;\rho,\nu)\,\frac{\partial\,\widetilde{Q}(\rho_1,\nu_1)}{\partial\nu_1}\,d\nu_1\,. \tag{B.11}$$

Applying the explicit expression for the Riemann function R (B.8) and performing the inverse transform of both the function and the variables allows us to rewrite equations (B.9), (B.10), and (B.11) in the original coordinate system. This yields the integral offset continuation operators in the $\{t_n,h,k\}$ domain

$$\widetilde{P}(t_n,h,k) = H(t_n)\left(\widetilde{P}_0(t_n,h,k) + t_n\,\widetilde{P}_1(t_n,h,k)\right)\,, \tag{B.12}$$

where

$$\widetilde{P}_0 = \frac{\partial}{\partial t_n}\int_{(h_1/h)t_n}^{t_n} \widetilde{P}_1^{(0)}(|t_1|,k)\,J_0\left(k\sqrt{\left(\frac{h^2}{t_n^2}-\frac{h_1^2}{t_1^2}\right)\left(t_n^2-t_1^2\right)}\right)dt_1\,, \tag{B.13}$$

$$\widetilde{P}_1 = \int_{(h_1/h)t_n}^{t_n} h_1\,\widetilde{P}_1^{(1)}(|t_1|,k)\,J_0\left(k\sqrt{\left(\frac{h^2}{t_n^2}-\frac{h_1^2}{t_1^2}\right)\left(t_n^2-t_1^2\right)}\right)\frac{dt_1}{t_1^2}\,, \tag{B.14}$$

$$\widetilde{P}_1^{(j)}(t_1,k) = \int P_1^{(j)}(t_1,y_1)\exp(-iky_1)\,dy_1\;(j=0,1)\,, \tag{B.15}$$

$$\widetilde{P}(t_n,h,k) = \int P(t_n,h,y)\exp(-iky)\,dy\;(j=0,1)\,. \tag{B.16}$$

The inverse Fourier transforms of equations (B.13) and (B.14) are reduced to analytically evaluated integrals (Gradshtein and Ryzhik, 1994) to produce explicit integral operators in the time-and-space domain

$$P(t_n,h,y) = \mathrm{sign}(h-h_1)\,\frac{H(t_n)}{\pi}\,\left(P_0(t_n,h,y)+t_n\,P_1(t_n,h,y)\right)\,, \tag{B.17}$$

where

$$P_0(t_n,h,y) = \frac{\partial}{\partial t_n}\iint_{\Sigma}\frac{P_1^{(0)}(|t_1|,y_1)\,dt_1\,dy_1}{\sqrt{\left(\frac{h^2}{t_n^2}-\frac{h_1^2}{t_1^2}\right)\left(t_n^2-t_1^2\right)-(y-y_1)^2}}\,, \tag{B.18}$$

$$P_1(t_n,h,y) = \iint_{\Sigma}\frac{\left(h_1/t_1^2\right)\,P_1^{(1)}(|t_1|,y_1)\,dt_1\,dy_1}{\sqrt{\left(\frac{h^2}{t_n^2}-\frac{h_1^2}{t_1^2}\right)\left(t_n^2-t_1^2\right)-(y-y_1)^2}}\,. \tag{B.19}$$

The range of integration Σ in (B.18) and (B.19) is defined by the inequality

$$\left(\frac{h^2}{t_n^2}-\frac{h_1^2}{t_1^2}\right)\left(t_n^2-t_1^2\right)-(y-y_1)^2>0\,. \tag{B.20}$$

Equations (B.17), (B.18), and (B.19) coincide with (**??**), (**??**), and (**??**) in the main text.

Appendix C

The kinematics of offset continuation

In this Appendix, I apply an alternative method to derive equation (**??**), which describes the summation path of the integral offset continuation operator. The method is based on the following considerations.

The summation path of an integral (stacking) operator coincides with the phase function of the impulse response of the inverse operator. Impulse response is by definition the operator reaction to an impulse in the input data. For the case of offset continuation, the input is a reflection common-offset gather. From the physical point of view, an impulse in this type of data corresponds to the special focusing reflector (elliptical isochrone) at the depth. Therefore, reflection from this reflector at a different constant offset corresponds to the impulse response of the OC operator. In other words, we can view offset continuation as the result of cascading prestack common-offset migration, which produces the elliptic surface, and common-offset modeling (inverse migration) for different offsets. This approach resemble that of Deregowski and Rocca (1981). It was also applied to a more general case of azimuth moveout (AMO) by Fomel and Biondi (1995). The geometric approach implies that in order to find the summation pass of the OC operator, one should solve the kinematic problem of reflection from an elliptic reflector whose focuses are in the shot and receiver locations of the output seismic gather.

In order to solve this problem , let us consider an elliptic surface of the general form

$$h(x) = \sqrt{d^2 - \beta\,(x-x')^2}\,, \tag{C.1}$$

where $0 < \beta < 1$. In a constant velocity medium, the reflection ray path for a given source-receiver pair on the surface is controlled by the position of the reflection point x. Fermat's principle provides a required constraint for finding this position. According to Fermat's principle, the reflection ray path corresponds to a stationary value of the travel-time. Therefore, in the neighborhood of this path,

$$\frac{\partial \tau(s,r,x)}{\partial x} = 0\,, \tag{C.2}$$

where s and r stand for the source and receiver locations on the surface, and τ is the reflection traveltime

$$\tau(s,r,x) = \frac{\sqrt{h^2(x)+(s-x)^2}}{v} + \frac{\sqrt{h^2(x)+(r-x)^2}}{v}\,. \tag{C.3}$$

Substituting (C.3) and (C.1) into (C.2) leads to a quadratic algebraic equation on the reflection point parameter x. This equation has the explicit solution

$$x(s,r) = x' + \frac{\xi^2+H^2-h^2+\mathrm{sign}(h^2-H^2)\sqrt{\left(\xi^2-H^2-h^2\right)^2-4\,H^2\,h^2}}{2\,\xi\,(1-\beta)}\,, \tag{C.4}$$

where $h=(r-s)/2$, $\xi = y-x'$, $y=(s+r)/2$, and $H^2 = d^2\left(\frac{1}{\beta}-1\right)$. Replacing x in equation (C.3) with its expression (C.4) solves the kinematic part of the problem, producing the explicit traveltime expression

$$\tau(s,r) = \left\{\begin{array}{ll} \dfrac{1}{v}\sqrt{\dfrac{4\,h^2-\beta\,(f+g)^2}{1-\beta}} & \text{for} \quad h^2 > H^2 \\ & \\ \dfrac{1}{v}\sqrt{\dfrac{4\,h^2+\beta\,(F+G)^2}{1-\beta}} & \text{for} \quad h^2 < H^2 \end{array}\right.\,, \tag{C.5}$$

where

$$f = \sqrt{(r-x')^2 - H^2} \quad , \quad g = \sqrt{(s-x')^2 - H^2} \, ,$$
$$F = \sqrt{H^2 - (r-x')^2} \quad , \quad G = \sqrt{H^2 - (s-x')^2} \, .$$

The two branches of equation (C.5) correspond to the difference in the geometry of the reflected rays in two different situations. When a source-and-receiver pair is inside the focuses of the elliptic reflector, the midpoint y and the reflection point x are on the same side of the ellipse with respect to its small semi-axis. They are on different sides in the opposite case (Figure C.1).

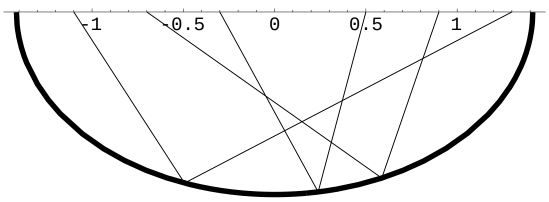

Figure C.1: Reflections from an ellipse. The three pairs of reflected rays correspond to a common midpoint (at 0.1) and different offsets. The focuses of the ellipse are at 1 and -1. appen-ell [CR]

If we apply the NMO correction, equation (C.5) is transformed to

$$\tau_n(s,r) = \left\{ \begin{array}{ll} \frac{1}{v}\sqrt{\frac{\beta}{1-\beta}}\sqrt{4h^2 - (f+g)^2} & \text{for} \quad h^2 > H^2 \\ & \\ \frac{1}{v}\sqrt{\frac{\beta}{1-\beta}}\sqrt{4h^2 + (F+G)^2} & \text{for} \quad h^2 < H^2 \end{array} \right. . \tag{C.6}$$

Then, recalling the relationships between the parameters of the focusing ellipse r, x' and β and the parameters of the output seismic gather (Deregowski and Rocca, 1981)

$$r = \frac{v\,t_n}{2} \, , \; x' = y \, , \; \beta = \frac{t_n^2}{t_n^2 + \frac{4h^2}{v^2}} \, , \; H = h \, , \tag{C.7}$$

and substituting expressions (C.7) into equation (C.6) yields the expression

$$t_1(s_1,r_1;s,r,t_n) = \begin{cases} \dfrac{t_n}{2h}\sqrt{4h_1^2-(f+g)^2} & \text{for} \quad h_1^2 > h^2 \\ \\ \dfrac{t_2}{2h}\sqrt{4h_1^2+(F+G)^2} & \text{for} \quad h_1^2 < h^2 \end{cases} , \tag{C.8}$$

where

$$f = \sqrt{(r_1-r)(r_1-s)}\,,\; g = \sqrt{(s_1-r)(s_1-s)}\,,$$
$$F = \sqrt{(r-r_1)(r_1-s)}\,,\; G = \sqrt{(s_1-r)(s-s_1)}\,.$$

It is easy to verify algebraically the mathematical equivalence of equation (C.8) and equation (**??**) in the main text. The kinematic approach described in this appendix applies equally well to different acquisition configurations of the input and output data. The source-receiver parameterization used in (C.8) is the actual definition for the summation path of the integral shot continuation operator (Schwab, 1993; Bagaini and Spagnolini, 1993, 1996). A family of these summation curves is shown in Figure C.2.

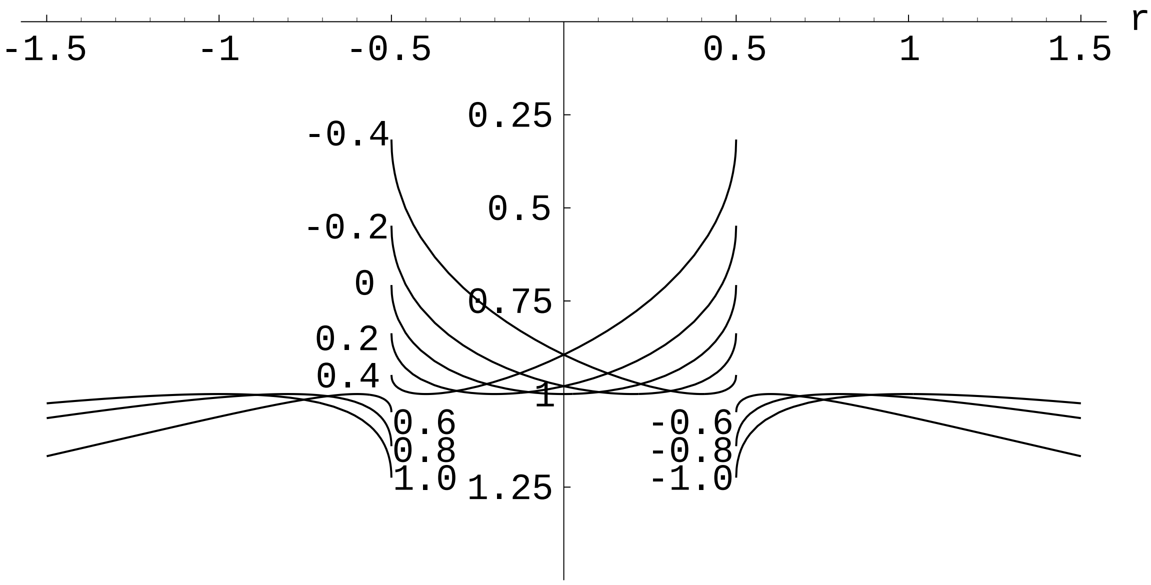

Figure C.2: Summation paths of the integral shot continuation. The output source is at -0.5 km. The output receiver is at 0.5 km. The indexes of the curves correspond to the input source location. appen-shc [CR]

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