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Modeling vector fields in regions with irregular boundaries

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INTRODUCTION

Geologists often see the Earth as homogeneous blocks separated by smooth curved boundaries. In contrast, computer modeling algorithms based on finite-difference schemes need to have the elastic constants of the Earth specified at the points of a regular grid. A common way of gridding a geological model is to lay the computer model grid down and use whatever elastic constants lie beneath each grid point. This naive method may result in artifacts; for example, a gently sloping interface will be modeled as a coarse staircase, which generates unwanted diffractions at the stair edges.

Elastic parameter interpolation scheme

Some sort of interpolation is called for, but what works best for elastic tensors? Schoenburg-Muir (hereafter S&M) theory tells us how to average a stack of flat elastic layers of arbitrary anisotropy exactly, and so suggests a good starting point.

We apply this method to the gridding problem in a straightforward way: lay the model grid down on the geological model as before, but this time look inside each square grid cell to find a stack of layers to be averaged using S&M theory.

Our model results show that this S&M averaging method works quite well, and works significantly better than simply averaging the elastic stiffness constants. The price we pay is an increase in model complexity. For example, although the geological model itself may be everywhere isotropic, interpolated grid cells at dipping layer boundaries will have transverse isotropic symmetry with the symmetry axis tilted normally to the dipping lithologic boundary.

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ACCURATE ELASTIC MODELING SCHEMES

Historically, the acoustic or elastic wave equation was solved on a discrete grid using first or second-order differencing in time and space (Kelly et al., 1976). The accuracy of the finite-difference method improves by using longer, more accurate spatial differencing operators, commonly 5-33 points long (Holberg, 1987). Taking this approach to its limit, the pseudospectral method (Kosloff et al., 1985) uses multiplication by ik in the wavenumber domain to compute spatial derivatives that are accurate to spatial Nyquist. In addition to the above improvements in the accuracy of spatial discretizations, Edwards et al. (1987) describe a new time-update method that eliminates errors due to time discretization.

These accurate modeling methods are efficient, especially in 3-D, because they allow the wave field and velocity model to be coarsely sampled. Unfortunately, there is also a drawback: if we sample the velocity model too coarsely it becomes difficult to represent curved or dipping boundaries accurately.

One solution is to use finite-element methods, which attack this problem by discretizing the model with elements that follow the boundaries. Fornberg (1988) introduced a mapping method that can be applied to pseudospectral finite-differences to convert curved-grid models to vertically layered models that can be more accurately represented on rectangular grids.

The main attraction of explicit finite-difference methods on rectangular grids is their efficiency and simplicity. Finite-element methods or other distorted-grid methods require complicated setup steps while the simpler rectangular-grid methods just need 2-D or 3-D arrays of values. So from a computational viewpoint, the real importance of the interpolation method we present in this paper is the ability to handle dipping interfaces without relinquishing regularly sampled grids.

IMPLEMENTATION

We now describe the steps necessary to implement our interpolation algorithm.

Smooth gridding

Figure **??** shows a medium consisting of two homogeneous regions separated by a gently dipping boundary. A 10 by 10 grid of cells has been laid down on the medium. We want to find the appropriate elastic constants to assign to each cell. There are three sorts of cells to consider; we have highlighted one of each kind.

The top highlighted cell is entirely within the top medium. Its elastic constants are therefore the same as that of the top (light-colored) homogeneous medium. To the right of the blown-up cell we show the P-SV impulse response for this medium. The bottom highlighted cell is entirely in the lower medium, and so its elastic constants are the same as that of the lower (dark-shaded) homogeneous medium. Again, to the right we show the P-SV impulse reponse.

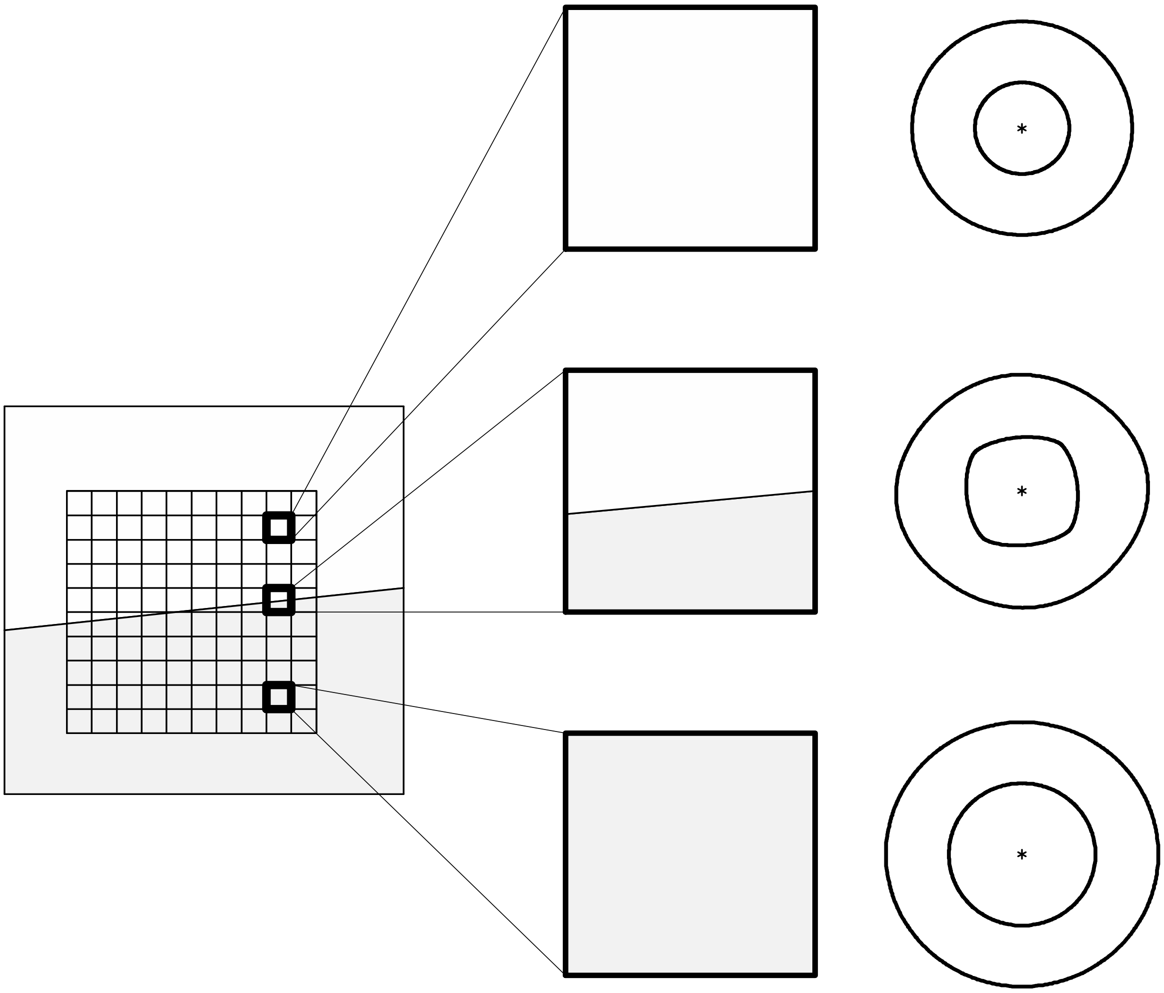

Figure 1: Left: A sampling grid laid down on a medium made up of two homogeneous blocks. Three representative grid cells have been highlighted. Center column: Expanded views of the three representative grid cells. Right column: P-SV impulse responses of the homogeneous equivalent medium for each expanded grid cell. francis3-tilt [NR]

The middle highlighted cell is the interesting one, since it contains a mixture of both media. Intuitively, the elastic constants for this cell should be some sort of average of the top and bottom media. We perform the average by considering the cell to be a bit of a tilted layered media, with the proportions of each of the two media set in accordance with the area of each in the grid cell. We then use S&M theory to find the homogeneous equivalent to this particular tilted layered media. To the right we show the resulting transversely-isotropic P-SV impulse response of this cell. Note the axis of symmetry is normal to the tilted layer boundary within the cell.

The only step left to specify in our interpolation algorithm is how to determine the effective elastic constants of a layered-medium grid cell using S&M theory.

An equivalent medium toolkit

We did this by developing a set of utilities for calculating the elastic properties of equivalent media. The toolkit enables the user to derive the elastic properties of layered and fractured media using S&M theory.

The calculation of equivalent media is based on the parameterisation of a layer by its thickness, density, and 6×6 (compressed subscript notation) stiffness tensor. The symmetric 6×6 matrix is rearranged to give three 3×3 submatrices. We call the 6×6 form the "parameter" form and the three 3×3 matrix form the "model" form. This change is merely a reordering of elements, no arrithmetic is performed.

Layers are combined in a "group domain" by simple addition of group elements. In the group domain the parameterisation takes the form of two scalars and three 3×3 matrices. The mapping from the layer to group is computed using only 3×3 matrix operations as is the inverse mapping from group to layer.

The other operation we wish to perform is rotation of the coordinate frame of the elastic constants of a layer. This is done by expanding the 6×6 stiffness matrix to its full $3 \times 3 \times 3 \times 3$ tensor form and then multiplying by a 3×3 transformation operator.

To find the elastic constants for the boundary grid cell, we add the two bounding media in the appropriate ratio, and then rotate the result so the layer dips the right amount.

EXAMPLE

For our example we use a fairly sophisticated finite-difference method to time-step forward our test wavefield. Spatial derivatives are calculated using a 2-D pseudo-spectral method; odd-order FFT's are used to obviate the need for special treatment of the spatial Nyquist. We use the accurate time integration method of Tal-Ezer (1986) to avoid losing in the time derivatives the accuracy we purchased by our careful calculation of the space derivatives. A non-staggered grid is employed to center all spatial derivatives at the grid points. This allows a complete set of 2-D elastic constants to be used. (Specifically: ρ, C_{11}, C_{13}, C_{15}, C_{33}, C_{35}, and C_{55}.) This finite-difference method can take huge time steps; the wavefield can and does travel many gridpoint lengths in one ΔT time-update step. The wavefield can also be sampled quite close to the spatial Nyquist while still generating accurate model results.

The model medium consists of one homogeneous isotropic medium over a somewhat faster one (in fact but for the exact placement of the interface the model is just

Block Model

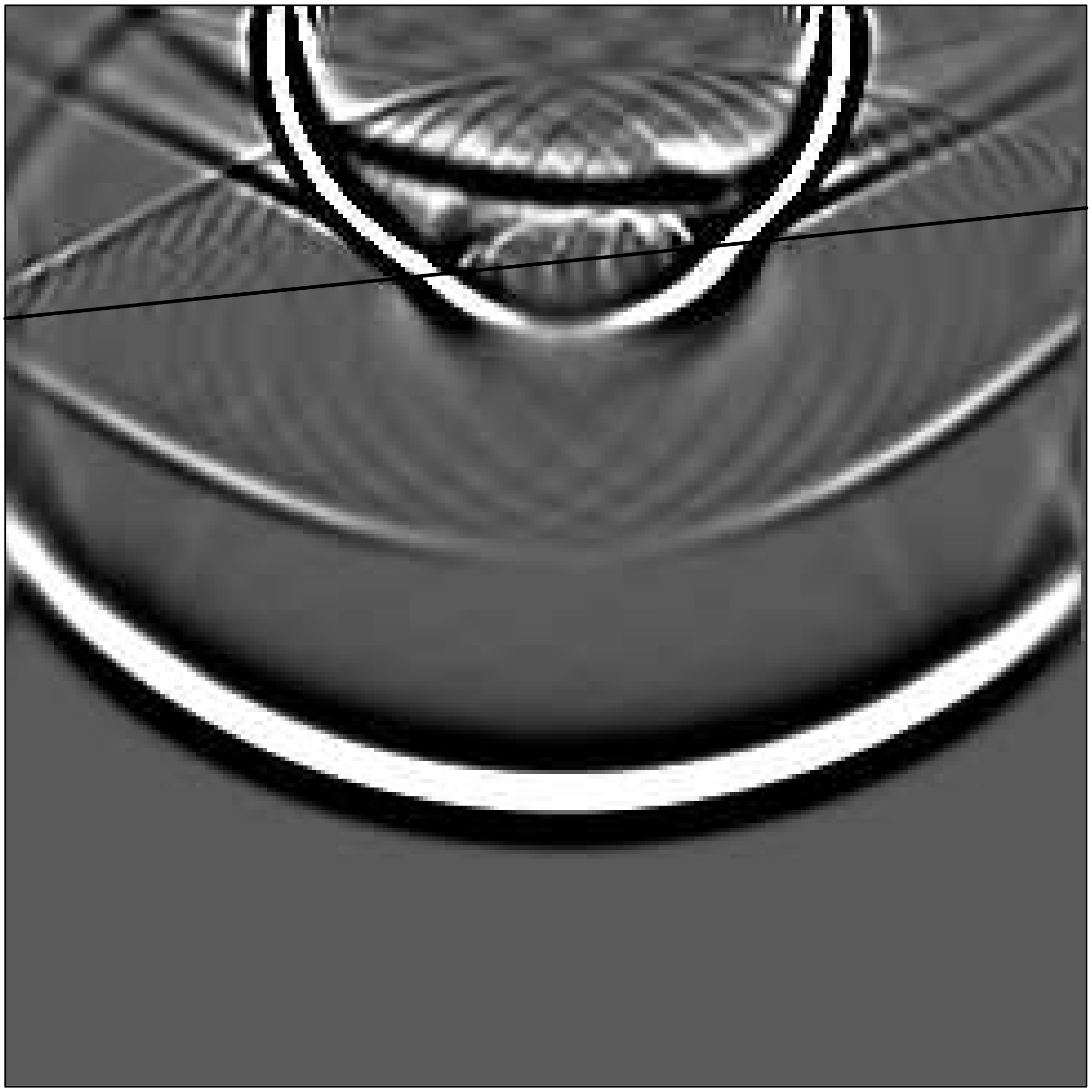

Figure 2: A snapshot from a finite-difference model showing the results of a wavefield hitting a gently sloping reflector. The source is a vertical point force located at the top center. The top edge of the model is a free surface, while all the other edges are absorbing boundaries. The stairsteps in the discretized sloping interface generate noticeable diffractions. The plot has been clipped at a level suitable for the P-S conversions, since the higher spatial frequency S waves display the diffractions better; as a result the primaries are strongly overclipped. francis3-block [NR]

S&M Model

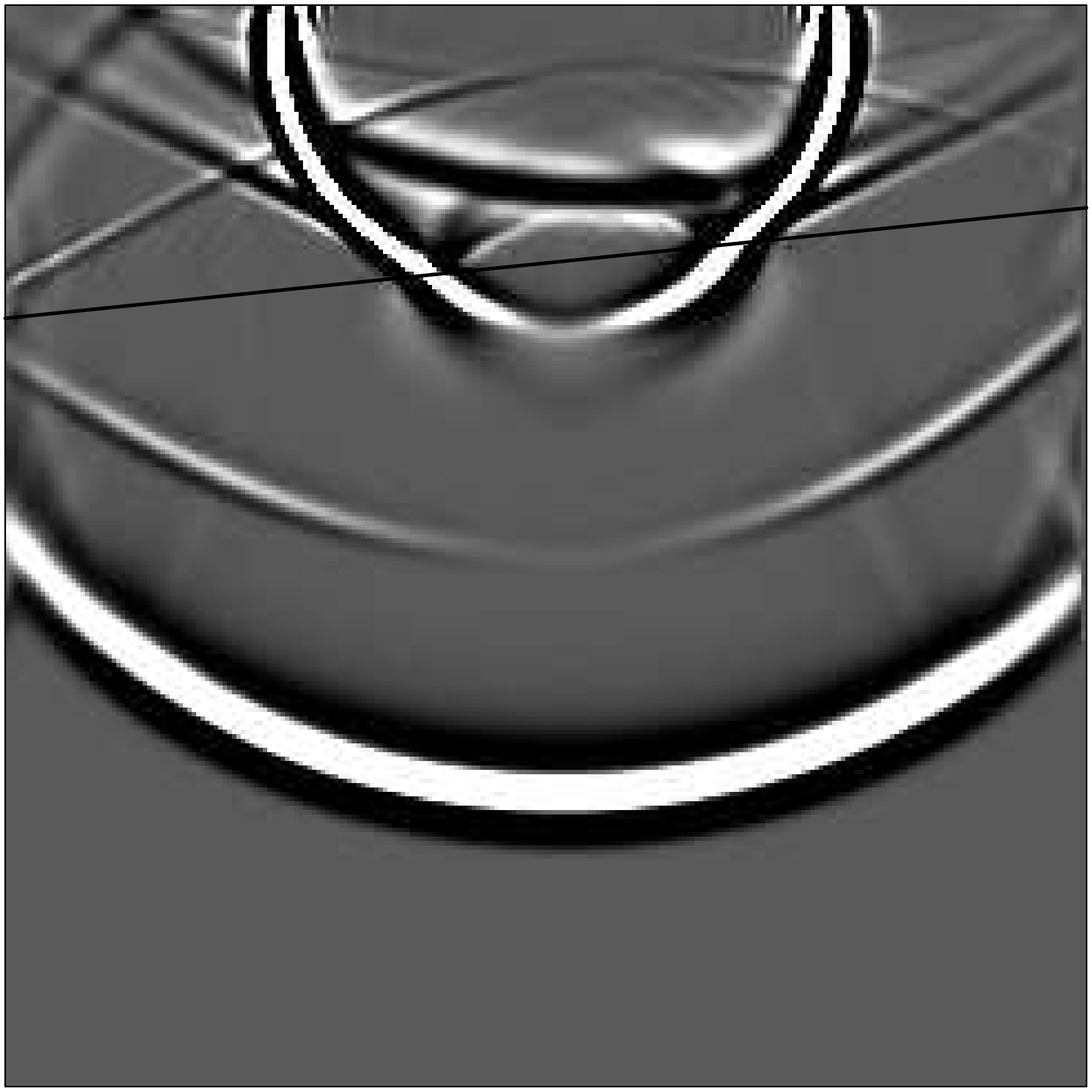

Figure 3: A snapshot from the same finite-difference model used in Figure 2, but this time Schoenburg-Muir theory has been used to interpolate the interface. The unwanted diffractions are almost completely avoided. (As a result, the remaining artifacts are more noticeable. The direct P wave can be seen to have slightly reflected off the absorbing boundaries, and to have slightly wrapped around. The artifact that can be hard to explain is the apparent semicircular wavefront attached to and just behind the direct P wave. This "artifact" is a laser-printer ghost of the strong direct S wave at the top of the model!) francis3-fuzz [NR]

the one depicted in Figure 1). The model grid consists of 243×243 gridpoints. The upper medium has elastic constants $\rho = 1.$, $C_{11} = 4.41$, and $C_{44} = .81$ (isotropic). The lower medium has elastic constants $\rho = 1.25$, $C_{11} = 8.45$, and $C_{44} = 2.45$ (isotropic). The slope of the interface is 1/10.

Figure 2 shows the results for the model if we follow the usual practice and fit a best-approximating set of stair steps to the interface. The step every 10 horizontal gridpoints makes very noticeable diffractions in the scattered wavefield. Figure 3 shows the same example using our Schoenburg-Muir interpolation scheme. As advertised, the S&M model nearly eliminates the stair-step scattering.

DISCUSSION

We have shown an example where S&M interpolation is considerably better than nearest-neighbor interpolation. As we have stated, there is an apparent drawback in that interpolated grid points near lithologic boundaries will usually require increased anisotropic complexity. In particular, an average of two differing isotropic materials will not be isotropic. The long-term solution is clear. There is no longer much justification for the idea that the Earth can be assumed isotropic at seismic scales. We should develop and preferentially use modeling schemes that allow for a general Hooke's Law relation with a complete set of elastic constants. In the short term there is an intermediate solution. S&M theory is based on conservation arguments, and we can validly ask which isotropic material best represents the elastic properties of the anisotropic composite.

In any case, it is clear the elastic constants should be interpolated as a unified tensor quantity. In our S&M example in Figure 1, for example, C_{15} is zero in both of the isotropic homogeneous media, but is distinctly nonzero in the tilted-axis transversely-isotropic medium necessary to represent the border grid cells. No interpolation method that considers each elastic constant independently of the others could interpolate a nonzero between two sets of zeroes. While we have not proven that our S&M-based method is the optimal tensor interpolation scheme for this application, it does have the advantage that it is simple to understand and easy to implement, and does seem to work well.

How might we do better? There are two areas we have not discussed. First, we have taken the non-overlapping tile as being a good representation of the region of influence of a point. By analogy with the sort of thing we do in geophone array design, we might consider using a tile of twice the linear dimension and tapered from the center out to the corners. These thick tiles would still uniformly cover the region, and might offer a better compromise between smooth transition and compact size. Second, our scheme is based on static arguments. As frequencies rise, so do the results deteriorate. Introducing viscous damping into the scheme may help, but the theory is not yet developed, and it would add double the number of parameters required to describe the elastic properties. Again, this may be a worthwhile complication to be building into our modeling, since heterogeneity lowers Q factor as well as increasing anisotropy.

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