Stanford Exploration Project, Report 67, January 13, 1998, pages 1–??

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A geometrical approach to seismic processing: the method of discontinuities

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keywords: *velocity, continuation, amplitudes, ray tracing, modeling, stacking, traveltime*

INTRODUCTION

What do we expect from a theory? Efficiency, prediction, and explanation. Certainly the first aim seems to be the most important. Many specialists apply a theory mainly with the purpose to construct effective algorithms. But I believe the prediction and explanation by no means are less important. Though we do use computers, we ourselves are not computers at all. And an investigator who possesses algorithms but does not understand why and how it works cannot estimate proper results of processing and realize all possibilities of algorithms.

There is no comprehensive theory. Any theory describes only a part of a whole picture from some particular point of view. It especially concerns theories pretending to give explanations. Any analysis orientated towards understanding a specific class of transformations is connected with the decomposition of a wave field under transformation into two parts: (1) the main part and, (2) all other oscillations. In many cases a set of individual waves is considered the main part of the field. Such an analysis can't be but an approximation. As a rule, formulas describing propagation of waves are asymptotic. But it is not bad! A brilliant example is the ray theory which represents a basement of our understanding of the wave propagation in a wide class of inhomogeneous media. Of course, the approximate character of the ray method also has a pragmatic significance since it generates a lot of simple algorithms for dynamic forward modeling. For many years this pragmatic aspect was estimated as the most essential. Many geophysicists believed that after some time (as a result of the appearance of big computers) finite-difference techniques would monopolize forward modeling. But this has not happened! If something did happen, it was that results of numerical modeling in many cases were analyzed with the help of the ray method because sometimes this is the only way to connect particularities of a wave field with particularities of a model.

A very important advantage of the ray theory is its geometrical character. All main notions of the theory — fronts, rays, caustics, critical angles, diffraction, etc.

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— are geometrical by nature. It is the geometry that stimulates our imagination and, therefore, improves our understanding of wave phenomena. Geometrical character of the ray theory is essentially connected with the fact that the theory factually describes propagation of wave field discontinuities and their interaction with discontinuities of functions describing media.

It is natural to use the same approach for the analysis of wave field transformations that we use in the seismic processing. The usual approach is to analyze these transformations in (f, k)-domain. But this technique is adequate in a case where if the transformation under investigation possesses stationarity with respect to variables that are not valid in some other cases. But that's not all. It is not so simple to analyze the location (travel-time curve) and amplitude of a wave after transformation in spectra terms. It is necessary to use a complex domain in the framework of the stationary phase technique. Meanwhile, discontinuities allow one to obtain a desirable result much more simple.

For the first time I used discontinuities about a decade ago while investigating the wave field continuation. I could see that the notion of wave field discontinuity allows one to study a wide class of problems without specifying a particular algorithm and therefore to pose the task of choosing the optimum one. Subsequently, it had become evident that this approach was effective in many other cases, especially when the transformation was expressed by the integral geometry operator.

Publication of these course lectures became possible thanks to Jon Claerbout who invited me to visit Stanford University and to the enthusiasm of the SEP participants who were not only my attentive listeners but helped very much in the preparation of this course. I am especially much obliged to Debbie Levoy and Lin Zhang.

1: WHAT IS A DISCONTINUITY?

Functions with discontinuities

Let us consider the functions listed in the left column of Table 1.

Table 1: Functions and discontinuities

function	order	A_0	A_1	A_2
$H(t)$	0	1	0	0
$e^{-\alpha t} \cdot H(t)$	0	1	$-\alpha$	α^2
$e^{-\alpha t} \sin \omega_0 t \cdot H(t)$	1	0	ω_0	$2\alpha\omega_0$
$te^{-\alpha t} \cos \omega_0 t \cdot H(t)$	1	0	1	-2α
$te^{-\alpha t} \sin \omega_0 t \cdot H(t)$	2	0	0	$2\omega_0$
$t^2 e^{-\alpha t} \cos \omega_0 t \cdot H(t)$	2	0	0	2
$e^{-\alpha\|t\|}$	1	0	-2α	$2\alpha^2$
$e^{-\alpha\|t\|} \sin \omega_0 t$	1	0	0	$-4\alpha\omega_0$

All these functions have one general property in common: a discontinuity (a jump of derivatives) at the point $t = 0$. The simplest example of a discontinuity is given by the step function $H(t)$ (Figure **??**). It has a discontinuity of order 0. We observe the same discontinuity for the second function $e^{-\alpha t}H(t)$. One can find the orders of the discontinuities of the other functions in column 2. In columns 3 etc., the amplitudes of derivative jumps are given:

$$A_k = f^{(k)}(0^+) - f^{(k)}(0^-) \tag{1}$$

where

$$f^{(k)}(0^+) = \lim_{t \to 0^+} f^{(k)}(t),$$

$$f^{(k)}(0-) = \lim_{t \to 0^-} f^{(k)}(t).$$

Definition: The function $f(t)$ has a discontinuity of order r at the point $t = 0$, if for all $k < r,\ A_k = 0$ and for $k = r,\ A_r \neq 0$. Values $A_k,\ (k \geq r)$ are called amplitudes of order k.

Fourier transformations

Very simple examples of Fourier transformations for functions with discontinuities are given Table 2.

Table 2: Fourier Transformations	
$e^{-\|t\|}$	$\frac{2}{1+\omega^2}$
$e^{-t}.H(t)$	$(1 - i\omega)/(1 + \omega^2)$
$H(t)$	$\frac{1}{i\omega}$
$te^{-t}.H(t)$	$1/(1 + i\omega)^2$

It is very simple to determine the Fourier transforms of all the functions in Table 1 using these simple properties of the Fourier transformation:

$$\begin{aligned} f(at) &\longleftrightarrow \frac{1}{|a|}F(\frac{\omega}{a}) \\ f(t)\sin(\omega_0 t) &\longleftrightarrow \frac{1}{2i}[F(\omega+\omega_0) - F(\omega-\omega_0)] \\ f(t)\cos(\omega_0 t) &\longleftrightarrow \frac{1}{2}[F(\omega+\omega_0) + F(\omega-\omega_0)]. \end{aligned}$$

We can see from the Table 2 that the Fourier transforms of functions with discontinuities of order 0 behave at $\omega \to \infty$ as $\frac{A_0}{i\omega}$. Meanwhile, in case of a discontinuity of order 1, $|F(\omega)|$ behaves asymptotically as $\frac{A_1}{\omega^2}$. Let us take, for instance, the third function from Table 1: $g(t) = \exp(-\alpha t)\sin\omega_0 t H(t)$. In this case, $g(t) = f(\alpha t)\sin\omega_0 t$ where $f(t) = \exp(-t)H(t)$. Then

$$G(\omega) = \mathbf{F}_t g(t) = \frac{1}{2i\alpha}\left[\frac{1 - i(\frac{\omega-\omega_0}{\alpha})}{1 + (\frac{\omega-\omega_0}{\alpha})^2} - \frac{1 - i(\frac{\omega+\omega_0}{\alpha})}{1 + (\frac{\omega+\omega_0}{\alpha})^2}\right]$$

and when $\omega \to \infty$

$$G(\omega) \simeq \frac{1}{2i\alpha}\frac{\frac{2i\omega_0}{\alpha}}{1 + \frac{\omega^2}{\alpha^2}} \simeq \frac{\omega_0}{\omega^2}.$$

More generally, it is simple to prove that, if $f^{(r-1)}(t) \in \mathrm{C}$ (i.e., is continuous), and if $f^{(r)}(t) \in \mathrm{L}_1$ (i.e., is summable), then $(i\omega)^r F(\omega) \in \mathrm{L}_1$. Consequently, when $\omega \to \infty$, $F(\omega) \to 0$, this convergence being not slower than $\frac{1}{(i\omega)^{r+1}}$.

We shall prove the following statement. Let $f(t)$ be a right-sided function, and, let's suppose that, for all $r = 0, 1, 2, \cdots$, $f^{(r)}(t) \to 0$ (when $t \to \infty$) in the same way as $\exp(-\alpha_k|t|)$. Then:

$$F(\omega) = \sum_{n=r}^{N}\frac{A_n}{(i\omega)^{n+1}} + o(\frac{1}{\omega^{N+1}}) \qquad (A_r = f^{(r)}(0^+)). \tag{2}$$

As $f(t)$ is a right-sided function, we have:

$$F(\omega) = \int_0^\infty f(t)e^{-i\omega t}dt. \tag{3}$$

Let us integrate this integral by parts:

$$\begin{aligned} F(\omega) &= -\frac{1}{i\omega}\int_0^\infty f(t)de^{-i\omega t} && (4)\\ &= -\frac{1}{i\omega}f(t)e^{(-i\omega t)}\Big|_{0^+}^{\infty} + \frac{1}{i\omega}\int_0^\infty f'(t)e^{-i\omega t}dt && (5)\\ &= \frac{A_0}{i\omega} + \frac{1}{i\omega}\int_0^\infty f'(t)e^{-i\omega t}dt. && (6) \end{aligned}$$

Repeating this N times we obtain

$$F(\omega) = \sum_{n=0}^{N}\frac{A_n}{(i\omega)^{n+1}} + \frac{1}{(i\omega)^{N+1}}\int_0^\infty f^{(N+1)}(t)e^{i\omega t}dt. \tag{7}$$

By definition of the order of discontinuity $A_n = 0$, $n = 0, 1 \ldots, r-1$. From the condition concerning the behavior of $f^{(k)}(t)$ at $t \to \infty$, the integral on the righthand side of equation (7) converges to 0 at $\omega \to \infty$. This means that the last equation is equivalent to equation (2). So the asymptotic behavior of the Fourier-transforms for high frequencies reflects the sharpness of the discontinuities of the (original) functions. Both notions (asymptotic behavior and discontinuities) are equivalent ones.

Equivalence of the order r

We shall write $f(t) \stackrel{r}{\sim} 0$ if $f^{(r)}(t)$ is continuous (that is, $f^{(r)} \in \mathrm{C}$). If $f_1 - f_2 \stackrel{r}{\sim} 0$ then $f_1 \stackrel{r}{\sim} f_2$. (It is not necessary that each of the functions f_1 and f_2 belongs to C^r). We don't distinguish functions whose difference is a smooth function. Each continuous function is equivalent (of order $r = 0$) to 0 and each polynomial $P_n(t)$ is equivalent to 0 with the order $r = \infty$.

- Example 1: $H(t) \stackrel{r}{\not\sim} 0$ (r=0,1,2,...) but $H^{(-1)}(t) = t_+ \equiv tH(t) \in \mathrm{C}$. We may introduce the order $r = -1$ and write $H(t) \stackrel{-1}{\sim} 0$.
- Example 2:

$$f_1(t) = e^{-\alpha t} \sin \omega_0 t \cdot H(t) \stackrel{0}{\sim} 0 \tag{8}$$

$$f_2(t) = te^{-\alpha t} \cos \omega_0 t \cdot H(t) \stackrel{0}{\sim} 0 \tag{9}$$

  but definitely $f_1 \stackrel{1}{\not\sim} 0$ and $f_2 \stackrel{1}{\not\sim} 0$.

Is it true that $f_1(t) \stackrel{0}{\sim} f_2(t)$? The answer is: yes. Is it true that $f_1(t) \stackrel{1}{\sim} f_2(t)$? The answer is: no, however $f_1(t) \stackrel{1}{\sim} \omega_0 f_2(t)$. (It easily may be derived that $f_1(t) - \omega_0 f_2(t)$ has a continuous derivative at the point $t = 0$).

2: THE STANDARD DISCONTINUITIES (OF INTEGER ORDER)

In fact we shall deal not with functions themselves but with classes of equivalent (of the fixed order q) functions, that is, with classes of equivalence. In this situation, it is convenient to choose one simple representative from each class of equivalence.

Definition

We shall call a standard discontinuity a function:

$$R_q(t) = \frac{t_+^q}{q!}, \qquad (q = 0, 1, 2, \ldots;\ t_+ = tH(t)).$$

We see that $H(t) \equiv R_0(t)$ and $t_+ \equiv R_1(t)$.

The denominator $q!$ is introduced in order to have a very simple (and very important!) relation:

$$\mathbf{D}R_q(t) = R_{q-1}(t), \qquad q = 1, 2, 3, \ldots$$

where $\mathbf{D} = \frac{d}{dt}$.

Since $\mathbf{D}R_0(t) = \delta(t)$, it is convenient to define

$$R_{-1}(t) = \delta(t)$$

(and $R_{-k}(t) = \delta^{(k)}(t), \quad k = 1, 2, \ldots$).

The simplest properties

Let $S(t) \in \mathrm{C}^q$, then:

$$R_q(t) + S(t) \overset{q}{\sim} R_q(t)$$

and

$$S(t)R_q(t) \overset{q}{\sim} S(0)R_q(t). \tag{10}$$

Let us prove the last formula. We must prove that $\mathbf{D}^q[S(t)R_q(t) - S(0)R_q(t)]$ is a continuous function:

$$\begin{aligned}
\mathbf{D}^q[\cdot] &= \mathbf{D}^q\{[S(t) - S(0)]R_q(t)\} && (11)\\
&= \mathbf{D}^{q-1} \cdot \mathbf{D}\{[S(t) - S(0)]R_q(t)\} && (12)\\
&= \mathbf{D}^{q-1}\{S^{'}(t)R_q(t)\} + \mathbf{D}^{q-1}\{[S(t) - S(0)]R_{q-1}(t)\}. && (13)
\end{aligned}$$

The last term has the sharpest discontinuity , so we needn't pay any attention to the first one. Repeating the procedure we get at the end the sharpest term:

$$[S(t) - S(0)]R_0(t)$$

which is continuous at the point $t = 0$.

Instead of equation (10), we may derive a more general formula:

$$S(t)R_q(t) \begin{cases} \overset{q}{\sim} & S(0)R_q(t) \\ \overset{q+1}{\sim} & S(0)R_q(t) + (q+1)S^{'}(0)R_{q+1}(t) \\ \overset{q+s}{\sim} & \sum_{r=0}^{s} \frac{(q+r)!}{r!q!} S^{(r)}(0)R_{q+r}(t). \end{cases}$$

Examples

$$\begin{aligned}
e^{-\alpha t}H(t) &\overset{0}{\sim} H(t) \equiv R_0(t)\\
&\overset{1}{\sim} R_0(t) - \alpha R_1(t)\\
&\overset{2}{\sim} R_0(t) - \alpha R_1(t) + \alpha^2 R_2(t)\\
&\cdots
\end{aligned}$$

$$\begin{aligned}
e^{-\alpha t}\sin(\omega_0 t)H(t) &\overset{0}{\sim} 0\\
&\overset{1}{\sim} \omega_0 R_1(t)\\
&\overset{2}{\sim} \omega_0 R_1(t) + 2\alpha\omega_0 R_2(t)\\
&\cdots
\end{aligned}$$

$$
\begin{aligned}
te^{-\alpha t}\cos(\omega_0 t)H(t) &\overset{0}{\sim} 0\\
&\overset{1}{\sim} R_1(t)\\
&\overset{2}{\sim} R_1(t) - 2\alpha R_2(t)\\
&\cdots
\end{aligned}
$$

From equation (10), it follows immediately that:

$$\mathbf{D}^p S(t)R_q(t) \overset{q-p}{\sim} S(t)R_{q-p}(t).$$

Indeed:

$$
\begin{aligned}
\mathbf{D}^p S(t)R_q(t) &\overset{q-p}{\sim} \mathbf{D}^p S(0)R_q(t) = S(0)\mathbf{D}^p R_q(t) = \cdots\\
&\cdots = S(0)R_{q-p}(t) \overset{q-p}{\sim} S(t)R_{q-p}(t).
\end{aligned}
$$

The operators $\mathbf{I}^p$ and $\mathbf{D}^{-p}$

It is obvious that the integration of R_q(t) gives R_{q+1}(t):

$$\mathbf{I}R_q(t) = R_{q+1}(t)$$

and in general:

$$\mathbf{I}^p R_q(t) = R_{q+p}(t) \qquad (p > 0).$$

Does it imply that: $\mathbf{D}^{-p}R_q(t) = R_{q+p}(t)$ (where $\mathbf{D}^{-p}$ is the inverse of $\mathbf{D}^p$)? If we apply the operator $\mathbf{D}^p$ to the sum $R_{p+q}(t) + P_n(t)$, we get for $n \leq p-1$:

$$\mathbf{D}^p[R_{q+p}(t) + P_n(t)] = R_q(t).$$

But $P_n(t) \overset{\infty}{\approx} 0$, so:

$$\mathbf{D}^{-p}R_q(t) \overset{\infty}{\approx} R_{q+p}(t). \tag{14}$$

Hilbert transformation

There is a very important case of discontinuity: $1/t$. It does not belong to our set of standard discontinuities. It will be useful to expand our set of discontinuities by introducing the Hilbert transformation:

$$R_{q,1}(t) = \mathbf{H}R_q(t).$$

We see that $R_{-1,1} = \mathbf{H}\delta(t) = \frac{1}{\pi t}$. But what about $R_{q,1}$ for $q > -1$? Unfortunately, the integral:

$$\mathbf{H}R_{q,1}(t) = \frac{1}{\pi q!}\int_0^\infty \frac{\tau^q}{(t-\tau)}d\tau$$

does not exist because of the non-integrable singularity at the point τ=0. But we may use the following trick: if $q > -1$, we apply the operator $\mathbf{D}^{q+1}$ and obtain the discontinuity R_{-1}. After that we may apply the Hilbert transformation $\mathbf{H}$ and then the inverse operator $\mathbf{D}^{-(q+1)}$.

Remembering the equation (14), we get:

$$R_{q,1}(t) \overset{\infty}{\sim} \mathbf{D}^{-(q+1)}\mathbf{H}\mathbf{D}^{q+1}R_q(t).$$

This is a definition of the Hilbert transformation for discontinuities. We define also:

$$R_{q,\nu}(t) = \left[\cos(\frac{\pi\nu}{2})\mathbf{E} + \sin(\frac{\pi\nu}{2})\mathbf{H}\right] R_q(t) \equiv \mathbf{H}^{\nu}R_q(t)$$

where $\mathbf{E}$ is the unit operator.

Fourier-transformation

As

$$R_q(t) = \mathbf{I}^{q+1}R_{-1}(t) = \mathbf{I}^{q+1}\delta(t)$$

then

$$\mathbf{F}_t R_q(t) = (\frac{1}{-i\omega})^{q+1} \cdot 1 = i|\omega|^{-q-1}\exp(i\frac{q\pi}{2}\mathrm{sgn}(\omega)). \tag{15}$$

Convolution

It follows directly from convolution theorem and Formula 15 that

$$aR_q(t) * bR_p(t) = abR_{q+p+1}(t).$$

Moreover, if $f_1(t) \overset{q}{\sim} aR_q(t)$ and $f_2(t) \overset{p}{\sim} bR_p(t)$, then

$$f_1(t) * f_2(t) \overset{q+p+1}{\sim} a \cdot bR_{q+p+1}(t). \tag{16}$$

Proof: Let us express Fourier-transformations of $f_1(t)$ and $f_2(t)$ according to equation (2), taking $N = r = q$ (for $f_1(t)$) and $N = r = p$ (for $f_2(t)$). Performing multiplication, we get:

$$F_1(\omega)F_2(\omega) = \frac{ab}{(i\omega)^{p+q+2}} + 0\left(\frac{1}{\omega^{p+q+2}}\right).$$

That is exactly the same that equation (16) expresses in temporal domain. The convolution

$$f(t) * R_q(t) = f_{q+1}(t)$$

is the q^{th} integration of the $f(t)$.

Discontinuities and high-frequency band filters

Let $L(\omega)$ be the spectral characteristics of a high frequency band filter. We suggest that $L(\omega) \simeq 0$ at $|\omega| < \omega_c$ and that at $|\omega| > \omega_c$, Fourier-transformation of an input signal may be described as

$$F(\omega) \simeq \frac{A_r}{(i\omega)^{r+1}}.$$

Then Fourier-transformation of the output signal $g(t)$ will be equal to $G(\omega) \simeq \frac{A_r L(\omega)}{(i\omega)^{r+1}}$. It means that

$$g(t) \simeq A_r l_{r+1}(t)$$

where $l_{r+1}(t)$ is $(r+1)^{th}$ integration of the impulse response. We have obtained two facts: (a) the only feature of the input signal that influences the shape of output signal is the order of its discontinuity, and (b) the amplitude of the discontinuity coincides with the amplitude of output signal. We conclude that the output signal is a response to the discontinuity. This conclusion may be interpreted from the point of view of the theory of ordinary differential equations. Let us assume that we have a filter which is described by the equation

$$a_n y^{(n)} + \ldots + a_0 y = 0. \tag{17}$$

If the input is $\sin \omega_0 t \cdot H(t)$, then the output is $y(t) = y_0(t) + y_1(t)$ where

$$y_1(t) = |L(\omega_0)| \sin(\omega_0 t - \varphi(\omega_0))$$

represents the particular solution with $L(\omega) = \frac{1}{a_n(i\omega)^n + \ldots a_0}$. $y_0(t)$ describes the free oscillation that is the solution to a Cauchy problem for the equation (17) with conditions

$$y^{(k)}(0+) = A_k$$

(here $A_0 = 0, A_1 = \omega_0, A_2 = 0, A_3 = -\omega_0^3, \ldots$). Therefore, we see that the free oscillations are a response to discontinuities.

Let us now assume that a right-sided quasi-sinusoidal signal $f(t)$ is the input of the filter L which has very short impulse characteristics $l(t)$. For times comparable with the duration of $l(t)$, the filter "does not know" that $f(t)$ differs from $\sin(\omega_0 t)$. Therefore one can expect that the response will contain two parts, the first part is $y_0(t)$, the reaction to the beginning of the signal (that is, to the discontinuity!) with the dominating frequency of the $l(t)$. The second part represents the response to the signal itself with the dominating frequency of the signal $f(t)$. A corresponding theory was proposed by S. Katz (1966) (see also my book (1974)).

3: WHY DISCONTINUITIES?

Discontinuities are connected with characteristics of hyperbolic equations in partial derivatives that describe wave phenomena in continuous media.

Characteristics

Let us consider the equation:

$$\mathbf{L}u - b(\mathbf{r})\frac{\partial^2 u}{\partial t^2} = 0 \tag{18}$$

where:

$$\mathbf{L} = \sum_{i,k=1}^{3} a_{ik}(\mathbf{r})\frac{\partial^2}{\partial x_i \partial x_k} + \dots$$

$(x_1 = x, x_2 = y, x_3 = z, \mathbf{r} = (x, y, z))$. The three dots represent terms of lower order.

Let $\psi(x, y, z, t) = 0$ be a surface of arbitrary shape and let the values of u, $\frac{\partial u}{\partial x_i}(i = 1, 2, 3)$ and $\frac{\partial u}{\partial t}$ be given on the surface. Is this data sufficient for the reconstruction of the unique solution of the equation (18)? Does the Cauchy problem for equation (18) with the initial data on ψ have a unique solution? The answer will be positive if it is possible to reconstruct all second derivatives on the surface ψ.

Let us introduce a system of functions $\omega_i(x, y, z, t)(i = 1, 2, 3, 4)$, with $\omega_1 \equiv \psi(x, y, z, t)$ and with the change of variables:

$$x'_s = \omega_s(\mathbf{r}, t),\ s = 1, 2, 3, 4.$$

On the surface $\omega_1 \equiv x'_1 = 0$ all values x'_2, x'_3, x'_4 do vary. That means the given data allows one to calculate all mixed derivatives $\frac{\partial^2 u}{\partial x'_i \partial x'_j}(i, j = 1, 2, 3, 4)$ and second derivatives $\frac{\partial^2 u}{\partial {x'_i}^2}(i = 2, 3, 4)$. Is it possible to calculate $\frac{\partial^2 u}{\partial {x'_1}^2}$?

Let us express the derivatives with respect to the old variables through the derivatives with respect to the new variables, omitting terms that are not interesting for us, such as:

$$u_{x_i} = u_{x'_1}\frac{\partial \omega_1}{\partial x_i} + \dots;\ u_{x_i x_k} = u_{x'_1 x'_1}\frac{\partial \omega_1}{\partial x_i}\frac{\partial \omega_1}{\partial x_k} + \dots$$

and so on.

The equation (18) can be rewritten in the form:

$$a'_{11}u_{x'_1 x'_1} + \dots = 0 \tag{19}$$

where:

$$a'_{11} = \sum_{i,k=1}^{3} a_{ik}\frac{\partial \omega_1}{\partial x_i}\frac{\partial \omega_1}{\partial x_k} - b\left(\frac{\partial \omega_1}{\partial t}\right)^2.$$

All terms denoted by dots are known on the surface $\omega_1 = 0$, so the only condition that supplies a possibility to determine $u_{x'_1 x'_1}$ is $a'_{11} \neq 0$. If $a'_{11} = 0$, then the Cauchy problem is unsolvable. In this case, surface $\psi = 0$ is a special one. And we call it a characteristic (or characteristic surface).

So

$$\sum_{i,k=1}^{3} a_{ik} \frac{\partial \psi}{\partial x_i} \frac{\partial \psi}{\partial x_k} - b\left(\frac{\partial \psi}{\partial t}\right)^2 = 0 \tag{20}$$

is a characteristic equation for the equation (18). But what does this have to do with discontinuities?

Characteristics and discontinuities

Let us assume that the solution $u(\mathbf{r}, t)$ of the equation (18) has a "weak discontinuity", (i.e., discontinuity of the order 2) on the surface $\omega(\mathbf{r}, t) = 0$. This means that

- $u(\mathbf{r}, t)$ and all first derivatives are continuous when crossing the surface ω.
- $u(\mathbf{r}, t)$ satisfies the equation (18) on each side of the surface ω.
- Some second derivatives have a jump when crossing the surface ω (they have two different values on both sides of the surface ω).

But the latter is only possible when ω is a characteristic! Otherwise, unique values of u and $\frac{\partial u}{\partial x_i}$ would determine the unique values of all second derivatives on the surface ω.

So a characteristic surface is the only possible location for a weak discontinuity if it is a solution of the equation (18). If this is valid for a weak discontinuity, it is also valid for discontinuities of all orders if they represent jumps of derivatives with respect to t. It follows through a simple consideration that: $u(\mathbf{r}, t), \int_0^t u(\mathbf{r}, t) dt, \dot{u}(\mathbf{r}, t), \ddot{u}(\mathbf{r}, t), \ldots$ satisfy the same equation (18), and we can always transform an arbitrary discontinuity into a weak one.

Eikonal equation

The characteristic equation $\omega(\mathbf{r}, t) = 0$ describes the propagation of a discontinuity in space (x, y, z). Let us assume that equation $\omega = 0$ can be solved with respect to t: $t = \tau(\mathbf{r})$ and $\omega \equiv t - \tau(\mathbf{r}) = 0$. Then we get from equation (20):

$$\sum_{i,j=1}^{3} a_{i,j} \frac{\partial \tau}{\partial x_i} \frac{\partial \tau}{\partial x_j} = b. \tag{21}$$

That is exactly the Eikonal equation for a medium with elliptic anisotropy. In the case of isotropy, it follows that $a_{ik} = a\delta_{ik}$ and equation (21) is reduced to

$$\left(\frac{\partial \tau}{\partial x_1}\right)^2 + \left(\frac{\partial \tau}{\partial x_2}\right)^2 + \left(\frac{\partial \tau}{\partial x_3}\right)^2 = \frac{1}{v^2} \tag{22}$$

$v = (\sqrt{\frac{a}{b}})$.

Let us consider, for example, the linear acoustic equation

$$\mathrm{div}(k\mathrm{grad}(u)) = \rho \frac{\partial^2 u}{\partial t^2}$$

or

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i}(k\frac{\partial u}{\partial x_i}) = \rho \frac{\partial^2 u}{\partial t^2}.$$

In this case $a_{ij} = k\delta_{ij}, b = \rho$ and we again get equation (**??**) at $v = (\sqrt{\frac{k}{\rho}})$. It is very important that the propagation of a discontinuity depends only on senior terms of the equation (18).

Elastic medium

The wave equation for elastic medium is a vector equation. It belongs to a class of equations of the type:

$$\mathbf{Lu} = b\frac{\partial^2 \mathbf{u}}{\partial t^2}$$

where $\mathbf{u}$ is a vector and $\mathbf{L}$ is a matrix operator:

$$\mathbf{L} = \sum_{i,j=1}^{3} \mathbf{A}_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

$\mathbf{A}_{ij}(\mathbf{r})$ - a matrix.

If we develop the same considerations as with the scalar case, we shall obtain instead of equation (19) the vector equation

$$\mathbf{Nu}_{x'_1 x'_1} + \ldots = 0$$

with a matrix

$$\mathbf{N} = \sum \frac{\partial \omega_1}{\partial x_i}\frac{\partial \omega_1}{\partial x_k}\mathbf{A}_{ik} - b(\frac{\partial \omega_1}{\partial t})^2 \mathbf{E}.$$

It is obvious that characteristics satisfy the equation

$$\det \mathbf{N} = 0. \tag{23}$$

In general, equation (23) has three solutions. This is due to the fact that in a continuous one-phase medium where wave phenomena are described by second order equations, only three types of body waves (not more, but eventually less!) may exist.

Let us check the Lame's equation (isotropic elasticity):

$$(\lambda+\mu)\mathrm{grad}(\mathrm{div})\mathbf{u}+\mu\Delta\mathbf{u}+\mathrm{grad}(\lambda\mathrm{div}\mathbf{u})+\sum_{i=1}^{3}\left[\mathrm{grad}\mu,\ \frac{\partial \mathbf{u}}{\partial x_i} + \mathrm{grad}u_i\right]\mathbf{e}_i = \rho\frac{\partial^2 \mathbf{u}}{\partial t^2}. \tag{24}$$

We may omit the terms with first derivatives of $\mathbf{u}$. So we have the system:

$$(\lambda+\mu)\frac{\partial}{\partial x_j}\left(\sum_i \frac{\partial u_i}{\partial x_i}\right)+\mu\sum_{i=1}^{3}\frac{\partial^2 u_j}{\partial x_i^2}+\ldots=\rho\frac{\partial^2 u_j}{\partial t^2},\ j=1,2,3.$$

It is easy to derive the correspondent system of characteristic equations. The first line is:

$$\left[(\lambda+\mu)\left(\frac{\partial\tau}{\partial x_1}\right)^2-\rho+\mu|\nabla\tau|^2\right]v_1+(\lambda+\mu)\frac{\partial\tau}{\partial x_1}\frac{\partial\tau}{\partial x_2}v_2+(\lambda+\mu)\frac{\partial\tau}{\partial x_1}\frac{\partial\tau}{\partial x_3}v_3+\ldots=0$$

where $v_i=\frac{\partial^2 u_i}{\partial x_1^{\prime 2}}$. The matrix of the equation system may be written in the following form:

$$\mathbf{N}=(\lambda+\mu)\mathbf{P}\mathbf{P}^T+(\mu|\nabla\tau|^2-\rho)\mathbf{E} \tag{25}$$

where $\mathbf{P}$ is a column vector with the components $\frac{\partial\tau}{\partial x_i}$.

Let us prove that the vector $\mathbf{P}$ is an eigenvector of $\mathbf{N}$:

$$\mathbf{NP}=(\lambda+\mu)\mathbf{P}\mathbf{P}^T\mathbf{P}+(\mu|\nabla\tau|^2-\rho)\mathbf{EP},$$

where $\mathbf{P}^T\mathbf{P}=|\nabla\tau|^2$, so

$$\begin{aligned}\mathbf{NP} &= [(\lambda+\mu)|\nabla\tau|^2+\mu|\nabla\tau|^2-\rho]\mathbf{P}\\ &= [(\lambda+2\mu)|\nabla\tau|^2-\rho]\mathbf{P}.\end{aligned} \tag{26}$$

Therefore, the first eigenvalue is:

$$(\lambda+2\mu)|\nabla\tau|^2-\rho.$$

Let us consider a vector $\mathbf{Q}\perp\mathbf{P}$:

$$\mathbf{P}^T\mathbf{Q}=0$$

$$\begin{aligned}\mathbf{NQ} &= (\lambda+\mu)\mathbf{P}\mathbf{P}^T\mathbf{Q}+(\mu|\nabla\tau|^2-\rho)\mathbf{Q}\\ &= (\mu|\nabla\tau|^2-\rho)\mathbf{Q}.\end{aligned} \tag{27}$$

The latter is valid for any vector $\mathbf{Q}\perp\mathbf{P}$. Consequently, this eigenvalue is degenerated and it follows that

$$|\mathbf{N}|=((\lambda+2\mu)|\nabla\tau|^2-\rho)(\mu|\nabla\tau|^2-\rho)^2.$$

So we may satisfy the equation (19) in both cases:

$$|\nabla\tau|^2=\frac{1}{{v_p}^2},\ v_p=\sqrt{\frac{\lambda+2\mu}{\rho}}$$

$$|\nabla\tau|^2=\frac{1}{{v_s}^2},\ v_s=\sqrt{\frac{\mu}{\rho}}.$$

This means that in isotropic elastic media, there are only two types of body waves which move with velocities v_p and v_s, respectively. In the general case of anisotropic elastic media, there are three families of characteristics and correspondently three body waves: quasi-P, quasi-SV, and quasi-SH. In Maxwell's equations of electrodynamics, there is only one characteristic which propagates with the velocity of light.

We may say that geometrical seismics (as well as geometrical optics and geometrical acoustics) describes the propagation of discontinuities. We have studied here only the pure geometrical aspect of wave propagation, but it will be shown that our results are also true for the amplitudes of waves.

4: CONNECTION WITH RAY METHOD

Ray method

In the framework of the ray method one looks for a solution of an equation using the following representation of a wave field:

$$\mathbf{u}(\mathbf{r}, t) = \mathrm{Re} \sum_{n=0}^{\infty} \mathbf{U}_n(\mathbf{r}) f_n(t - \tau(\mathbf{r})) \tag{28}$$

where $\mathbf{U}_n(\mathbf{r})$ are complex vector amplitudes: $\mathbf{U}_n = U_n^{(1)} + U_n^{(2)} + U_n^{(3)}$, $f_n(t)$ are "complex forms" which satisfy the relation

$$f_{n-1}(t) = \frac{d}{dt} f_n(t) = \mathbf{D} f_n(t). \tag{29}$$

If we put expression (28) into the equation

$$\mathbf{L}\mathbf{u}(\mathbf{r}, t) = \rho \frac{\partial^2 u}{\partial t^2} \tag{30}$$

(it can be, for instance, Lame's equation; see Chapter 3, equation (24)), we derive, after some manipulating, that function $\tau(\mathbf{r})$ satisfy eiconal's equation

$$|\Delta \tau|^2 = \frac{1}{v_Q^2} \tag{31}$$

(Q stands for P or S), and complex amplitudes satisfy the recurrent relations:

$$\mathbf{N}\mathbf{U}_n = \mathbf{M}\mathbf{U}_{n-1} - \mathbf{L}\mathbf{U}_{n-2}, \;\; n = 0, 1, 2, \ldots, \;\; \mathbf{U}_{-2} = \mathbf{U}_{-1} = 0, \tag{32}$$

where $\mathbf{L}$ is the same operator as in equation (30), $\mathbf{N}$ is the same matrice that is in Chapter 3 (see equation (23)), and $\mathbf{M}$ is a first order differential operator:

$$\begin{aligned} \mathbf{M}\mathbf{U}_n \;=\; & (\lambda + \mu)[\mathrm{div}\mathbf{U}_n \nabla\tau + \nabla\tau(\mathbf{U}_n \cdot \nabla\tau)] + \\ & \mu[\mathbf{U}_n \Delta\tau + 2\sum_{i=1}^{3} (\nabla\tau \cdot \nabla\mathbf{U}_n^{(i)})\mathbf{e}_i] + \\ & \nabla\lambda(\mathbf{U}_n \cdot \nabla\tau) + (\nabla\mu \cdot \mathbf{U}_n \nabla\tau + (\nabla\mu \cdot \nabla\tau)\mathbf{U}_n. \end{aligned} \tag{33}$$

We see that "ray waves" propagate with the same eiconal as discontinuities. It is not accidental because standard discontinuities satisfy the same relation (29) as "complex forms" $f_n(t)$.

Moreover, if a source function has a beginning at the $t = 0$ (as it usually occurs), then the solution of the equation (30) will inevitably have discontinuities that propagate with eiconal's satisfying equation (31). This means that all complex forms $f_n(t)$ must have a discontinuity at the point $t = 0$. Therefore, despite the choice of a complex form $f_0(t)$, any "ray series" descriptions of the wave field are equivalent in the sense that was introduced earlier.

Of course, a choice of $f_0(t)$ influences accuracy of wave field approximation behind the wave front $\tau = \tau(\mathbf{r})$, but in any case, convergence of the series (28) can be guaranteed only in a neighborhood of wave front (Babich, 1961).

If $f_0(t) = R_{q_0}(t)$ then series (28) represents ordinary Tailor's expansion in a neighborhood of the front (in the case of the "weak discontinuity", $q_0 = 2$).

In the general case

$$f_0(t) \overset{q_0}{\sim} R_{q_0,0}(t) + iR_{q_0,1}(t)$$

where q_0 can vary depending on the location of the point $\mathbf{r}$. We observe this dependence in a neighborhood of caustics and other special domains. But in this case we have to learn how to manipulate with discontinuities of noninteger order.

We see that usage of discontinuities is absolutely equivalent to ray series approach. Both approaches give the same geometrical law of wave propagation and the same behavior of wave amplitudes during propagation.

If the senior discontinuity has the order q_0, then the equivalence of the order q_0 corresponds to the zero-approximation of ray series; equivalence of the order $q_0 + 1$ corresponds to the first-order ray-series approximation and so on.

Usually the value of q_0 does not matter. So it will be convenient to write $f_1(t) \overset{(k)}{\sim} f_2(t)$, if there is such an order q_0 that $f_1(t) \overset{q_0+r}{\sim} f_2(t)$ for all $r = 0, 1, \ldots, k$, but not for $r < 0$. With this agreement, zero-approximation corresponds to (0)-equivalence, first-order ray-series approximation corresponds to (1)-equivalence, and so on. Usually we shall omit symbol (0) (so $\sim \equiv \overset{(0)}{\sim}$).

Zero-approximation.

Let us consider the equation (32) for $n = 0$:

$$\mathbf{N}\mathbf{U}_0 = 0.$$

With accordance to equation (23), this is a linear algebraic equation with degenerate matrices. It may be solved only in the case when $\mathbf{U}_0$ is an eigenvector. We restrict ourselves with the situation when the eiconal equation for P-waves is valid. In this case, the eigenvector is $\mathbf{P} = \nabla \tau_P$ so vector $\mathbf{U}_0$ can be expressed in the form $\mathbf{U}_0 = A_0 \nabla \tau_P$. In order to specify the amplitude A_0 let us consider the equation (32) for $n = 1$:

$$\mathbf{N}\mathbf{U}_1 = \mathbf{M}(A_0 \nabla \tau_P) \tag{34}$$

But any vector $\mathbf{U_1}$ can be decomposed into the sum $\mathbf{U_1} = A_1 \nabla \tau_P + \mathbf{Q}$ where $\mathbf{Q} \perp \nabla \tau_P$. Both vectors $\nabla \tau_P$ and $\mathbf{Q}$ are eigenvectors connected with eigenvalues 0 and $\mu \mid \nabla \tau_P \mid^2 - \rho$ correspondently, therefore,

$$\mathbf{N}\mathbf{U}_1 \equiv \mathbf{N}(A_1 \nabla \tau_P + \mathbf{Q}) \equiv (\mu \mid \nabla \tau_P \mid^2 - \rho)\mathbf{Q}$$

We see that the left side of the equation (34) is orthogonal to the vector $\nabla \tau_P$. It means that equation (33) can be resolved if and only if

$$\nabla \tau_P \cdot M \ (A_0 \nabla \tau_P) = 0 \tag{35}$$

Inserting expression in equation (33) into the last equation, we use the property of the operator $\nabla \tau_P$, if s is the natural parameter of a ray $\mathbf{r} = \mathbf{r}(s)$, then for any scalar function $\psi(\mathbf{r})$

$$\nabla \tau_P \cdot \nabla \psi = \frac{1}{v_P} \frac{d\psi}{ds}$$

where $d[\cdot]/ds$ means differentiation along the ray that intersects the given point. After simple manipulation we derive from equation (35) the transient equation for the amplitude $A = A_0/v$:

$$\frac{dA}{ds} + \frac{1}{2} A[v \triangle \tau_P + \frac{d(\ln \rho {v_P}^2)}{ds}] = 0 \tag{36}$$

which have solution

$$A(s) = \frac{A(0)}{v_P \sqrt{\rho}} e^{-\frac{1}{2} \int_0^s v_P \triangle \tau_P ds}$$

The well-known notion of geometrical spreading $L = \sqrt{J}$ is connected with $\triangle \tau$ by relation (S.V. Goldin, 1986)

$$v \triangle \tau = d(\ln J/v)/ds$$

consequently,

$$A(s) = A(0)/\sqrt{\rho v_P J} \tag{37}$$

Analogous formula is true for S-waves.

The equation (37) describes how the amplitude of the senior discontinuity of a wave is changing along a ray path.

If the value of J is positive in all points of a given ray $\mathbf{r} = \mathbf{r}(s)$, then along the ray

$$\mathbf{U} \sim \frac{A(0)}{\sqrt{\rho v_P J}} R_{q_0}[t - t(s)]\mathbf{t} \tag{38}$$

where $t(s) = \tau_P[\mathbf{r}(s)]$ and $\mathbf{t} = v_P \nabla \tau_P$ is the tangent-vector of the ray. And if $J < 0$, then

$$\mathbf{U} \sim \frac{A(0)}{\sqrt{\rho v_P J}} R_{q_0,1}[t - t(s)]\mathbf{t}$$

The situation when $J = 0$ means that $u \sim A' R_{q',v'}$ at $q' < q_0$, $0 \le v' \le 1$.

Fundamental solutions

As is well known, any solution $\mathbf{u}(\mathbf{r}, t)$ of the equation (30) in case of point source can be expressed in the form

$$\mathbf{u}(\mathbf{r}, t) = f(t) * \mathbf{g}(\mathbf{r}, t) \tag{39}$$

where $f(t)$ is the source function (the function that describes the temporal behavior of a source) and $\mathbf{g}(\mathbf{r}, t)$ is the fundamental solution of a correspondent boundary value problem for the equation (30).

Fundamental solution is a response of a medium to the δ-function as the source. It is easily understood that (in case of piecewise smooth media with restricted numbers of interfaces) $\mathbf{g}(\mathbf{r}, t)$ consists of some singularities (discontinuities) that propagates with correspondence to eiconal's equation (31) and relations (32) and some very smooth field $\rho(\mathbf{R}, t)$.

Let us consider a very simple example: a fundamental solution that corresponds to a point vertical force in a homogeneous medium (Figure **??**) is

$$\mathbf{g}(\mathbf{r}, t) = \mathbf{U}_{\mathrm{I}}(\mathbf{r}, t) + \mathbf{U}_{\mathrm{II}}(\mathbf{r}, t) \tag{40}$$

where

$$\begin{aligned}
\mathbf{U}_{\mathrm{I}}(\mathbf{r}, t) &= C\left[\frac{\delta\left(t - \frac{R}{V_P}\right)}{V_P^2 R} + \frac{2H\left(t - \frac{R}{V_P}\right)}{V_P R^2} + \frac{2\left(t - \frac{R}{V_P}\right)_+}{R^3}\right]\cos\theta \mathbf{e}_R \\
&\quad + C\left[\frac{H\left(t - \frac{R}{V_P}\right)}{V_P^2 R} + \frac{\left(t - \frac{R}{V_P}\right)_+}{R^3}\right]\sin\theta \mathbf{e}_\theta, \quad C = \frac{1}{4\pi\rho}, \quad R = |\mathbf{r}|
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{U}_{\mathrm{II}}(\mathbf{r}, t) &= -C\left[\frac{\delta\left(t - \frac{R}{V_S}\right)}{R V_s^2} - \frac{H\left(t - \frac{R}{V_s}\right)}{R^2 V_s} - \frac{\left(t - \frac{R}{V_s}\right)_+}{R^3}\right]\sin\theta \mathbf{e}_\theta \\
&\quad - 2C\left[\frac{H\left(t - \frac{R}{V_s}\right)}{R^2 V_s} + \frac{\left(t - \frac{R}{V_s}\right)_+}{R^3}\right]\cos\theta \mathbf{e}_R.
\end{aligned}$$

As we see, the wave field entirely consists of standard discontinuities! Figure **??** shows radial (a) and angular (b) component for some distance R and direction θ.

Let us perform the convolution in equation (39), bearing in mind that

$$f(t) * R_q(t) = I^q f(t) = f_q(t) \tag{41}$$

At $R \to \infty$ we shall have

$$\begin{aligned} u(\mathbf{r}, t) \quad &= \quad \frac{C}{V_P^2 R} f\left(t - \frac{R}{V_P}\right) \cos\theta bf e_R \\ &+ \frac{C}{V_P R^2} f_1\left(t - \frac{R}{V_P}\right) \sin\theta bf e_\theta \\ &- \frac{C}{V_s^2 R} f\left(t - \frac{R}{V_s}\right) \sin\theta bf e_\theta \\ &- \frac{2C}{V_s R^2} f_1\left(t - \frac{R}{V_s}\right) \cos\theta bf e_\theta \end{aligned}$$

The first term of the last equation is the main component of the P-wave. The second one is an additional component of the P-wave. The third and forth terms are main and additional components of the S-wave. It is interesting to note that at $\theta = \pi/2$, the P-wave has only additional (transverse!) component. In addition, at $\theta = 0$ the S-wave has only components in the ray direction. But what is indeed important for convolution of the amplitude of discontinuity of fundamental solution appears to be an amplitude of the signal! This is the real reason why discontinuities describe not only the arrival of a signal, but also the signal itself: discontinuity is an adequate tool to investigate fundamental solutions.

5: KINEMATICALLY EQUIVALENT OPERATORS OF WAVE-FIELD CONTINUATION

Let $u_0(\mathbf{r}_0, t)$ be the observed field ($\mathbf{r}_0 = x$ or $\mathbf{r}_0 = (x, y)$) that is given on the surface of observation $\Sigma = \{\mathbf{r}_0\}$ and $\mathbf{P}(v)$ is a linear operator that transforms the function $u_0(\mathbf{r}_0, t)$ into the field $u(\mathbf{r}, t)$ in the adjacent domain $\mathbf{D} = \{(\mathbf{r}_0, z), \;\; z \geq 0\}$

$$u(\mathbf{r}, t) = \mathbf{P}(v) u_0(\mathbf{r}_0, t).$$

The fields $u_0(\mathbf{r}_0, t)$ and $u(\mathbf{r}, t)$ can be scalar or vectorial. The coordinates (x, y) can be curvilinear and the surface of observation can also be curved.

The action of the operator $\mathbf{P}(v)$ depends on the choice of the function $v = v(\mathbf{r})$ which in turn is a parameter in a fixed eikonal equation

$$\Psi(\nabla\tau(\mathbf{r}); v(\mathbf{r})) = 1, \qquad \mathbf{r} \in \mathbf{D}. \tag{42}$$

The simplest example of eikonal equation is the one for an isotropic medium

$$v^2(\mathbf{r})|\nabla\tau|^2 = 1.$$

In an anisotropic medium $v(\mathbf{r})$ is a vector-function. For instance, in a medium with elliptic anisotropy, we have

$$\nabla\tau \cdot \mathbf{G}\nabla\tau = 1$$

where "$\cdot$" stands for the scalar product and $\mathbf{G}$ is a positive matrix.

We don't demand that the eikonal equation (42) is a characteristic equation of some hyperbolic differential operator. It is given, that's all!

Let us suppose that the field $u_0(\mathbf{r}_0, t)$ contains a discontinuity τ_0 (of some order q) with a travel time curve $t = \tau_0(\mathbf{r}_0)$. It means that in a neighborhood of the surface $t = \tau_0(\mathbf{r}_0)$

$$u_0(\mathbf{r}_0, t) \sim A_0(\mathbf{r}_0) R_q[t - \tau_0(\mathbf{r}_0)].$$

Generally speaking then, the wave field $u(\mathbf{r}, t)$ will contain one or several discontinuities (with orders not necessarily equal to q) and at least one of them will coincide with τ_0 at $\mathbf{r} = (\mathbf{r}_0, 0)$. If the position of this discontinuity is described by a function $t = \tau(\mathbf{r})$, then

$$\tau(\mathbf{r})|_{\mathbf{r}\in\Sigma} = \tau_0(\mathbf{r}_0). \tag{43}$$

We shall call the operator $\mathbf{P}(v)$ a kinematically-equivalent operator of wave field continuation (on short K-operator, KO) if the field $u(\mathbf{r}, t)$ necessarily contains at least one discontinuity with eikonal $t = \tau(\mathbf{r})$, that satisfies equations (42) and (43) simultaneously.

The notion of K-operator may be expanded to the case when both fields $u_0(\mathbf{r}_0, t)$ and $u(\mathbf{r}, t)$ are vectorial fields. In that case we have two equations (isotropic case) or three equations (anisotropic case):

$$\Psi_i(\nabla\tau, \mathbf{v}) = 0 \tag{44}$$

depending on the polarization of the vector-amplitude of the discontinuity. And if the field $u_0(\mathbf{r}_0, t)$ has a discontinuity with i-th polarization at $t = \tau_0(\mathbf{r}_0)$, then application of a K-operator $\mathbf{P}(v)$ will result in the presence of discontinuity of the field $u(\mathbf{r}, t) = \mathbf{P}(v)u_0(\mathbf{r}_0, t)$ which:

- has the same polarization,
- propagate with accordance to equation (44); and
- satisfies initial condition $\tau|_\Sigma = \tau_0(\mathbf{r}_0)$.

The question of existence of a K-operator corresponding to a given equation (42) can be simply answered (in a positive sense) if equation (42) is a characteristic equation for some equation in partial derivatives:

$$\mathbf{L}u = \rho\frac{\partial^2 u}{\partial t^2}. \tag{45}$$

Let us consider the boundary value problem for equation (45) with condition

$$u|_{\mathbf{r}\in\Sigma} = u_0 \tag{46}$$

and any additional condition

$$\mathbf{A}u|_{\mathbf{r}\in\Sigma} = 0 \tag{47}$$

that supplies a unique solution for the problem (45) - (47).

It is obvious that the solution contains at least one discontinuity which at $\mathbf{r} = (\mathbf{r}_0, 0)$ coincides with τ_0 and propagates with the eikonal satisfying equation (42).

Later we shall show that KO exists for any equation (42) if there is a solution for the Cauchy problem for equation (42) with the initial condition (43). As a matter of fact there is a whole family $\mathbf{P}_\Psi$ of K-operators that corresponds to the particular eikonal equation (42).

K-equivalence is a notion which is much wider than q-equivalence or (k)-equivalence. For instance, if $\mathbf{P}_1(v), \mathbf{P}_2(v)$ belong to the same family $\mathbf{P}_\Psi$, then it is not necessary that

$$\mathbf{P}_1(v)u_0(\mathbf{r}_0, t) \sim \mathbf{P}_2(v)u_0(\mathbf{r}_0, t).$$

Fields $u_1 = \mathbf{P}_1(v)u_0$ and $u_2 = \mathbf{P}_2(v)u_0$ can have different orders and amplitudes of discontinuities (and even their number) which satisfy the equations (42) and (43).

Let us consider K-operators for the classical eikonal equation

$$|\nabla\tau|^2 = \frac{1}{v^2}. \tag{48}$$

Let us pose for that equation the Cauchy problem with initial condition

$$\tau(\mathbf{r})|_{\mathbf{r}\in\Sigma} = \tau_0(\mathbf{r}_0). \tag{49}$$

It is easy to show that if for all $\mathbf{r}_0$

$$|\nabla_\Sigma \tau_0(\mathbf{r}_0)| < \frac{1}{v_0(\mathbf{r}_0)}$$

(here $v_0(\mathbf{r}_0) \equiv v(\mathbf{r}_0, 0)$), then in a neighbourhood of the surface Σ there are two solutions for this problem: $\tau^+(\mathbf{r})$ and $\tau^-(\mathbf{r})$. In order to prove this we shall search for solution of the problem (48) - (49)) using characteristics of equation (48). Characteristics of the equation (48) (that is rays, or characteristics of the proper wave equation (45)) satisfy to the first order differential equation

$$\begin{cases} \frac{d\mathbf{r}}{ds} = v\mathbf{p} \\ \frac{d\mathbf{p}}{ds} = \nabla(\frac{1}{v}), \end{cases} \qquad \mathbf{p} = \nabla\tau. \tag{50}$$

In order to get a unique solution of the equation (50) it is necessary (and sufficient) to have a definite value $\mathbf{p} = \mathbf{p}_0$ in a starting point $\mathbf{r}_0$. We can express the vector $\mathbf{p}_0$ in the form of a sum

$$\mathbf{p}_0 = pr_\Sigma \nabla\tau|_{\mathbf{r}=\mathbf{r}_0} + a\mathbf{n}$$

where $\mathbf{n}$ is a normal to Σ at the point $\mathbf{r}_0$, and $a = \mathbf{p}_0 \cdot \mathbf{n}$ (instead of $\mathbf{r} \in \Sigma$ we shall write $\mathbf{r} = \mathbf{r}_0$). The value $pr_\Sigma \nabla\tau(\mathbf{r}_0)$ can be expressed through the derivatives of the function $\tau_0(x, y)$. If Σ is a horizontal plane and x and y are rectangular coordinates, then

$$pr_\Sigma \nabla\tau(\mathbf{r}_0) = \frac{\partial \tau_0}{\partial x}\mathbf{e}_x + \frac{\partial \tau_0}{\partial y}\mathbf{e}_y \equiv \nabla_\Sigma \tau_0$$

where ∇_Σ means gradient in the plane Σ and $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit vectors.

Let us consider the case when Σ is a curved surface and x and y are a curvilinear system of coordinates. We propose that the axis z coincides with the direction of a vector $\mathbf{n}$. Let us determine a tangent plane P to Σ in the point $\mathbf{r}_0$ and let's introduce the rectangular coordinates ξ and η in P, then

$$pr_\Sigma \nabla\tau(\mathbf{r}_0) = \frac{\partial \tau_0}{\partial \xi}\mathbf{e}_\xi + \frac{\partial \tau_0}{\partial \eta}\mathbf{e}_\eta$$

where

$$\frac{\partial \tau_0}{\partial \xi} = \left(\frac{\partial \tau_0}{\partial x}\right)\frac{\partial x}{\partial \xi} + \left(\frac{\partial \tau_0}{\partial y}\right)\frac{\partial y}{\partial \xi}$$

and

$$\frac{\partial \tau_0}{\partial \eta} = \left(\frac{\partial \tau_0}{\partial x}\right)\frac{\partial x}{\partial \eta} + \left(\frac{\partial \tau_0}{\partial y}\right)\frac{\partial y}{\partial \eta}.$$

This construction we will also denote as $\nabla_\Sigma \tau_0$. So,

$$pr_\Sigma \nabla\tau(\mathbf{r}_0) = \nabla_\Sigma \tau_0(\mathbf{r}_0).$$

In accordance with eikonal equation (48)

$$|\mathbf{p}_0|^2 = \frac{1}{v_0^2(\mathbf{r}_0)},$$

$(v_0(\mathbf{r}_0) \equiv v(\mathbf{r}_0, 0) \equiv v(\mathbf{r})|_{\mathbf{r}=\mathbf{r}_0})$, then

$$|\nabla_\Sigma \tau_0|^2 + a^2 = \frac{1}{v_0^2(\mathbf{r}_0)}$$

and

$$a = \pm\sqrt{\frac{1}{v_0^2(\mathbf{r}_0)} - |\nabla_\Sigma \tau_0(\mathbf{r}_0)|^2}.$$

We have obtained that

$$\nabla\tau|_{\mathbf{r}=\mathbf{r}_0} = \nabla_\Sigma \tau_0(\mathbf{r}_0) \pm \mathbf{n}\sqrt{\frac{1}{v_0^2(\mathbf{r}_0)} - |\nabla_\Sigma \tau_0(\mathbf{r}_0)|^2}.$$

The sign '+' determines the solution $\tau^+(\mathbf{r})$ which will be called *forward eikonal continuation* . The sign "−" determines reverse eikonal continuation $\tau^-(\mathbf{r})$, (the relationship of these continuations is shown in Figure **??**).

It is easily understood that $\tau^{\pm}(\mathbf{r}) = \tau_0(\mathbf{r}_0) \pm T(\mathbf{r}_0, \mathbf{r})$ where $T(\mathbf{r}_0, \mathbf{r})$ is the time of propagation from $\mathbf{r}_0$ to $\mathbf{r}$. We shall call K-operator $\mathbf{P}(v)$ as the K-operator of forward (reverse) wave field continuation and denote it as $\mathbf{P}^{(+)}(v)$ (similarly $\mathbf{P}^{(-)}(v)$) if in the neighbourhood of Σ the field

$$u^{(+)}(\mathbf{r}, t) = \mathbf{P}^{(+)}(v) u_0(\mathbf{r}_0, t)$$

(similarly $u^{(-)} = \mathbf{P}^{(-)}(v) u_0$) contains a discontinuity with eikonal τ^+ (τ^-) and does not contain a discontinuity with the opposite eikonal τ^- (τ^+) .

Each operator $\mathbf{P}^{(+)}$ can be used for determination of the operator $\mathbf{P}^{(-)}$

$$\mathbf{P}^{(-)} = \mathbf{I}_T^{-1} \mathbf{P}^{(+)} \mathbf{I}_T \tag{51}$$

where $\mathbf{I}_T u(\mathbf{r}, t) = u(\mathbf{r}, T - t)$. And vice versa:

$$\mathbf{P}^{(+)} = \mathbf{I}_T^{-1} \mathbf{P}^{(-)} \mathbf{I}_T.$$

There are, of course, such K-operators that contain (in a neighbourhood of Σ) discontinuities with both eikonals τ^+ and τ^- (mixed type operators). This classification of K-operators may be expanded for many other eikonal equations.

The classes of this classification are still very wide. If operators $\mathbf{P}_1(v)$ and $\mathbf{P}_2(v)$ belong to the same class (forward, reverse or mixed) of K-operators and for some k

$$\mathbf{P}_2(v) u_0 \sim \mathbf{D}^k \mathbf{P}_1(v) u_0 \tag{52}$$

then we shall call these operators amplitude-equivalent (KA-operators). The most important case of the amplitude equivalence is the situation when the amplitude $A(\mathbf{r})$ is a solution of a transport equation for the definite wave equation, for instance at $\mathbf{L} \equiv \triangle$ (classical wave equation). In the latter case discontinuity's amplitudes change with accordance to geometrical spreading (as ray's amplitudes). If in equation (52) $k = 0$ we have dynamic equivalence.

6: WHY SO MANY ALGORITHMS FOR MIGRATION ?

The method of time fields (Common Source Pattern)

During World War II the Soviet geophysicist J. Riznichenko proposed the following method for reconstruction of a reflector in a medium with a given velocity $v(\mathbf{r})$. Let the source be located at point $\mathbf{s}$ and let $z = h(\mathbf{r})$ be a surface that shows the location of a reflector R. In a medium D between surfaces Σ and R two eikonals can be introduced:

- eikonal of the direct wave: $\tau_d(\mathbf{r})$

- eikonal of the wave reflected from R: $\tau_r(\mathbf{r})$.

Both eikonals satisfy the eikonal equation

$$|\nabla\tau|^2 = \frac{1}{v^2(\mathbf{r})}. \tag{53}$$

The initial condition for τ_d is

$$\tau_d(\mathbf{s}) = 0; \qquad \tau_d(\mathbf{r}_0) \neq 0, \quad \mathbf{r}_0 \neq \mathbf{s}; \tag{54}$$

initial condition for τ_r is

$$\tau_r(\mathbf{r})|_R = \tau_1(\mathbf{r}_R), \tag{55}$$

where

$$\tau_1(\mathbf{r}_R) = \tau_d(\mathbf{r})|_R \tag{56}$$

$\mathbf{r}_R$ are curvilinear coordinates on the surface R. τ_d and τ_r are determined as the direct eikonal's continuations.

It is by definition that eikonals τ_d and τ_r have the same values on the reflector R:

$$\tau_d|_R = \tau_r|_R. \tag{57}$$

In fact the surface R is a geometrical collection of points $\mathbf{r}$ that satisfies equation

$$\tau_d(\mathbf{r}) = \tau_r(\mathbf{r}). \tag{58}$$

We suggest that travel-times $\tau_0(r_0)$ for a common source are given on the surface Σ. Since the velocity-function $v(\mathbf{r})$ is proposed to be known, we can determine the reverse eikonal continuation $\tau^{(-)}(\mathbf{r})$. Both functions $\tau^{(-)}(\mathbf{r})$ and $\tau_r(\mathbf{r})$:

- satisfy the same equation (53)
- coincide with $\tau_0(\mathbf{r}_0)$ on Σ
- approach the surface Σ from below.

This means that

$$\tau^{(-)}(\mathbf{r}) = \tau_r(\mathbf{r}), \mathbf{r} \in D \tag{59}$$

and the reflector's location is determined by

$$\tau^{(-)}(\mathbf{r}) = \tau_d(\mathbf{r}). \tag{60}$$

In fact, Riznichenko proposed not only this theoretical scheme but also a graphical scheme. Huygens principle founded method of reconstructing $\tau^{(-)}$ and τ_d in the 2D case. Nowadays this second scheme has lost its significance since we have a great deal of different computer-flexible techniques to solve equation (53). Due to the principle of reciprocity, absolutely the same scheme works when, instead of $\tau_0(\mathbf{r}_0)$, we have $\tau_0(s, r = const)$: travel-times for common receiver patterns with source location s.

It is easy to extend this technique to converted waves PS or SP: in this case we must use different velocities for reconstructing τ_d and $\tau^{(-)}$.

Zero-offset case

Unfortunately equation (59) is valid only for common source and common receiver gathers. Let us take, for instance, the common offset pattern in the 2D case :

$$s = X - \frac{l}{2}$$

$$r = X + \frac{l}{2}$$

at fixed l. In this case we have datum

$$\tau_0(X) = \tau(X - \frac{l}{2}, X + \frac{l}{2})$$

where $\tau(s,r)$ is the arrival time of a reflected wave propagating from source s to receiver r. Then

$$\frac{\partial \tau_0}{\partial X} = \frac{\partial \tau}{\partial s} + \frac{\partial \tau}{\partial r}.$$

We can't separately determine values $\frac{\partial \tau}{\partial s}$ and $\frac{\partial \tau}{\partial r}$ and, consequently, can't determine initial conditions for eiconal τ. We see that the reverse eikonal's continuation with datum τ_0 can't reconstruct the true location of the seismic rays. The problem simplifies greatly when $l = 0$ (zero offset pattern). In this case incidence and reflected rays coincide, and due to the principle of reciprocity,

$$\frac{\partial \tau}{\partial s} = \frac{\partial \tau}{\partial r}$$

so

$$\frac{\partial \tau_0}{\partial X} = 2\frac{\partial \tau}{\partial r}$$

and

$$\frac{\partial(\frac{1}{2}\tau_0)}{\partial X} = \frac{\partial \tau}{\partial r}. \tag{61}$$

This means that the solution $\tau^{(-)}$ of equation (61) with initial condition

$$\tau|_{\sum} = \frac{1}{2}\tau_0 \tag{62}$$

will give the true ray pattern although the travel-times will be different. This is not bad. Since the time of propagation from the reflector R to surface Σ along the normal ray equals $\frac{\tau_0}{2}$, the condition

$$\tau^{(-)}(\mathbf{r}) = 0 \tag{63}$$

determines the location of the reflector. We could obtain the same result by solving the half-velocity eikonal equation

$$|\nabla \tau|^2 = \frac{1}{(\frac{v}{2})^2} \tag{64}$$

with initial datum

$$\tau|_\Sigma = \tau_0(X). \tag{65}$$

Usually this scheme is obtained from some thought experiment (i.e., the exploding reflector hypothesis). Let us find the forward eikonal's continuation in the medium with velocity $\frac{v(\mathbf{r})}{2}$ and initial condition

$$\tau|_R = 0. \tag{66}$$

This corresponds to a thought experiment with an impulsive source that is distributed along the surface R at $t = 0$). It is easy to understand that the solution $\tau^+(\mathbf{r})$ of this problem will coincide on the surface Σ with $\tau_0(X)$. It is clear that the reverse eikonal's continuation $\tau^{(-)}(\mathbf{r})$ of $\tau_0(X)$ in a medium with half-velocity has the isochron $\tau^{(-)} = 0$ on the surface R.

Migration

Since discontinuities of the wave field $u^{(-)}(\mathbf{r},t)$ move with eikonal $\tau^{(-)}(\mathbf{r})$ it is easy to reconstruct the reflector's location (without picking seismic events) with the help of an operator $\mathbf{P}^{(-)}(v)$. In the case of common source patterns:

$$u^{(-)}(\mathbf{r},t) = \mathbf{P}^{(-)}(v)u_0(\mathbf{r}_0,t) \tag{67}$$

and the image of the reflector is a correspondent discontinuity of the field

$$A(\mathbf{r}) = u^{(-)}(\mathbf{r},\tau_d(\mathbf{r})). \tag{68}$$

This formula follows directly from equation (60). In order to get a better understanding, let us propose that the discontinuity of the field $u^{(-)}(\mathbf{r},t)$ has an order q. If we differentiate the field $u^{(-)}(\mathbf{r},t)$ $q+1$ times, this discontinuity will become a δ-function: $A(\mathbf{r})\delta(t-\tau^{(-)}(\mathbf{r}))$. If instead of t we substitute $t = \tau_d(\mathbf{r})$, then the support of δ-function will satisfy equation 60.

In the zero offset case

$$u^{(-)}(\mathbf{r},t) = \mathbf{P}^{(-)}(\frac{v}{2})\ u_0(X,t) \tag{69}$$

where $u_0(x,t)$ is a set of seismic records with $s = r = X$. The image of the reflector coincides with the discontinuity of the field

$$A(\mathbf{r}) = u^{(-)}(\mathbf{r},0). \tag{70}$$

In both cases depth migration consists of two steps:

- reverse wave continuation

- imaging.

This scheme is applicable only in the cases of common source, common receiver, or zero offset gathers. It does not mean that migration cannot be performed for other patterns. On the contrary: we shall show that migration may be done for any arbitrary pattern. But in the previously mentioned gathers, it may be done with these two steps: wave field continuation and imaging. I don't take into consideration possible approximate approaches. Now we can answer the question: why do we have so many algorithms of migration? For this purpose, we can use any kinematically-equivalent operation: all of them are useful for imaging reflectors.

Classical time migration is a result of the substitution $z = \overline{v}t$:

$$M(x,t) = A(x, \overline{v}t) \tag{71}$$

where $\overline{v}$ is an average velocity along a vertical line and $a(x,z) = a(\mathbf{r})$.

7: GEOMETRICAL PROBLEMS OF WAVE-FIELD CONTINUATION

Interference analysis

If we have two waves with different apparent velocities that interfere with each other, then it is reasonable to apply wave-field continuation into the homogeneous model because these waves can be separated as a result of propagation in a continued field. The consideration is very simple. If both waves are observed in the interval $-a \leq x \leq a$ and the k-th one has travel time $\tau_k(x) = t_0 + x/v_K$ ($k = 1, 2,\ v_2 < v_1$, see Figure **??**a), then as the result of forward continuation into the medium with velocity, $v_c <$min$\{v_1, v_2,\}$ we obtain waves with eiconals

$$\tau_K^{(+)}(x,z) = t_0 + \frac{z}{v_c}\sqrt{1-\eta_k^2} + \frac{x-d_k}{v_k}, \quad \mid x - d_k \mid \leq a$$

where $\eta_k = v_c/v_k,\ d_u = z\eta_k/\sqrt{1-\eta_k^2}$, see Figure **??**b. It is easy to check that $\tau_k^{(+)}$ satisfies to the eiconal equation and to the boundary condition $\tau_k^{(+)}(x,0) = \tau_k(x)$.

The choice of velocity v_c does not depend on the velocity in the true model of the medium, but only on the condition of wave resolution. We have time resolution if $\tau_2^{(+)}(-a+d_2,z) > \tau_1^{(+)}(-a_2+d_2,z) + 2T_0$ (where T_0 is the duration of the pulse) and spatial resolution if $-a+d_2 \geq a+d_1$. It is very easy to derive proper conditions in terms of z, a, v_1, v_2 and V_c. For example, we have spatial resolution if $z/a > 2\sqrt{1-\eta^2}/\eta$ where $\eta = V_0/V_n,\ V_0 = V_c^{-1} - V_1^{-1},\ V_n = V_2^{-1} - V_1^{-1}$. But we shall not develop these considerations since there are more powerful solution. Let us return to the condition $V_c <$min$\{V_1, V_2\}$. Does it have a sense to refuse from the condition?

Let one of the waves be nondesirable (for example, one which has less value of apparent velocity: $v_2 < v_1$). Then the best choice of the velocity for continuation is

$v = v_c$:$v_2 < v_c < v_1$. It is well known that apparent velocity is always more than the velocity in the medium. When $v = v_c$, then the wave with velocity $v_2 < v_c$ will not be continued into a medium at all! To be exact, it will propagate in the model as an evanescent wave (along the surface Σ) whose amplitude very quickly (exponentially) diminishes with increasing $|z|$.

In fact, the operators $\mathbf{P}^{\pm}(v_c)$ act as fan filters for the fan of velocities

$$|v| > v_c.$$

In order to improve the quality of resolution, we can introduce preliminary time shifts $\triangle t = -x/v_1$. Afterward waves have new apparent velocities $v_1' = \infty$ and $v_2' = v_n$. Now we may apply the operator $P^{\pm}(v_c)$ at $v_c < v_n$. The big advantage of this approach is that it is not necessary that apparent velocities v_1 and v_2 (or v_1' and v_2') not be constant along a profile. The only essential condition is $v_1(x) < v_c < v_2(x)$.

The Figure **??** illustrates interference analysis for the situation when V_1 changes in the interval (5 km/s, 6.3 km/s) and V_2 changes in the interval (3 km/s, 4.1 km/s). Figure **??**a shows the original record, Figure **??**b shows the result of standard fan filtration in the velocity fan (4.6 km/s, 7.1 km/s), and Figure **??**c shows the result of the forward continuation with $V_c = 4.9$ km/s.

Untying loops

If, at direct propagating of a wave in a medium with velocity $v = v(\mathbf{r})$, one observes appearance of loops, then at reverse continuation of the wave into the same medium these loops will be untied. But usually the velocity function $v(\mathbf{r})$ is unknown. Moreover, we have to untie loops just in order to obtain good conditions for velocity estimation. So the question arises: how many continuation velocity v_c can be different from real velocity without the loss of the untying effect?

I've obtained the following conditions:

1. The observed field $u_0(\mathbf{r}_0, t)$ $(r \in \Sigma)$ is not complicated by loops if

$$K > \frac{\cos\alpha}{2}\left[\frac{\cos(\alpha-\phi)}{h} + \frac{\cos(\alpha+\phi)}{h}\right]$$

holds for any ray, where K, h and ϕ are a curvature, a depth and a slope of the interface at the reflection point accordingly; α is an angle of incident.

2. The condition for the absence of loops in the field $u^{(-)}(\mathbf{r}; t)$ continued (with respect to receivers) onto the horizontal plane $z = h_c$ with continued velocity $v_c \neq v$ becomes weaker:

$$K > \frac{\cos\alpha}{2}\left[\frac{\cos(\alpha-\phi)}{h} + \frac{\cos(\alpha+\phi)}{h-\kappa h_c}\right] \tag{72}$$

where

$$\kappa = \frac{v_c}{v}\cos^3(\alpha+\phi)\left[1 - \frac{{v_c}^2}{v^2}\sin^2(\alpha+\phi)\right]^{-\frac{3}{2}}.$$

3. In the case of double (with respect to receivers and sources) continuation we have the weakest condition:

$$K > -\frac{\cos\alpha}{2}\left[\frac{\cos(\alpha-\phi)+\cos(\alpha+\phi)}{h-\kappa h_c}\right].$$

4. For small reflection angles the condition (72) may be simplified:

$$0 \le \frac{3}{2}\frac{h_c v_c}{hv}\left(1 - \frac{{v_c}^2}{v^2}\right)\tan^2\phi_{\max} \le \min\left\{\begin{array}{l} \epsilon - (1 - \frac{h_c v_c}{hv}) \\ 1 - \frac{h_c v_c}{hv} \end{array}\right.$$

where

$$\epsilon = \frac{1}{2\mu - 1},\ \mu = -\min\frac{Kh}{\cos\phi} > 0.$$

This condition is shown in Figure **??**. An example of untying loops is shown in Figure **??** where data of physical modeling are represented.

We can interpret inverse wave-field continuation into the model with velocity $v = v_c$ as a continuation in a fictitious layer above the surface $z = 0$ (the layer with known velocity).

The fact that at $z > 0$ we have direct propagation and at $z < 0$ reverse continuation causes the modification of Snell's law at the interface $z = 0$. This modification is shown in Figure **??**.

$$\lambda_q = (-1)^{r_q} v_q^{-1}\mathrm{grad}\tau(N_q), \quad q = 1,2, \quad N_q \in D_q$$

$$r_q = \left\{\begin{array}{ll} 0 & \text{at forward propagation at } D_q \\ 1 & \text{at reverse propagation at } D_q \end{array}\right.$$

Geometrical analysis of imaging

The notion of K-equivalence allows us to use a very simple approach to analysis of geometrical distortions on migrated sections that originated from a difference between the real medium and the model used at wave-field continuation. This difference may be caused not only by wrong velocities but also to use another Eiconal equation (when, for example, instead of anisotropic medium one uses the model of an isotropic one). The procedure of analysis is very simple:

- First step: solution of direct kinematical problem in the correct model; calculation of $\tau_0(\mathbf{r}_0)$

- Second step: the reverse Eiconal's continuation
- Third step: imaging in a new model

We shall illustrate this procedure using a very simple example of a routine processing when one performs the time migration with wrong velocity and time-to-depth migration with correct velocity.

A. Homogenous layer with a planar reflector

Let equation

$$z = x \tan\phi + h_0$$

describe the location of a reflector.

First step: calculation of two-way zero-offset travel time in a true model:

$$t_0 = \frac{2\sin\phi}{v}x + \frac{2h_0\cos\phi}{v}.$$

Second step: reverse Eiconal's continuation of $t_0(x)$ with wrong velocity ($v_c = \frac{v'}{2}$, $v' \neq v$):

$$\tau^{(-)}(x,z) = \frac{2h_0\cos\phi}{v} + \frac{2\sin\phi}{v}x - \frac{2z}{v'}\sqrt{1-\left(\frac{v'}{v}\right)^2\sin^2\phi}. \tag{73}$$

Third step: imaging from the condition

$$\tau^{(-)}(x,z) = 0. \tag{74}$$

Inserting equation (73) into equation (74) and solving with respect to z, we derive location of the image in depth-section:

$$z = \frac{(\frac{v'}{v})\cos\phi}{\sqrt{1-(\frac{v'}{v})^2\sin^2\phi}}(h_0 + x\tan\phi).$$

The reflection image in time section we receive after substitution $z = tv'$ is:

$$t = \frac{1}{v}\frac{\cos\phi}{\sqrt{1-(\frac{v'}{v})^2\sin^2\phi}}(h_0 + x\tan\phi).$$

Time-to-depth migration in this situation is depth scaling with true velocity ($t = \frac{z}{v}$):

$$z = \frac{\cos\phi}{\sqrt{1-(\frac{v'}{v})^2\sin^2\phi}}(h_0 + x\tan\phi).$$

We see that depth scaling does not supply correct imaging, although the error is not big: it is proportional to $(v_c - v)\sin^2\phi$.

B. Two-layer media

Let the first layer with horizontal bedding $z = h_1$ have the velocity v_1, and the second layer with the dipping bed

$$z = h_1 + h_2 + x\tan\phi$$

have velocity v_2. Time-migration is performed in homogeneous model with velocity $v_c = \frac{v'}{2}$.

First step:

$$t_0(x) = \frac{2h_2\cos\phi}{v_2} + \frac{2h_1}{v_1}\sqrt{1-(\frac{v_1}{v_2})^2\sin^2\phi} + \frac{2\sin\phi}{v_2}x.$$

Second step: $(v_c = \frac{v'}{2})$:

$$\tau^{(-)}(x,z) = \frac{2h_2\cos\phi}{v_2} + \frac{2h_1}{v_1}\sqrt{1-(\frac{v_1}{v_2})^2\sin^2\phi} + \frac{2\sin\phi}{v_2}x - \frac{2z}{v'}\sqrt{1-(\frac{v'}{v_2})^2\sin^2\phi}.$$

Third step: imaging (from the condition (74)):

$$z = \frac{(\frac{v'}{v_2})\cos\phi}{\sqrt{1-(\frac{v'}{v_2})^2\sin^2\phi}}\left[h_2 + h_1\frac{\sqrt{1-(\frac{v_1}{v_2})^2\sin^2\phi}}{(\frac{v_1}{v_2})\cos\phi} + x\tan\phi\right]$$

After substitution $z = v't$, we obtain location of reflector image in the time section:

$$t = \frac{\cos\phi}{v_2\sqrt{1-(\frac{v'}{v_2})^2\sin^2\phi}}\left[h_2 + h_1\frac{\sqrt{1-(\frac{v_1}{v_2})^2\sin^2\phi}}{(\frac{v_1}{v_2})\cos\phi} + x\tan\phi\right]. \qquad (75)$$

Let us perform time-to-depth migration in true two-layer model with the help of Hubral's technique (Hubral, 1977). This technique consists of transmission of amplitudes along image rays, which satisfy Snell's law and approach the surface Σ vertically. But in this simple example all image rays are vertical in both layers. So the location of the reflector's image in depth-section will be derived from the equation:

$$t - \frac{h_1}{v_1} - \frac{(z-h_1)}{v_2} = 0$$

where t must be expressed according to equation (75). After simple manipulation we derive

$$z = \frac{\cos\phi}{\sqrt{1-(\frac{v'}{v_2})^2\sin^2\phi}}\left[h_2 + h_1\frac{\sqrt{1-(\frac{v_1}{v_2})^2\sin^2\phi}}{(\frac{v_1}{v_2})\cos\phi} + x\tan\phi\right] - h_1\cdot\left(\frac{v_2}{v_1} - 1\right)$$

that differs from true position of the reflector.

C. 15-degree algorithm in one-layer model

This example shows distortions that are connected with application of a wave-field continuation operator which is not kinematically equivalent to the classical Eiconal equation (see the next chapter).

First step: The time t_0 in a true model is the same as in the example:

$$\tau_0 = \frac{2\sin\phi}{v}x + \frac{2h_0\cos\phi}{v}.$$

Second step: If we apply 15-degree algorithm with the continuation velocity $v_c = \frac{v'}{2}$, then discontinuities propagate in accordance to the Eiconal equation

$$\tau_x^2 - \frac{4}{v'}\tau_z = \frac{4}{v'^2}$$

(see equation (98) from Chapter 8 at $v = v_c = \frac{v'}{2}$).

The solution of this equation which satisfies boundary condition $\tau|_{z=0} = t_0$ is:

$$\tau^{(-)}(x,z) = \frac{2h_0\cos\phi}{v} + \frac{\sin\phi}{v}x - \left[1 - \frac{1}{2}\left(\frac{v'}{v}\right)^2 \sin^2\phi\right]\frac{2z}{v'}.$$

Third step: From the condition $\tau^{(-)} = 0$ we find location of the reflector image

$$z = \frac{(\frac{v'}{v})\cos\phi}{1 - \frac{1}{2}(\frac{v'}{v})^2 \sin^2\phi}(h_0 + x\tan\phi)$$

when $v' = v$

$$z = \frac{\cos\phi}{1 - \frac{1}{2}\sin^2\phi}(h_0 + x\tan\phi).$$

The factor

$$\frac{\cos\phi}{1 - \frac{1}{2}\sin^2\phi} \simeq 1 - \frac{1}{8}\sin^4\phi$$

so the depth distortion is very small for subhorizontal reflectors.

D. CSP-depth migration in one-layer medium

We suggest that a reflector is flat: $z = H$.

First step: We represent travel-time dependence for common source gathered in a true model in the following form:

$$t(\theta) = \frac{2H}{v\cos\theta}$$

$$x(\theta) = 2H \tan\theta \tag{76}$$

where θ is incident angle.

Second step: Reverse Eiconal's continuation can be determined with help of ray's equations

$$\xi(l) = x(\theta) - l \sin\theta' \tag{77}$$

$$\zeta(l) = l \cos\theta' \tag{78}$$

where θ' is incident angle of a ray which is correspondent to reverse Eiconal's continuation:

$$\sin\theta' = \frac{v'}{v}\sin\theta, \;\; \cos\theta' = \sqrt{1 - \left(\frac{v'}{v}\right)^2 \sin^2\theta}, \tag{79}$$

l is a length of the ray from the point $(x(\theta), 0)$ to the point (ξ, ζ). So

$$\tau^{(-)} = t(\theta) - \frac{l}{v'}.$$

Third step: The location of the reflector's image can be found from the condition $\tau^{(-)} = \tau_d$, that is

$$t(\theta) - \frac{l}{v'} = \frac{1}{v'}\sqrt{\xi^2 + \zeta^2}.$$

Having usage of equations (76), (77), (78), and (79), we derive

$$l = \frac{v'}{v}\frac{H}{\cos^3\theta} \cdot \left[1 - \left(\frac{v'}{v}\right)^2 \sin^2\theta\right].$$

Now we substitute this formula into equations (77) and (78) and finally get parametric descriptions of the reflector image:

$$\xi(\theta) = 2H \tan\theta \left[1 - \frac{\left(\frac{v'}{v}\right)^2 - \sin^2\theta}{2\cos^2\theta}\right]$$

$$\zeta(\theta) = \frac{v}{v'} H \cdot \frac{\left(\frac{v'}{v}\right)^2 - \sin^2\theta}{\cos^3\theta}\sqrt{1 - \left(\frac{v'}{v}\right)^2 \sin^2\theta}.$$

8: CLASSIFICATION OF K-OPERATORS

The general classification of K-operators describes the scheme as shown in Figure **??**, where IKO stands for "integral K-operators", DKO for "differential K-operators", and PDKO for "pseudo differential K-operators".

This scheme does not describe numerical methods but only the way in which a K-operator is introduced. The fact that the circles intersect, shows that different definitions can determine the same operator.[2] Due to the three numerical representations of the operators, we have three classes of algorithms: stacking(integration), finite difference techniques in the $(\mathbf{r}, t)$-domain, and continuation in the $(\omega, \mathbf{k})$-domain.

Differential K-operators

The DKO is determined as a solution of the equation

$$\mathbf{L}u = \rho\frac{\partial^2 u}{\partial t^2}$$

with boundary conditions that supply the uniqueness of the solution.
We have two problems to solve:

I. The choice of the operator $\mathbf{L}$.

II. The choice of the boundary conditions.

I. Let us take the classical eikonal equation

$$|\nabla\tau|^2 = \frac{1}{v^2(\mathbf{r})}.$$

This is a characteristic equation for any hyperbolic equation of the type:

$$a\Delta u + \sum b_i\frac{\partial u}{\partial x_i} + c\frac{\partial u}{\partial t} + hu = \rho\frac{\partial^2 u}{\partial t^2}, \tag{80}$$

at $v = \sqrt{\frac{a}{\rho}}$. The classical wave-equation

$$\Delta u = \frac{1}{v^2(\mathbf{r})}\frac{\partial^2 u}{\partial t^2}$$

is the simplest hyperbolic equation of this type. But is it also the best choice?

1. Absence of wave dispersion. Let the coefficient of the equation (80) be constant. After substitution

$$u = \overline{u}\exp\left(-\frac{1}{2}\sum\beta_i x_i + \frac{1}{2}\alpha t\right)$$

where $\beta_i = b_i/a$ and $\alpha = c/a$. The equation is reduced to

$$\triangle\overline{u} + \overline{h}\overline{u} - \frac{1}{v^2}\frac{\partial^2\overline{u}}{\partial t^2} = 0$$

[2]We include differential operators into the class pseudodifferential in the same sense in which rational numbers are the particular case of irrational ones.

where $\overline{h} = h/a + v^2\alpha^2/4 + \sum \beta_i^2/4$.

Looking for the solution in the form

$$\exp[i(\omega t - \sum_{i=1}^{3} k_i x_i)],$$

we obtain the dispersion equation

$$\omega_{1,2} = \pm v\sqrt{\sum k_i^2 + \overline{h}^2}$$

Since in nondispersive media $\sum k_i^2 = v^2\omega^2$, only in the case when $\overline{h} = 0$ or

$$h = -\frac{(cv^2 + \sum \frac{b_i^2}{a})}{4}.$$

2. Amplitude equivalence. This term means that while discontinuities propagate, amplitudes A change along rays s following the transport equation (see Chapter 4)

$$2\frac{dA}{ds} + A\left[v\Delta\tau + \frac{d}{ds}\ln(\rho v^2)\right] = 0$$

which is valid for P and S waves. Using the standard technique of ray theory, it is easy to show that in the case of amplitude equivalence

$$c = 0$$

and

$$b_i = \frac{\partial a}{\partial x_i}.$$

For homogeneous media the wave equation is the only one that is nondispersive and amplitude-equivalent simultaneously.

3. Nonscattering vertical propagation. If in the amplitude-equivalent equation

$$\sum \frac{\partial}{\partial x_i}(\rho v^2 \frac{\partial u}{\partial x_i}) = \rho\frac{\partial^2 u}{\partial t^2}, \tag{81}$$

the following is valid: $v = v(z)$, $\rho = \rho(z)$ and $v(z)\rho(z) = const.$, then plane waves propagate in z-direction without scattering. This is easy to show if one inserts

$$u = Af\left(t - \int_0^z \frac{d\xi}{v(\xi)}\right)$$

into equation (81).

II. Boundary conditions. There are few types of conditions that can be used for the wave-field continuation with the help of equation (80).

1. Mixed problem with boundary and initial conditions

$$u|_{z=0} = u_0 \;\; (or \;\; \tfrac{\partial u}{\partial z}|_{z=0} = u_0)$$

$$u|_{t=t^\pm} = \tfrac{\partial u}{\partial t}|_{t=t^\pm} = 0$$

where $t^+ = 0$ (for forward continuation), $t^- = T$ (for reverse continuation), $T > t_{max}$, (where t_{max} is the duration of the seismic record).

This problem has a unique and stable solution (in a restricted domain). It is restricted at $t \to +\infty$ (for direct continuation) and $t \to -\infty$ (for inverse continuation) if $h > 0$ and $+c \geq 0$ (or $-c \geq 0$).

Representations of the solution:

- Finite-difference technique which can be applied for any inhomogeneous media.
- Kirchhoff's type integral:

$$u^{(\pm)}(\mathbf{r}, t) = \int\limits_{I_\pm} d\tau \iint\limits_{\sum} u_0(\mathbf{r}_0, t) \frac{\partial g^{(\pm)}(\mathbf{r}, \mathbf{r}_0, t - \tau)}{\partial z} d\mathbf{r}_0 \tag{82}$$

where $\sum : z = 0$, $I_+ = (0, t)$, $I_{(-)} = (T - t, T)$, $g^{(\pm)}(\mathbf{r}, \mathbf{r}_0, t)$-Green's function which is the solution of the equation

$$\mathbf{L}u - \rho \frac{\partial^2 u}{\partial t^2} = \delta(\mathbf{r} - \mathbf{r}_0)\delta(t) \tag{83}$$

with the zero initial condition at $t = t^\pm$ and $u|_{z=0} = 0$.

For inhomogeneous media Green's function usually is unknown (although one can calculate ray zero-approximation of $g^{(\pm)}(\mathbf{r}, \mathbf{r}_0, t)$). In the case of wave equation Green's function for homogeneous medium is

$$g^{(\pm)}(\mathbf{r}, \mathbf{r}_0, t) = \frac{\delta\left(t \mp \frac{R_+}{v}\right)}{4\pi R_+} - \frac{\delta\left(t \mp \frac{R_-}{v}\right)}{4\pi R_-} \tag{84}$$

where $R_\pm = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z \mp z_0)^2}$. It gives

$$\begin{aligned} u^{(\pm)}(\mathbf{r}, t) = \;& \tfrac{1}{2\pi} \int\limits_{-\infty}^{\infty}\!\!\int \tfrac{\cos\theta}{vR} \Big[\tfrac{v}{R} u(\mathbf{r}_0, t \mp \tfrac{R}{v}) \\ & \mp \mathbf{D}u(\mathbf{r}_0, t \mp \tfrac{R}{v}) \Big] d\mathbf{r}_0 \end{aligned} \tag{85}$$

where $R = \mid \mathbf{r} - \mathbf{r_0} \mid, \cos\theta = z/R$.

At big values of R the first term in square brackets can be neglected:

$$u^{\pm}(\mathbf{r}, t) \simeq \frac{1}{2\pi} \int\limits_{\infty}^{-\infty}\!\!\int \frac{\cos\theta}{vR} \mathbf{D}u(\mathbf{r}_0, t \mp \frac{R}{v}) d\mathbf{r}_0 \tag{86}$$

These expressions are valid only in 3D case. In 2D case Green's function is

$$g^{+}(\mathbf{r},\mathbf{r}_0,t) = \frac{1}{2\pi}\left[\frac{H(t-\frac{R_+}{v})}{\sqrt{t^2-(\frac{R_+}{v})^2}} - \frac{H(t-\frac{R_-}{v})}{\sqrt{t^2-(\frac{R_-}{v})^2}}\right] \tag{87}$$

Inserting this into the equation (82) we obtain for the reverse continuation

$$\begin{aligned} U^{(-)}(\mathbf{r},t) = \; & \tfrac{1}{\pi v}\int\limits_{-\infty}^{\infty}\cos\theta\, dx_0 \int\limits_{T-\frac{R}{v}}^{t}\Big[\tfrac{H(\tau-t)u(x_0,\tau+\frac{R}{v})}{\sqrt{\tau-t}[\tau-t+\frac{2R}{v}]^{3/2}} \\ & + \tfrac{H(\tau-t)}{\sqrt{\tau-t}}\tfrac{\dot{u}(x_0,\tau+\frac{R}{v})}{\sqrt{\tau-t+\frac{2R}{v}}}\Big] dx_0 \end{aligned} \tag{88}$$

where $\dot{u} = \mathbf{D}u$.

The operator

$$I^{\frac{1}{2}} f(t) = \int\limits_{-\infty}^{t}\frac{H(t-\tau)}{\sqrt{\pi(\tau-t)}} f(t)\, d\tau \tag{89}$$

is by definition the operator of integration of the order 1/2. It is easy to notice that the operator

$$\int\limits_{t}^{\infty}\frac{H(\tau-t)}{\sqrt{\pi(\tau-t)}} f(\tau)\, d\tau$$

is the same operator but acting in reverse time. We can denote this constriction as $\mathbf{I}^{1/2}_{(-)}$. Now we may also introduce $\mathbf{D}^{1/2}_{(\pm)} = \mathbf{D}_{(\pm)}\mathbf{I}^{1/2}_{(\pm)}$ where signs + and − show what time (usual or reverse) is used.

Let the field $u_0(x_0,t)$ contain a wave at $t=t_0$. The field $u^{(-)}$ will contain the wave at $t = t_0 - R/v$. Then the main contribution in the integral (88) is made by the values of the field u_0 at $\tau + R/v \simeq t_0$, that is, $\tau - t + 2R/v \simeq t_0 - R/v - t_0 + R/v + 2R/v \simeq 2R/v$.

Taking into account all these considerations and neglecting the term with fast attenuating at $R \to \infty$, we derive from equation (88):

$$U^{(-)}(\mathbf{r},t) \simeq \frac{1}{\sqrt{2\pi}}\int\frac{\cos\theta}{\sqrt{vR}}\mathbf{D}^{1/2}_{(-)} u_0\left(x_0, t+\frac{R}{v}\right) dx_0 \tag{90}$$

• Spectral representation in (ω,k)-domain (the wave equation, homogeneous media):

$$u^{(-)}(\mathbf{r},t) = \mathbf{F}^{-1}_{x,y,t}\left\{\tilde{u}_0 e^{ik_z^{\pm}z}\right\} \tag{91}$$

where

$$k_z^{\pm} = \left\{\begin{array}{ll} \mp|k_z| sgn(\omega) & k_z^2 \geq 0 \\ i|k_z| & k_z^2 < 0 \end{array}\right.$$

$$k_z^2 = \frac{\omega^2}{v^2} - k_x^2 - k_y^2$$

2. The Cauchy problem with respect to z in the $(\omega, \mathbf{k})$ domain for the equation

$$\frac{d^2\tilde{u}}{dz^2} + k_z^2 \tilde{u} = 0 \tag{92}$$

with the conditions,

$$\tilde{u}(\omega, k_x, k_y; z = 0) = \tilde{u}_0$$

$$(\tfrac{\partial \tilde{u}}{\partial z})_{z=0} = ik_z^{\pm}\tilde{u}_0$$

yields in homogeneous medium to the spectral form

$$\tilde{u}^{(\pm)} = \tilde{u}_0 e^{ik_z^{\pm} z}.$$

This gives a stable (regular) solution, but the finite-difference solution of equation (92) is stable only in the restricted domain $z < z^*$ and unstable at $z \to \infty$. We have the same problem with respect to the Cauchy problem for the wave equation with the conditions:

$$u|_{z=0} = u_0$$

$$\tfrac{\partial u}{\partial z}|_{z=0} = \mathbf{F}^{-1}\left\{ik_z^{\pm}u_0\right\}$$

The natural way to stabilize Cauchy's problem is to cut off all frequencies at $k_z^2 < 0$.

3. Reconstruction-type wave-field extrapolation. In this case we pose the conditions

$$u|_{z=0} = \tilde{u}_0$$

and

$$\frac{\partial u}{\partial z}|_{z=0} = \sigma_0 \tag{93}$$

where condition (93) is connected with the source. The solution of this problem gives mixed-type continuation which is nonregular (at $z \to \infty$) in both the $(\mathbf{r}$,t) and the $(\omega, \mathbf{k})$ domain. The simplest regularization is the same as above.

There is only one reverse type regular continuation into homogeneous medium which satisfies the condition:

$$u|_{z=0} = u_0 \tag{94}$$

Of course we have the whole family of continuation methods if we use instead of equation (94) the condition:

$$\left[\alpha u + (1-\alpha)\frac{\partial u}{\partial z}\right]_{z=0} = u_0$$

for any $\alpha, 0 < \alpha < 1$.

In inhomogeneous media different conditions produce different solutions. Figure **??** shows an example of different solutions: (a) is the reversed continuation of a single event at $t = \tau_0$ for mixed conditions, and (b) for the Cauchy problem.

In Figure **??**a we use reverse time $-t$ instead of t (we can do it with accordance with relation (51) in Chapter 5) because it promotes better understanding of reflection-refraction pattern at the interface $z = d$. In reverse time we have the usual process of wave propagation. Dashed lines show some ghost waves.

One-way propagation (PDKO)

Let us consider the equation:

$$\left(\frac{d}{dz} - ik_z\right)\tilde{u} = 0, \tag{95}$$

at

$$k_z = k_z(z).$$

For homogeneous medium this equation can be obtained by splitting the wave equation operator $\frac{d^2}{dz^2} + k_z^2$. Equation (95) has the solution:

$$\tilde{u}_0 \ \exp\left\{i\int_0^z k_z(\xi)d\xi\right\}$$

which describes upgoing waves. The operator

$$u(\mathbf{r}, t) = \mathbf{F}^{-1}_{x,y,t}\left\{\tilde{u}_0 e^{i\int_0^z k_z(\xi)d\xi}\right\} \tag{96}$$

is kinematically equivalent (in the sense of the classical eikonal equation), but there is no partial derivative equation in space-time domain that has equation (96) as a solution. This operator is an example of PDKO (Pseudo-Differential K-Operator). But we can find a sequence of equations

$$\mathbf{L}_j^{(-)}u = 0$$

for which:

- each equation permits only one-way propagation along the z-axis.
- the characteristic equation for $\mathbf{L}_j^{(-)}$ approximates the classical eikonal equation.
- The accuracy of the approximation increases with the number j.

The way to perform it is the expansion

$$ik_z = \frac{i\omega}{v}\left(1 - \frac{(ik_x)^2 + (ik_y)^2}{2\frac{(i\omega)^2}{v^2}} + \ldots\right)$$

If we substitute the nth approximation of ik_z into equation (95) and return to the ($\mathbf{r}$,t) domain, remembering the correspondence

$$\begin{array}{l}(i\omega)^r \leftrightarrow \frac{\partial^r}{\partial t^r}\\ (-ik_x)^r \leftrightarrow \frac{\partial^r}{\partial x^r}\end{array}$$

we get the equation which we looked for. The first approximation:

$$\frac{\partial^2 u}{\partial z \partial t} + \frac{1}{v}\frac{\partial^2 u}{\partial t^2} - \frac{v}{2}\frac{\partial^2 u}{\partial x^2} = 0 \tag{97}$$

yields to Jon Claerbout's famous 15-degree migration algorithm. With accordance to the formula (24) (in Chapter 3), the characteristic equation which corresponds to equation (97) is as follows:

$$\tau_x^2 - \frac{2}{v}\tau_z = \frac{1}{v^2}. \tag{98}$$

It differs from classical eiconal's equation for isotropic medium.

9: INTEGRAL OPERATORS OF WAVE-FIELD CONTINUATIONS

Kirchoff's type representations of the DKO for the homogeneous medium prompts the general form of an integral KO:

$$u^{(\pm)}(\mathbf{r},t) = \int_\Sigma w(\mathbf{r},\mathbf{r}_0)\mathbf{D}^n_{\pm t} u_0(\mathbf{r}_0, t \mp T(\mathbf{r},\mathbf{r}_0))d\mathbf{r_0}, \tag{99}$$

where $w(\mathbf{r},\mathbf{r}_0)$ - negative weight function, $T(\mathbf{r},\mathbf{r}_0)$ - propagation time from $\mathbf{r}$ to $\mathbf{r}_0$ with accordance with a given eiconal equation

$$\Psi(\nabla\tau; v) = 1. \tag{100}$$

Without any lack of generality we can put $n = 0$. We propose that

$$u_0(\mathbf{r}_0, t) \overset{q}{\sim} A_0(\mathbf{r}_0) R_q[t - \tau_0(\mathbf{r}_0)].$$

then

$$\mathbf{D}^{q+1} u_0(\mathbf{r}_0, t) \overset{-1}{\sim} A_0(\mathbf{r}_0)\delta[t - \tau_0(\mathbf{r}_0)]$$

and

$$U^{(\pm)}(\mathbf{r},t) = \mathbf{D}^{q+1} u(\mathbf{r},t) \sim \int w(\mathbf{r},\mathbf{r}_0) A_0(\mathbf{r}_0)\delta[t - \tau_0(\mathbf{r}_0) \mp T(\mathbf{r},\mathbf{r}_0)]d\mathbf{r}_0. \tag{101}$$

It is well known that

$$\int_X f(x)\delta(x-x_0)dx = \begin{cases} f(x_0) & x_0 \in X \\ 0 & x_0 \notin X. \end{cases}$$

It is easy to show that when point $\mathbf{r}$ is placed behind the front $\tau^{(-)} = t^*$ (where $\tau^{(-)}$ is the correspondent solution of equation (100) with condition $\tau|_\Sigma = \tau_0$), then argument of δ-function at $t = t^*$ is always positive:

$$\begin{aligned} t^* - \tau_0(\mathbf{r}_0) + T(\mathbf{r},\mathbf{r}_0) &> t^* - \tau_0(\mathbf{r}_0) + T(\mathbf{r}^*,\mathbf{r}_0) \\ &= t^* - \tau_0(\mathbf{r}_0) + (\tau_0(\mathbf{r}_0) - t^*) \\ &= 0 \end{aligned}$$

(see Figure **??**). It means that in this case, $U^{(-)}(\mathbf{r},t^*) = 0$. It is also can be shown that when $\mathbf{r}$ is between the front $\tau^{(-)} = t^*$ and the surface Σ, that argument of δ-function at $t = t^*$ takes both negative and positive meanings. So in this case, $U^{(-)}(\mathbf{r},t^*) \neq 0$. This proves kinematically equivalence of the operator (99).

Formulas for amplitudes (3D case)

We shall propose that points $\mathbf{r}^*$, $\mathbf{r}$ and $\mathbf{r}_0^*$ lie on the same ray γ of eiconal $\tau^{(-)}$ (Figure **??**). Then

$$\tau_0(\mathbf{r}_0) - T(\mathbf{r},\mathbf{r}_0) = t^* + \epsilon + \frac{1}{2}(\mathbf{r}_0 - \mathbf{r}_0^*)^T(\mathbf{B}_{\mathbf{r}_0^*} - \mathbf{A}_\mathbf{r})(\mathbf{r}_0 - \mathbf{r}_0^*),$$

$$\epsilon = T(\mathbf{r}^*,\mathbf{r}_0^*) - T(\mathbf{r},\mathbf{r}_0^*),$$

$$\mathbf{A}_\mathbf{r} = \frac{\partial^2 T(\mathbf{r},\mathbf{r}_0)}{\partial x_i \partial x_k} \qquad i,k = 1,2$$

$$\mathbf{B}_{\mathbf{r}_0^*} = \frac{\partial^2 \tau_0(\mathbf{r}_0)}{\partial x_i \partial x_k} \qquad i,k = 1,2$$

After substituting into the equation (101) at $\mathbf{r} = \mathbf{r}^*$ and $t = t^*$ and performing rather rigorous transformations, we obtain that

$$\lim_{\mathbf{r}\to\mathbf{r}^*} U^{(\pm)}(\mathbf{r},t) = \frac{2\pi w^* A_0^*}{\sqrt{D_*^{(\pm)}}} \equiv A^{(\pm)}(\mathbf{r}^*)$$

where $w^* = w(\mathbf{r}^*, \mathbf{r}_0^*), A_0^* = A_0(\mathbf{r}_0^*)$,

$$D_*^{(\pm)} = \det(\mathbf{A}_{\mathbf{r}^*} \pm \mathbf{B}_{\mathbf{r}_0^*}).$$

In Figure **??** different locations of fronts $\tau^{(-)} = t_k$ $(k = 1, 2, t_2 > t^* > t_1)$ are shown for reverse (a) and direct (b) continuation.

It is easy to derive from the pictures that

$$U^{(\pm)}(\mathbf{r}^*, t) \stackrel{0}{\sim} A^{(\pm)}(\mathbf{r}^*) H[\pm(t - t^*)]$$

and finally

$$u^{(\pm)}(\mathbf{r}, t) \stackrel{q+1}{\sim} \pm A^{(\pm)}(\mathbf{r}) R_{q+1}[\pm(t - \tau^{(\pm)}(\mathbf{r}))].$$

2D case

If we use the same type of considerations as in the 3D case, we receive

$$U^{(-)}(\mathbf{r}, t) = \frac{1}{\sqrt{\pi d^{(-)}}} \frac{w A_0 H[\tau(\mathbf{r}) - t]}{\sqrt{\tau^{(-)}(\mathbf{r}) - t}} + \psi(\mathbf{r})$$

where $\psi(\mathbf{r})$ is a smooth function,

$$d^{(-)} = \frac{\partial^2 T(\mathbf{r}, x)}{\partial x} - \frac{\partial^2 \tau_0(x)}{\partial x}.$$

Let us expand the class of discontinuities

$$R_q^{(+)}(t) = \frac{t_+^q}{\Gamma(q+1)} \qquad q \neq -1$$

with arbitrary (noninteger) q, and

$$R_q^{(-)}(t) = R_q^{(+)}(-t).$$

We must also expand the notion of q-equivalence (which was introduced in the Chapter 1) to a noninteger q: $f_1(t) \stackrel{q}{\sim} f_2(t)$ if $\mathbf{D}^q(f_1(t) - f_2(t)) \in$ C. The operator of noninteger differentiation $\mathbf{D}^q$ was considered above for $q = 1/2$ (see Chapter 8). For arbitrary q it can be defined by spectra response $(i\omega)^q = |\omega|^q \exp(-\frac{i\pi}{2} q \mathrm{sgn}(\omega))$.

It can be shown that

$$R_q^{(-)}(t) \stackrel{q}{\sim} -R_{q,-2q}^{(+)} \quad ; \quad R_q^{(+)}(t) \stackrel{q}{\sim} -R_{q,2q}^{(-)}(t)$$

$$\mathbf{D}_{\pm t}^p R_q^{\pm}(t) \stackrel{q-p}{\sim} R_{q-p}^{(\pm)}(t)$$

$$\mathbf{D}^p_{(\beta)} R^{(\alpha)}_{q,\nu} \overset{q-p}{\sim} \alpha\beta R^{(\beta)}_{q-p,\nu+q(\alpha-\beta)}(t)$$
$$\overset{q-p}{\sim} R^{(\alpha)}_{q-p,\nu+p(\alpha-\beta)}(t).$$

It is easy to see that

$$U^{(-)}(\mathbf{r},t) \sim \frac{wA_0}{\sqrt{d^{(-)}}} R^{(-)}_{-\frac{1}{2}}[t-\tau^{(-)}(\mathbf{r})]$$

and

$$u(\mathbf{r},t) \sim \frac{wA_0}{\sqrt{d^{(-)}}} R^{(-)}_{q+\frac{1}{2}}[t-\tau^{(-)}(\mathbf{r})]$$

(it is proposed that $d^{(-)} > 0$).

If $d^{(-)} < 0$, then

$$U^{(-)} \sim \frac{wA_0}{\sqrt{|d^{(-)}|}} R^{(+)}_{-\frac{1}{2}}[t-\tau^{(-)}(\mathbf{r})]$$

and general formula

$$U^{(-)} \sim \kappa \frac{wA_0}{\sqrt{|d^{(-)}|}} R^{(-)}_{-\frac{1}{2},\frac{1-\kappa}{2}}[t-\tau^{(-)}]$$

where $\kappa = \operatorname{sgn}(d^{(-)})$.

Let $d^{(-)} = 0$. We introduce the order of touching of curves $T(\mathbf{r},x)$ and $\tau_0(x)$: the order = p if

$$\frac{\partial^k T}{\partial x^k} - \frac{\partial^k \tau_0}{\partial x^k} = 0, \qquad k = 0,1,\ldots,p$$

and

$$d^{(-)}_p = \frac{\partial^{p+1} T}{\partial x^{p+1}} - \frac{\partial^{p+1}\tau_0}{\partial x^{p+1}} \neq 0.$$

If the order p is even, then

$$u^{(\pm)} \sim A_p[R^{(+)}_{q+\frac{1}{p+1}}(t-\tau^{(-)}) + R^{(-)}_{q+\frac{1}{p+1}}(t-\tau^{(-)})]$$

$$A_p \sim \frac{1}{|d_p|^{\frac{1}{p+1}}}$$

If the order is uneven

$$u^{(\pm)} \sim A_p R^{(-)}_{q+\frac{1}{p}}(t-\tau_{(-)}).$$

The point $\mathbf{r}$ is a special one if, for given $\mathbf{r}$ and $\mathbf{r}^*_0 = \omega(\mathbf{r})$, $\quad d^{(-)} = 0$. Each special point of the order $p = 2$ is the point on caustics.

Amplitude equivalence

IKO is amplitude equivalent if the amplitude $A^{(\pm)}(r)$ satisfies transport equation of the ray method (see equation (36), Chapter 4). It follows from the definition that equations (86) and (90) describe amplitude equivalent operators for homogeneous 3D and 2D media. And it is important to notice that the omitted terms (compare equations (85) and (86) as well (88) and (86)) do not influence the behavior of the senior discontinuity.

In order to construct the amplitude equivalent IKO for continuous inhomogeneous media, we may use the integral representation (82) of the solution of equation (81). It was mentioned above that the explicit expression of Green's function for inhomogeneous media is unknown. But it is sufficient to find the senior discontinuity of the Green's function by the ray method.

Ray approximation of Green's function for the equation (81) with mixed boundary conditions is

$$
\begin{aligned}
g^{(\pm)}(\mathbf{r}, \mathbf{r}_0, t) \overset{q}{\sim} \; & \frac{1}{4\pi v(\mathbf{r})} \sqrt{\frac{\rho(\mathbf{r}_0) v(\mathbf{r}_0)}{\rho(\mathbf{r}) v(\mathbf{r})}} \frac{R_q[t \mp T(\mathbf{r},\mathbf{r}_0)]}{\sqrt{J(\mathbf{r},\mathbf{r}_0)}} \\
& + \frac{1}{4\pi v(\mathbf{r})} \sqrt{\frac{\rho(\mathbf{r}_0^-) v(\mathbf{r}_0^-)}{\rho(\mathbf{r}) v(\mathbf{r})}} \frac{R_q[t \mp T(\mathbf{r},\mathbf{r}_0^-)]}{\sqrt{J(\mathbf{r},\mathbf{r}_0^-)}}
\end{aligned}
$$

where $q = -(n-1)/2$, n-dimension of the problem ($n = 2$ *or* 3), $\sqrt{J(\mathbf{r}, \mathbf{r}_0)}$-geometrical spreading, $\mathbf{r}_0^{(-)}$–the point which is symmetrical to $\mathbf{r}_0$ relatively to plane $z = 0$. The medium at $z < 0$ is symmetrical to the medium at $z > 0$.

Finally one can receive that the weight function of the amplitude equivalent IKO is

$$
w(\mathbf{r}, \mathbf{r}_0) = w_E(\mathbf{r}, \mathbf{r}_0) \equiv \frac{\cos\theta}{v(\mathbf{r})} \sqrt{\frac{\rho(\mathbf{r}_0) v(\mathbf{r}_0)}{\rho(\mathbf{r}) v(\mathbf{r})}} \frac{1}{\sqrt{J(\mathbf{r}, \mathbf{r}_0)}}
$$

where θ is the angle between the ray from $\mathbf{r}$ to $\mathbf{r}_0$ and the vertical axis.

Very simple but important property

If $w = w_1 w_2$ and $A_1(\mathbf{r})$ is a discontinuity amplitude of the field

$$
U_1(\mathbf{r}, t) = \mathbf{P}_{w_1}^{(\pm)}(v) u_0(\mathbf{r}_0, t)
$$

(where $\mathbf{P}_{w_1}^{(-)}$ is the KO with the kernel w_1) then the discontinuity of the field

$$
U(\mathbf{r}, t) = \mathbf{P}_{w}^{(\pm)}(v) u_0(P\mathbf{r}_0, t)
$$

has the amplitude

$$
A(\mathbf{r}) = A_1(\mathbf{r}) w_2.
$$

In particular, it means that if

$$
w = w_E \Psi(\mathbf{r}, \mathbf{r}_0)
$$

and A_E is the amplitude of the field

$$u_E = \mathbf{P}^{(-)}_{w_E}(v)U_0(t),$$

then for field $u = \mathbf{P}^{(-)}_{w} u_0$ has discontinuity amplitude:

$$A(\mathbf{r}) = A_E \Psi(\mathbf{r}, \mathbf{r}^*_0), \qquad \mathbf{r}^*_0 = \omega(\mathbf{r}).$$

Here, dependence $\omega(\mathbf{r})$ means that $\mathbf{r}^*_0$ belongs to the same ray of eiconal $\tau^{(-)}(\mathbf{r})$.

10: EXTREMAL VELOCITIES

We know how to determine the image of a reflector's location on a migrated section. This location, of course, depends on the choice of the continuation velocity v_c. Being independent of a weight function $w(\mathbf{r}, r)$, the location of caustics in the x, z-plane is determined by the system of equations

$$\frac{\partial T(x,z,r)}{\partial r} - \frac{d\tau_0}{dr} = 0 \tag{102}$$

and

$$\frac{\partial^2 T(x,z,r)}{\partial r^2} - \frac{d^2\tau_0}{dr^2} = 0. \tag{103}$$

If at some velocity $v_c = v_{ex}$ the caustic and the image cross each other at a special point $\mathbf{r}$, we call this the extremal velocity v_{ex}. Usually we have a continuous interval of extremal velocities: $v_{ex} \in V_{ex}$. Typically for each $v_{ex} \in V_{ex}$ one can observe two or more special points. For some value $v_{ex} = v^*_{ex}$ called the true extremal velocity, these special points unify in one point that appears as a *bright spot*.

Example: If the true model represents a homogeneous medium with velocity v and a planar reflector at depth h, the location of the reflector's image after CSP-migration is described by the system of equations (see Chapter 7):

$$\begin{aligned}
\xi(\theta) &= [1 - \frac{(\frac{v_c}{v})^2 - \sin^2\theta}{2\cos^2\theta}]2h\tan\theta \\
\eta(\theta) &= h\frac{v}{v_c}\frac{(\frac{v_c}{v})^2 - \sin^2\theta}{\cos^3\theta}\sqrt{1 - (\frac{v_c}{v})^2\sin^2\theta}
\end{aligned}$$

To find the location of caustics, we substitute $\tau_0(r) = \frac{1}{v}\sqrt{r^2+4h^2}$ and $T(x,z,r) = \frac{1}{v_c}\sqrt{(r-x)^2+z^2}$. into system equations (102) and (103). So we have the system of equations

$$\begin{aligned}
\frac{r}{v\sqrt{r^2+4h^2}} &= \frac{r-x}{v_c\sqrt{(r-x)^2+z^2}} \\
\frac{4h^2}{v(r^2+4h^2)^{\frac{3}{2}}} &= \frac{1}{v_c}\frac{z^2}{((r-x)^2+z^2)^{\frac{3}{2}}}.
\end{aligned}$$

If for some θ and r we have $\xi = x$ and $\eta = z$, then the reflector's image contains the special point (ξ, η). Calculations show that at $v_c < v$ there is no special point on the image. At $v < v_c < \sqrt{2}$ we have two symmetrical special points which merge together into the one bright point when $v_c = \sqrt{2}v \equiv v_{ex}^*$ (as shown in Figure **??**).

Instead of a complete investigation of the situation, we can choose some particular ray of the eikonal $\tau^{(-)}$. It is not difficult to determine the meaning of $\frac{\partial^2 T(\mathbf{r},r)}{\partial r^2}$ for the given velocity $v = v_c$ and the point $\mathbf{r}'$ belonging to the image and the ray γ; we can easily check condition (103). Let us again consider CSP-migration with the true model containing a curved reflector in a homogeneous medium (Figure **??**, solid line). Let us choose the ray γ_s which approaches the surface Σ vertically. This ray is connected with a stationary point r_s of the function $\tau_0(r)$:

$$\frac{d\tau_0}{dr}|_{r=r_s} = 0$$

and we shall call this ray stationary. We shall denote h the true depth of the reflector and h_m the image depth for the $x = r_s$. If $\tau_0(r_s) = t_s$, then h_m satisfies

$$\frac{h_m}{v_c} + \frac{\sqrt{h_m^2 + r_s^2}}{v_c} = t_s. \tag{104}$$

It is easy to show that

$$\left.\frac{\partial^2 T}{\partial r^2}\right|_{r=r_s,\mathbf{r}=\mathbf{r}'} = \frac{1}{v_c h_m} \tag{105}$$

The point $\mathbf{r}' = (r_s, h_m)$ belongs to a caustic if

$$\frac{1}{v_c h_m} = \frac{d^2\tau_0}{dr^2}|_{r=r_s} \tag{106}$$

Using equation (104), we have

$$h_m = \frac{t_s^2 v_c^2 - r_s^2}{2t_s v_c}. \tag{107}$$

Combining (106) and (107) we receive

$$v_c = v_{ex} = \sqrt{\frac{2}{t_s \frac{d^2\tau_0}{dr^2}} + (\frac{r_s}{t_s})^2} \tag{108}$$

It is easy to derive (see Figure **??**)

$$r_s = h\tan(2\phi), \qquad t_s = \frac{2\cos^2\phi}{\cos(2\phi)}\frac{h_m}{v_c}. \tag{109}$$

As for $\frac{d^2\tau_0}{dr^2}$ we can use standard techniques based on the continuation of wave front curvatures along the ray. We shall omit all these calculations and give the final result:

$$\frac{d^2\tau_0}{dr^2}|_{r=r'} = \frac{1}{vh}\frac{\cos\phi\ \cos(2\phi) + 2kh}{2(\cos^3\phi + kh)} \tag{110}$$

where k is the reflector curvature at point $\mathbf{r}$. Substitution of equations (109) and (110) into (108) gives

$$v_{ex} = \frac{\sqrt{2}v}{\cos\phi}\sqrt{\frac{\cos(2\phi)\ (\cos^3\phi + kh)}{\cos\phi\ \cos(2\phi) + 2kh} + \frac{1}{2}\sin^2\phi} \tag{111}$$

If the reflector is planar ($k = 0$), then

$$v_{ex} = v\frac{\sqrt{1 + \cos^2\phi}}{\cos\phi} \tag{112}$$

If the point $\mathbf{r}'$ is stationary for the reflector ($\phi = 0$), then

$$v_{ex} = \sqrt{2}v\sqrt{\frac{1 + kh}{1 + 2kh}} \tag{113}$$

If the reflector is planar and horizontal ($k = 0$ and $\phi = 0$), then

$$v_{ex} = \sqrt{2}v \tag{114}$$

and, at last, if we have a point reflector ($k = \infty$), then

$$v_{ex} = v \tag{115}$$

Is this velocity v_{ex} the true extremal velocity? It is in all particular situations expressed by formulas (112) - (115).

In the zero offset case for arbitrary rays

$$v_{ex} = v\sqrt{\frac{1 + kh\cos\phi}{1 + kh\cos\phi\ \sin^2\phi}}$$

and at $\phi = 0$

$$v_{ex} = v\sqrt{1 + \frac{1}{kh}}$$

If $k = 0$ (planar reflector) we formally receive $v_{ex} = \infty$, but in this case the amplitude does not actually depend on the meaning of the velocity v_c.

In the common mid-point case travel-time curves for a point reflector and for a flat reflector are the same. It means that in this situation $v_{ex} = v$.

11: TRUE AMPLITUDE MIGRATION

Our task will be to choose a kernel $w(\mathbf{r}, r)$ that supplies the proportionality of image amplitude and reflection coefficient.

We shall neglect: 1) directional characteristics of the source, 2) refraction coefficients, 3) influence of the free boundary, and 4) variation of the density ρ. With these restrictions the amplitude of a wave on the surface Σ is:

$$A_0(r) = \frac{c \cdot R}{\sqrt{J(s,r)}} \sqrt{\frac{v(s)}{v(r)}} \tag{116}$$

where s and r are the source and receiver locations, $\sqrt{J(s,r)}$ describes the geometrical spreading of a wave propagating along the ray $s\mathbf{r}r$ (see Figure 1), and R is the reflection coefficient at the point $\mathbf{r}$.

Equation (116) is good for marine surveys. For land surveys we usually measure the vertical component

$$\tilde{A}_0\ (r) = A_0(r)\cos\theta.$$

CSP migration, 3D case

To develop true-amplitude formulas for the CSP case, we shall use the simple fact that if $A_E(\mathbf{r})$ is the amplitude of the discontinuity received as a result of wave-field continuation with the amplitude-equivalent operator $\mathbf{P}_E^{(-)}(v)$ with the kernel

$$w_E(\mathbf{r}, r) = \frac{\cos\theta}{v(\mathbf{r})} \sqrt{\frac{v(r)}{v(\mathbf{r})}} \frac{1}{\sqrt{J(\mathbf{r}, r)}}$$

and if the kernel of the operator $\mathbf{P}_w^{(-)}(v)$ is

$$w(\mathbf{r}, r) = w_E(\mathbf{r}, r) \cdot \Psi(\mathbf{r}, r),$$

then as a result of applying the operator $\mathbf{P}_w^{(-)}(v)$ we receive the amplitude

$$A(\mathbf{r}) = A_E(\mathbf{r}) \cdot \Psi(\mathbf{r}, r^*), \tag{117}$$

(note that the star means that r^* belongs to the same ray of eiconal $\tau^{(-)}$ as $\mathbf{r}$).

But the operator $\mathbf{P}_E^{(-)}(v)$ acts inversely (in the sense of q-equivalence). This means that at the point $\mathbf{r}$ on the reflector,

$$A_E(\mathbf{r}) = \frac{c \cdot R}{\sqrt{J(s, \mathbf{r}^{(-)})}} \sqrt{\frac{v(s)}{v(\mathbf{r})}}$$

where $\mathbf{r}^{(-)}$ means the point $\mathbf{r}$ before the reflection.

According to equation (117),

$$A(\mathbf{r}) = \frac{c \cdot R}{\sqrt{J(s, \mathbf{r}^{(-)})}} \sqrt{\frac{v(s)}{v(\mathbf{r})}} \Psi(\mathbf{r}, r^*).$$

Now we apply the condition

$$A(\mathbf{r}) = c \cdot R$$

and receive

$$\Psi(\mathbf{r}, r^*) = \sqrt{\frac{v(\mathbf{r})}{v(s)} J(s, \mathbf{r})}.$$

So the kernel we are looking for is

$$w_{tr}(\mathbf{r}, r) = \frac{\cos\theta}{v(\mathbf{r})} \sqrt{\frac{v(r)}{v(s)} \frac{J(s, \mathbf{r})}{J(\mathbf{r}, r)}}$$

or in homogeneous media[3]

$$w_{tr}(\mathbf{r}, r) = \frac{\cos\theta}{v} \sqrt{\frac{(s-x)^2 + z^2}{(r-x)^2 + z^2}} = \frac{\cos\theta}{v} \frac{\rho_0}{\rho}.$$

For land surveys, in the general case we have

$$\tilde{w}_{tr} = \frac{1}{v(\mathbf{r})} \sqrt{\frac{v(r)}{v(s)} \frac{J(s, \mathbf{r})}{J(\mathbf{r}, r)}}$$

and in homogeneous media

$$\tilde{w}_{tr} = \sqrt{\frac{(s-x)^2 + z^2}{(r-x)^2 + z^2}} = \frac{\rho_0}{\rho}.$$

Zero-offset migration

There is a remarkable relationship between the amplitude $B(\mathbf{r})$ of a discontinuity on zero-offset migrated sections and the amplitude $A(\mathbf{r})$ of the corresponding discontinuity on a CSP section for the point R that belongs to the same normal ray:

$$A(\mathbf{r}) = 4B(\mathbf{r}).$$

This means that we can use the formulas for CSP patterns at s=r; the constant factor of 4 doesn't play any role. If we take into account the equality

$$v(r) \cdot \sqrt{J(r, \mathbf{r})} = v(\mathbf{r}) \sqrt{J(\mathbf{r}, r)},$$

[3]Here s, r, and x are two-dimensional vectors.

we receive

$$w_{tr} = \frac{const}{v(r)} \cdot \cos\theta$$

and

$$\tilde{w}_{tr} = \frac{const}{v(r)}.$$

The operator $\mathbf{P}^{(-)}_{w_{tr}}(\frac{v}{2})$ is not unique. If the medium is homogeneous then there is a whole family of operators $\mathbf{P}^{(-)}_{\overline{w}_{tr}}(\frac{v}{2}) \cdot t^m (m = 0, 1, 2, \ldots)$ that acts according to the formula:

$$U^{(-)} = \mathbf{P}^{(-)}_{\overline{w}_m}(\frac{v}{2})[t^m U_0],$$

and

$$\overline{w}_m = \frac{\cos\theta}{\rho^m}.$$

In the 2D case the amplitude of the image depends on the radius of transverse curvature of the reflector. But if the reflector is planar or cylindrical, we have a family of true-amplitude operators of the same type as

$$\overline{w}_m = \frac{\cos\theta}{\rho^m}\sqrt{\rho}.$$

12: OPERATOR OF INTEGRAL GEOMETRY (IGO)

We have connected migration with the idea of wave-field continuation. The great advantage of this approach is the possibility to introduce different "continuous techniques" to particular finite difference algorithms. Although this approach was successful in the case of both CSP and zero-offset patterns, it failed in other situations. However, it is not necessary to base migration on wave-field continuation even for the above particular patterns. We have integral transformations

$$a_{csp}(\mathbf{r}) = \int w(\mathbf{r}, r)\mathbf{D}^n_{-t} u_0(r, t = T(s, \mathbf{r}) + T(\mathbf{r}, r))dr \tag{118}$$

and

$$a_{zop}(\mathbf{r}) = \int w(\mathbf{r}, r)\mathbf{D}^n_{-t} u_0(r, t = 2T(\mathbf{r}, r))dr$$

which we can perform in one step.

One may guess that equation (118) is a general form of the transformation of seismic records into the depth (migrated) section. It pushes us to investigate integral transformations of the form:

$$u(\xi, \eta) = \int_X w(x; \xi, \eta) u_0(x; y = \theta(x; \xi, \eta))dx \equiv \mathbf{P}u_0(x, y). \tag{119}$$

This kind of transformation is typical not only in the problem of migration, but also in several other cases:

- slant analysis, when

$$y = t, \quad \theta(x;\xi,\eta) = \eta + \xi x, \quad w \equiv 1$$

- parabolic filtration, when

$$y = t, \quad \theta(x;\xi,\eta) = \eta + a(\xi - x)^2, \quad w \equiv 1$$

- seismic tomography, when

$$u_0(x,y) = s(x,y) - s_0(x,y)$$

(here $s(x,z)$ is a slowness, $y = \theta(x;\xi,\eta)$ is the equation of a ray with the starting point ξ and end point η, and $w(x;\xi,\eta) = \sqrt{1 + |\theta'|^2}$).

It is also possible to find other examples. Integral transform (119) belongs to the class of integral geometry operators.

In the K-operator case, the method of discontinuities was only one of the possible alternatives; another was the stationary phase technique. For the IGO operator, the method of discontinuities is the only approach that allows us to apply a relatively elementary mathematical concept.

Let us presume that

$$u_0(x,y) \sim A(x) R_q^{(+)}[y - \tau(x)].$$

Again the main idea is that differentiation of the field $u(\xi,\eta)$, with respect to η, allows us to transform a discontinuity of the field $u_0(x,t)$ into a δ-function. But in this situation it is not so simple because we have essentially used properties of discontinuities of arbitrary order and index. (Some of them were mentioned above.) Now I will add one more: Let t_0 be the only solution of the equation $\phi(t) = 0$, then

$$\mathbf{D}^p_{(\beta),t} R^\alpha_{q,\nu}[\phi(t)] \overset{q-p}{\sim} |\phi_0^{'}|^p R^{(\alpha')}_{q-p,\nu+p(\alpha'-\beta)}[\phi(t)]$$

where $\alpha' = \alpha \;\operatorname{sgn}(\phi_0^{'})$, and $\phi_0^{'} = \phi^{'}(t_0)$. After that we get

$$U(\xi,\eta) = \mathbf{D}^{q+1} u(\xi,\eta) \sim \int_\infty^\infty w(x;\xi,\eta) A(x) |\theta_\eta|^{q+1} \delta[x(x;\xi,\eta) - \tau(x)] dx$$

Let's summarize the main concepts behind this technique. We shall consider the function $\theta(x;\xi,\eta)$ as a family of curves $\theta_M(x)$ in the plane (x,y) , that depends on the point $M = (\xi,\eta)$ of the plane (ξ,η) (see Figure **??**).

The curve $y = \theta_M(x)$ is called a stacking line. We shall suggest that a given family of stacking lines is regular. It means that for any point $N = (x,y)$ and any value $p \in \mathrm{P}$ the system of equation

$$\theta_M(x) = y, \; \frac{d\theta_M}{dx} = p \tag{120}$$

with respect to M has one and only one solution (P is a given open set). In other words, through the given point N in the given direction p runs only one stacking line.

If we consider only the first of the equations (120)

$$\theta_M(x) = y \tag{121}$$

(with fixed x and y) then we have a continuous line of points M satisfying the equation (121). This line is called a dual line (see Figure **??**) and denoted as $\eta = \bar{\theta}_N(\xi)$.

A discontinuity of wave-field $u_0(x, y)$ located along the line $y = \tau(x)$ influences the appearance of several discontinuities of the field $u(\xi, \eta)$. We shall call the main discontinuity a discontinuity of the field $u(\xi, \eta)$ with minimal order $q^{'}$.

Geometry of the main discontinuity

• The main discontinuity of the field is located along the line $\eta = \mu(\xi)$ if any of the following conditions is fulfilled (see Figure **??**):

1. The curve μ is an envelope of a family of dual lines $\overline{\theta}_N$ under $N \in \tau$.
2. The curve τ is an envelope of a family of stacking lines θ_M under $M \in \mu$.

The curve μ satisfies the equations

$$f(x; \xi, \eta) = 0, \quad \frac{df(x; \xi, \eta)}{dx} = 0 \tag{122}$$

where $f(x; \xi, \eta) = \theta_M(x) - \tau(x)$.

• Each operator $\mathbf{P}$ defines a one-to-one mapping $\mathbf{K}(P) : \{\tau\} \rightleftharpoons \{\mu\}$ of a set $\{\tau\}$ of curves in the plane (x, y) into the set $\{\mu\}$ of curves in the plane (ξ, η). This mapping does not depend on the kernel $w(x; \xi, \eta)$ and is determined only by the family of stacking lines $\{\theta_M\}$.

• There is a one-to-one correspondence between points of curves τ and μ(Figure **??**):

$$\omega : \quad M \in \mu \rightleftharpoons N \in \tau$$

• The curve θ_M is a special curve of a mapping $\mathbf{K}(P)$:

$$\mathbf{K}(P)\theta_M = M$$

• The operator $\bar{\mathbf{P}}$, with dual lines as stacking lines, produces the mapping inverse to $\mathbf{K}(\bar{P})$:

$$\mathbf{K}(P) = (\mathbf{K}(P))^{-1}$$

that follows immediately from the symmetry of the two conditions in the first statement.

• If the curve τ ends at the point N_0, then the field $u(\xi,\eta)$has an edge discontinuity that is located along dual line $\eta = \bar{\theta}_{N_0}(\xi)$ (Figure **??**).

• If the aperture X has the end a that does not depend on ξ,η and belongs to the interval of determination of the curve τ, then the field $u(\xi,\eta)$ has an edge discontinuity that is located along the dual line $\eta = \bar{\theta}_{N_0}(\xi)$ at $n_0 = (a, \tau(a))$.

• If the value a depends on values ξ and η, then the field $u(\xi,\eta)$ contains a discontinuity on the line λ satisfying the equation

$$\theta[a(\xi,\eta);\xi,\eta] - \tau[a(\xi,\eta)] = 0,$$

(see Figure **??**).

13: AN APPROACH TO A GENERAL THEORY OF SEISMIC IMAGING

Let equation

$$s = \psi_1(\lambda), \quad r = \psi_2(\lambda) \tag{123}$$

determine an arbitrary "continuous" pattern of seismic records and

$$t = \tau(\lambda) = t[s = \psi_1(\lambda), r = \psi_2(\lambda)]$$

be a traveltime curve of the reflected wave for this pattern and a given reflector. Does an IGO exist that (at given velocities) transforms the discontinuity on the line $t = \tau(\lambda)$ into a discontinuity coinciding with the position of the reflector?

Let us suggest that at the given velocity $v(x,t)$ travel time inversion for the pattern (123) has a unique solution.[4] This suggestion makes sense if it embraces situations such as CSP-pattern ($s = const., r = \lambda$), CRP-pattern ($s = \lambda, r = const.$) or zero-offset ($s = r = \lambda$). We shall show that for homogeneous media a sufficient (but unnecessary!) condition is

$$\psi_1'(\lambda) \geq 0\ ,\ \psi_2'(\lambda) \geq 0. \tag{124}$$

Let us fix a parameter meaning λ, $t = \tau(\lambda)$ and $\frac{d\tau}{d\lambda}$. It is evident that the reflection point lies on the the reflection isochron

$$\sqrt{(x-s)^2 + z^2} + \sqrt{(x-r)^2 + z^2} = vt.$$

The point we are looking for is the point on the isocron for which

$$\frac{\partial t}{\partial s}\psi_1'(\lambda) + \frac{\partial t}{\partial r}\psi_2'(\lambda) = \frac{dt}{d\lambda}. \tag{125}$$

[4]This pattern we shall call acceptable.

Let us parameterize the isochron by the angle ϕ (see Figure **??**). Then

$$\frac{dt}{dr} = f(\psi) \text{ and } \frac{dt}{ds} = -f(\pi - \psi)$$

where $f(\psi)$ is a monotonically decreasing function. Then at condition (124) the sum

$$-f(\pi - \psi)\, \psi_1' \ + \ f(\psi)\, \psi_2'$$

is also a monotonic function and it means that equation (125) has a unique solution. The condition (124) is essential as it follows from the example of CMP-pattern ($s = X - \lambda/2, r = X + \lambda/2, X = const.$), when there are exactly two symmetrical solutions to equation(125). It is easy to notice that in the case of common-offset pattern ($s = \lambda - l/2, r = \lambda + l/2, l = const.$) , $\psi_1' \ = \ \psi_2' \ = \ 1 \ > \ 0$.

Uniqueness of travel time inversion means that there is a mapping of a set of travel time curves into the set of reflectors (at a given $v(\mathbf{r})$). We pose the following general problem. Let $\mathbf{K}$ be an arbitrary one-to-one mapping of an arbitrary set of lines $\{\tau\}$ in the plane (x, z) into a set of lines $\{\mu\}$ in the plane (ξ, η). Does the IGO $\mathbf{P}$ exist for which $\mathbf{K} = \mathbf{K}(P)$? If this operator exists, we shall say that operator $\mathbf{P}$ realizes the mapping $\mathbf{K}$. Of course we must determine only the family of stacking lines $\{\theta_M\}$: the kernal of the operator $\mathbf{P}$ can be chosen arbitrarily.

It can be proved that there are only three alternatives:

1. Operator $\mathbf{P}$ does not exist.
2. One and only one operator $\mathbf{P}$ exists.
3. A continuum of operators $\mathbf{P}$ exists.

Of course the second alternative is the most important for us.

- We have the second alternative if a common point N of curves τ_1 and τ_2 is mapped into a common point of curves μ_1 and μ_2 when, and only when, τ_1 touches τ_2 at N (see Figure **??**).
- The family of special curves of mapping $\mathbf{K}$ originates the family of stacking lines:
$$\mathbf{K}\hat{\tau} = \hat{M} \text{ , then } \hat{\tau} \ \equiv \ \theta_{\hat{M}}.$$
- If $\mathbf{K}$ is realized then $\mathbf{K}^{-1}$ is realized too and $\overline{\theta}_N$ is a family of stacking lines.

Let us give an example of the second alternative. We assume that τ is the traveltime curve for an acceptable pattern and μ is a reflector on a depth section. It is easy to notice that the ray geometry depends only on the location of the reflector and the slope of the tangent (see Figure **??**). It immediately follows from this that all conditions for the existence of the second alternative are fulfilled, so there is an operator of integral geometry (at $\xi = x$ and $\eta = y$) that realizes the mapping of traveltime curves into reflector images.

It is evident that special lines of this mapping are traveltime curves for point reflectors.

$$\theta(\lambda; x, z) = T(s(\lambda); x, z) + T(x, z; r(\lambda))$$

We shall continue listing some general statements.

- If mappings $\mathbf{K}_1$ and $\mathbf{K}_2$ are realized, then the mapping of $\mathbf{K} = \mathbf{K}_2 \cdot \mathbf{K}_1$ is realized too.

Example. Let us answer the following question: Can an acceptable pattern (123) be transformed into a time section that is geometrically equivalent to t_0?

Let us define $\mathbf{K}_1$ and $\mathbf{K}_2$ such that

$$\mathbf{K}_1 \;:\; \tau(\lambda) \to \mu$$

$$\mathbf{K}_2 \;:\; \mu \to t_0$$

where μ is the position of a reflector on the depth section. We have shown above that $\mathbf{K}_1$ is realized. The possibility of realizing the transformation of depth section into the time section $t_0(x)$ is obvious so the operator $\mathbf{K}$ is realized. Stacking lines of the corresponding operator $\mathbf{P}$ are travel time curves from the circular reflector with center at x and radius $\rho = vt_0$, because this reflector gives a point on the zero offset time (non-migrated) section, and consequently originates a special curve of mapping $\mathbf{K}$. Another example: cascaded migration. Let $\mathbf{P}_1$ be determined by the system of stacking lines:

$$\theta_1 = \sqrt{\eta_1^2 + \frac{4(x - \xi_1)^2}{v_1^2}}$$

and $\mathbf{P}_2$ by the system of stacking lines:

$$\theta_2 = \sqrt{\eta_2^2 + \frac{4(\xi_1 - \xi)^2}{v_2^2}}.$$

The system of stacking lines for the operator $\mathbf{P} = \mathbf{P}_1 \cdot \mathbf{P}_2$ in the general case is determined by the formula

$$y = \theta(x; \xi, \eta) = \theta_1[x; \xi_1, \theta_2(\xi_1; \xi, \eta)]$$

at $\xi_1 = \Psi(x, \xi, \eta)$ where the function Ψ is derived from the equation

$$\frac{d\theta_1}{d\xi_1} : \frac{\partial\theta_1}{\partial\eta_1} + \frac{\partial\theta_2}{\partial\xi_1} = 0.$$

Application of all these equations to cascaded migration gives the system of stacking lines:

$$\theta(x;\xi_1,\eta) = \sqrt{\eta^2 + \frac{4(\xi - x)^2}{v_1^2 + v_2^2}}.$$

Now we shall give an example of the first alternative: the case of head waves (Figure **??**). It is easy to show that in the case of the CSP-pattern there is a unique solution of travel time inversion for head waves (for a given v_1 and v_2). But touching of refractors does not guarantee touching of traveltime curves. So the mapping $\mathbf{K}$ is not generally realized.

Nevertheless, it can be done for some special cases. For example, when $\{\mu\}$ is a set of planar refractors, this produces an operator $\mathbf{P}$ which approximately realizes the mapping $\mathbf{K}$ for refractors with small curvature.

In Chapter 7 we have testified that image ray technique does not guarantee exact time-to-depth migration if time migration was performed in a simpler model of medium. Does the correct time-to-depth migration algorithm in this case exist? The answer is positive. The mapping $\mathbf{K}$ we are looking for is a product of three mappings: scaling operator $\mathbf{S}$ that, with help of substitution $t = \frac{z}{v_c}$ (v_c - continuation velocity for time migration), connects reflector images $\{\mu_t\}$in time-sections with reflector images $\{\mu_z\}$ in depth section; mapping $\mathbf{K}_1$ that connects $\{\mu_z\}$ with zero-offset travel-time curves $\{t_0\}$ in a simpler model of medium; and mapping $\mathbf{K}_2$ that connects to reflectors μ in a true model. So $\mathbf{K} = \mathbf{K}_2\mathbf{K}_1\mathbf{S}$. Mappings $\mathbf{K}_1$ and $\mathbf{K}_2$ are realizable, so their product is realizable too.

Finally, an example of the third alternative: CMP stacking. Traveltime $\tau(l)$ is always an even function so any family of even curves $\{\theta(l;\tau_0, v_s(\tau_0))\}$ guarantees touching at $l = 0$ and $t = \tau_0$ (Figure **??**). This means that CMP stacking always gives a section that is geometrically equivalent to $t_0(x)$.

14: DYNAMIC ASPECTS OF IGO AND THEIR APPLICATION TO TOMOGRAPHY

Dynamics of the main discontinuity

Our task is to describe the profile of the main discontinuity of the field

$$u(\xi,\eta) = \mathbf{P}_w u_0(x,y) \equiv \int_X w(x;\xi,\eta) u_0(x, y = \theta(x;\xi,\eta))dx \qquad (126)$$

along the line $\xi = const$. I remind you that

$$u_0(x,y) \overset{q}{\sim} A_0(x) R^{(+)}_{q,\nu}[y - \tau(x)].$$

Applying the operator $\mathbf{D}^{q+1}\mathbf{H}^{-\nu}$ we obtain the field

$$U(\xi,\eta) \sim \int w(x;\xi,\eta)|\theta_\eta|^{q+1}A_0(x)\delta[\theta_M(x)-\tau(x)]dx.$$

Then we fix a line in the plane (ξ,η):

$$\frac{d\theta_M}{dx} = p\ (x = x^* = const,\ p = const),$$

which we shall call a pseudoray. For each point M' belonging to the pseudoray, we apply the expansion of the difference $\theta_{M'}(x)-\tau(x)$ into Taylor's series in a neighborhood of the point x^*:

$$\theta_{M'}(x)-\tau(x) = \epsilon - \left(\frac{d}{2}\right) l^2 + 0(l^2)$$

where $\epsilon = y' - y^*$, $y^* = \theta_M(X^*)$, $l = x - x^*$ and

$$d = \left[\frac{d^2\tau}{dx^2} - \frac{d^2\theta_M}{dx^2}\right]_{x=x^*}.$$

The above equations are absolutely the same as those that we have already done for the KO. As a result, we will receive a very similar formula describing the field $U(M')$:

$$U(M') \sim \left[\sqrt{\frac{2\pi}{|d|}}|\theta|_\eta^{q+1} w A_0\right]_\omega R^{(\kappa)}_{-\frac{1}{2}}(\epsilon) \tag{127}$$

where $\kappa = \mathrm{sgn}(d)$.

But here we have some important differences that we have to take into account: Formula (127) describes the profile of the discontinuity along the pseudoray but not along a line $\xi = const$. There is a formalism that connects different profiles of discontinuity. I shall omit this consideration and give you only the final result under $\theta_\eta > 0$:

$$u(\xi,\eta) \overset{q'}{\sim} (\mathbf{D}^{q+1}\mathbf{H}^{-\nu})^{-1} U \overset{q'}{\sim} \kappa \left[\sqrt{\frac{2\pi}{|d|}}\theta_\eta^{q+\frac{1}{2}} w A_0\right]_\omega R^{(+)}_{q',\nu'}[y-\tau(x)] \tag{128}$$

where $q' = q + \frac{1}{2}$, $\nu' = \nu + \frac{\kappa - 1}{2}$. I consider this formula to be the main result of the developing theory.

Application to tomography

Integral (126) describes the situation in the linear tomography when rays can be parametricized as a function of $x : y = \theta(x;\xi,\eta)$ and

$$w = (x;\xi,\eta) = \sqrt{1+|\theta'_x|^2}. \tag{129}$$

The general problem of tomography is the reconstruction of the field $u_0(x,y)$ using the given field $u(\xi,\eta)$. We shall reduce this problem to a simpler one: to reconstruct the discontinuities of the field $u_0(x,y)$. The possibility of this reduction follows from the fact that discontinuities are a model for the most vivid particularities of the object.

So our task is to find such an operator $\mathbf{P}^{(-)}$, that

$$\mathbf{P}^{(-)}u(\xi,\eta) \overset{q}{\sim} u_0(x,t).$$

Let us introduce a dual IG-operator

$$u_1(x,y) = \hat{\mathbf{P}}_{w'}u(\xi,\eta),$$

which has the family of dual lines $\{\hat{\theta}_N(\xi)\}$ as a system of stacking lines and an arbitrary kernel $w' = w'(M,N)$. It follows from the results of Chapter 12 that for any kernel w', the operator $\hat{\mathbf{P}}_{w'}$ gives geometrical reconstruction of the original discontinuity of the field $u_0(x,y)$ since this operator realizes the mapping $\tau = K^{-1}\mu$. It is clear that the operator $\hat{\mathbf{P}}_{w'}$ transforms the dynamics of a discontinuity of the field $u(\xi,\eta)$ in the same way as the operator $\mathbf{P}_w$ did. Then, according to formula (126)

$$U_1 \sim \hat{\mathbf{P}}_{w'}u \sim \kappa' \left[\sqrt{\frac{2\pi}{|d'|}}\hat{\theta}_y^{q'+\frac{1}{2}}w'A'\right]_\omega R^{(+)}_{q'',\nu''}[y-\tau(x)] \tag{130}$$

where $q'' = q' + \frac{1}{2}$, $\nu'' = \nu' - \frac{\kappa'-1}{2}$

$$d' = \left(\frac{d^2\mu}{d\xi^2} - \frac{d^2\hat{\theta}_N}{d\xi^2}\right)_{\xi=\xi^*}, \ \kappa' = sgn(d').$$

We have taken into account that

$$\hat{\theta}_y = \frac{1}{\theta_y} > o.$$

Now, combining formulae (128) and (130), we have:

$$u_1(x,y) \sim \kappa\kappa' \left[\sqrt{\frac{2\pi}{|dd'|}}\theta_\eta^{-\frac{1}{2}}w'wA_0\right]_\omega R^{(+)}_{q'',\nu''}[y-\tau(x)]$$

where

$$q'' = q+1, \tag{131}$$

$$\nu'' = \nu - \frac{\kappa-1}{2} - \frac{\kappa'-1}{2}.$$

The operator $\mathbf{P}_w$ acts like $\mathbf{D}^{-\frac{1}{2}}_{(\kappa)}$ and the operator $\hat{\mathbf{P}}_{w'}$ acts like $\mathbf{D}^{-\frac{1}{2}}_{(\kappa')}$. This means that in order to reconstruct the order and the index of discontinuities we shall apply the correction filter $\mathbf{D}^{\frac{1}{2}}_{(\kappa)}\mathbf{D}^{-\frac{1}{2}}_{(\kappa')}$. It can be shown that independently on the curve $y=\tau(x)$

$$\kappa' = -\kappa, \tag{132}$$

consequently, the correction filter is $\mathbf{D}^{\frac{1}{2}}_{(+)}\mathbf{D}^{\frac{1}{2}}_{(-)} = |\mathbf{D}| \equiv \mathbf{DH}$. It follows from equations (131) and (132) that

$$\nu^{''} = \nu + 1 \text{ and } \kappa\kappa^{'} = -1.$$

But

$$\mathbf{DH}[-R_{q+1,\nu+1}] \sim -R_{q,\nu+2} \sim -\mathbf{H}^2 R_{q,\nu} \sim R_{q,\nu}$$

since $\mathbf{H}^2 = -\mathbf{E}$.

The kernal $w^{'}$ must be chosen from the condition

$$\frac{2\pi w w^{'}}{\sqrt{\theta_\eta |d d^{'}|}} = 1.$$

I have proven that

$$d d^{'} = F G$$

where

$$F = \theta_{x\xi}\theta_\eta - \theta_{x\eta}\theta_\xi, \; G = \hat{\theta}_{x\xi}\hat{\theta}_y - \hat{\theta}_{y\xi}\hat{\theta}_x.$$

Consequently,

$$w^{'} = \sqrt{\frac{\theta_\eta |FG|}{2\pi w}}. \tag{133}$$

So we have shown that

$$\mathbf{P}^{(-)} = \hat{\mathbf{P}}_{w'} |\mathbf{D}| \tag{134}$$

where $w^{'}$ is expressed by Formula (133).

For example, let

$$\theta = \eta + \frac{a}{2}(\xi - x)^2, \tag{135}$$

then

$$\hat{\theta} = \eta - \frac{a}{2}(\xi - x)^2, \; w w^{'} = \frac{a}{2\pi}.$$

If equation (135) is an equation of rays in ray tomography, then according to equation (129)

$$w = \sqrt{1 + a^2(\xi - x)^2}$$

and

$$w^{'} = \frac{a}{2\pi\sqrt{1 + a^2(x - \xi)^2}}.$$

Figures **??**-**??** give an illustration of reconstruction of the step-function

$$u_0(x, y) = H(y - \frac{1}{2})$$

given at the square $-\frac{1}{2} \le x \le \frac{1}{2}$), $0 \le y \le 1$ (equation (126)). Figure **??**a shows the field $u(\xi, \eta)$. Figure **??**b is a result of application of the simplest operator $\hat{\mathbf{P}}_{w'}$

at $w' \equiv 1$. Figure **??**c is a result of application of the operator $\hat{\mathbf{P}}'_w$ with w' chosen according to equation(135), And Figure **??**d gives the field

$$u_1(x,y) = \hat{\mathbf{P}}_{w'}|\mathbf{D}|u(\xi,\eta).$$

It is interesting that Kirchhoff's operators of the forward and reverse wave-field continuation originate a symmetrical pair of asymptotically inverse operators $\mathbf{P}_w\mathbf{D}^{\frac{1}{2}}_{(+)}$ and $\hat{\mathbf{P}}_{\hat{w}}\mathbf{D}^{\frac{1}{2}}_{(-)}$ with

$$w = \sqrt{\frac{|F|}{2\pi}} \equiv w_E \text{ and } \hat{w} = \sqrt{\frac{|G|}{2\pi}} = w_E.$$

In the n-dimensional case we have the following analog of formula (134)

$$\mathbf{P}^{(-)} = \hat{\mathbf{P}}_{w'}|\mathbf{D}|^{\frac{m}{2}},\ m = n-1 \tag{136}$$

where the kernel w' is expressed by the same formula (133) but F and G are determinants of the correspondent matrices.

The formula (111) and (113) were published first by Gr. Beylkin (1984) who applied a very different approach connected with the theory of pseudodifferential operators.

15: COMPARISON OF PRE- AND POST-STACK MIGRATION

For any single fold gather

$$s = \Phi_I(\lambda),\ r = \Psi_I(\lambda)$$

there is only one family of stacking lines forming the IG-operator which transforms the field $u_0(s(\lambda), r(\lambda), t)$ into the depth migration section (2D case) or volume (3D case). This operator has a structure $\mathbf{P}_w\mathbf{D}^k_{(-)}$ where w is a weight function that may satisfy some additional condition, $k = \frac{(n-1)}{2}$, where n is the dimension of the problem. But when we have a multi-fold observational system there are different ways to perform depth migration. Any continuous system of observations may be considered as a family of basic gathers

$$s = \Phi_\alpha(\lambda),\ r = \Psi_\alpha(\lambda) \tag{137}$$

where α is a continuous parameter. The choice of a basic gather determines parametrization of a multi-fold observational system. If, for instance, basic gather is a common offset pattern, then

$$s = X - h,\ r = X + h$$

where h plays the role of α and X plays the role of λ. Exchanging roles of X and h gives common-mid-point parametrization. If we consider multi-fold systems as a family of common source patterns, then $\alpha = s$ and $r = s + \lambda$ and so on.

For any such description of the observational system, we define prestack migration as

$$M_I(x,z) = \int M(x,z,\alpha)d\alpha \tag{138}$$

where

$$M(x,z,\alpha) = \mathbf{P}_w^\lambda \mathbf{D}_{(-)}^k u_0(\lambda,\alpha,t) \tag{139}$$

and

$$u_0(\lambda,\alpha,t) = u_0(\Phi_\alpha(\lambda),\Psi_\alpha(\lambda),t)$$

The operator $\mathbf{P}_w^\lambda$ in equation (140) acts on variable λ and has the following stacking lines:

$$\theta_\alpha(\lambda,x,z) = T(\Phi_\alpha(\lambda),x,z) + T(\Psi_\alpha(\lambda),x,z).$$

We may shortly describe this scheme by the symbol sequence $\sum_\alpha \mathbf{P}_w^\lambda \mathbf{D}_{(-)}^m$, where $\sum_\alpha$ is the operator of simple stacking (without time or depth shifts).

It is natural to consider r and s as primary coordinates of observations. Then any other description of equation (137) is a change of the system of coordinates. In order to stress this, we can rewrite equation (137) in the form

$$s = \Phi(\lambda,\alpha),\ r = \Psi(\lambda,\alpha).$$

If $w_s(x,z,r)$ is a weight function of the operator $\mathbf{P}_{w_s}^r$ transforming common-shot-point records to a depth migrated section (volume) and we choose:

$$w_\alpha(x,z,\lambda) = \frac{D(s,r)}{D(\lambda,\alpha)} w_{\Phi_\alpha(\lambda)}(x,z,\Psi_\alpha(\lambda))$$

then

$$\Sigma_\alpha \mathbf{P}_{w_\alpha}^\lambda u_0(\lambda,\alpha,t) = \Sigma_s \mathbf{P}_{w_s}^r u_0(s,r,t).$$

More fully, it is:

$$\begin{aligned}
\Sigma_s \mathbf{P}_{w_s}^r u_0(s,r,t) &= \int\int w_s(x,z,r) u_0(s,r,T(s,x,z)+T(r,x,z))dsdr \\
&= \int\int w_{\Phi_\alpha(\lambda)}(x,z,\Psi_\alpha(\lambda)) \frac{D(s,r)}{D(\lambda,\alpha)} u_0(\Phi_\alpha(\lambda), \\
&\quad \Psi_\alpha(\lambda),\theta_\alpha(\lambda,x,z))d\lambda d\alpha) \\
&= \Sigma_\alpha \mathbf{P}_{w_\alpha}^\lambda u_0(\lambda,\alpha,t).
\end{aligned}$$

The proved relation means that under a proper choice of weight function, migration does not depend on the choice of a basic data pattern. Of course, this statement does not take into account some practical aspects that can be connected with numerical realizations of operators $\mathbf{P}_{w_\alpha}^\lambda$ and $\mathbf{P}_{w_s}^r$.

A particular case of the scheme (138)-(139) is the situation where the operator $\mathbf{P}_{w_\alpha}^\lambda$ is split into the sequence of two or more operators. For example, when

$$\mathbf{P}^{\lambda}_{w_\alpha} = \mathbf{P}^{x}_{w_0}\mathbf{Q}^{\lambda}_{\hat{w}_\alpha}$$

where $\mathbf{P}^{x}_{w_0}$ realizes transformation of zero offset time section into the depth migrated section and $\mathbf{Q}^{\lambda}_{\hat{w}_\alpha}$ is IG-operator that transform basic gather into the zero offset time section. Let the basic pattern be a common-offset gather. Then the operator $\mathbf{Q}^{\lambda}_{\hat{w}_\alpha}$ may be represented in a form

$$\mathbf{Q}^{\lambda}_{\hat{w}_\alpha} = DMO \cdot NMO$$

But this representation determines the same prestacking migration. It means that without the loss of generality, we can restrict ourselves by consideration of the operator $\sum_s \mathbf{P}^{r}_{w_s}\mathbf{D}^{k}_{(-)}$, having chosen common-source gather as a basic pattern.

In principle, another way to provide depth migration is the routine scheme (post-stack migration):

$$M_{\Pi}(x,z) = \mathbf{P}^{x}_{w_0}\mathbf{D}^{k}_{(-)}\sum_h \cdot NMOu_0(h,x,t)$$

(where $x = \frac{(r+s)}{2}\, h = \frac{(r-s)}{2}$). The operator sequence $\sum_h \cdot NMO$ describes common-midpoint stacking.

Geophysicists know advantages and disadvantages of both schemes. It is known, for example, that post-stack migration is a rather robust procedure with respect to a prior knowledge of velocity model, but it fails when the velocity model is complicated. On the other hand, prestack migration needs good information about the velocity distribution in the subsurface. But once we have this information we can perform correct imaging even in rather complicated situations.

Is it possible to connect this judgment with some quantitative characteristics? Let us denote q as an order of discontinuity in primary wave field $u_0(s,r,t)$ and $q^{'}$ an order of discontinuity after migration. We remind the reader that a discontinuity is a model of reflected wave.

Criteria for comparison:

Smaller values of $q^{'}$ result in higher quality images. At equal values of $q^{'}$, it wins over the others by providing predominance of the discontinuity's amplitude over the intensity of noise. We shall restrict ourselves only to consideration of orders $q^{'}$ for different situations. Earlier we introduced the order of touching of a stacking line and a travel-time curve in the case of a wave-field continuation operator. The same definition is valid for an arbitrary IG-operator. If r is the order of touching, then

$$q^{'} = q + \frac{1}{r+1}. \tag{140}$$

In order to avoid some uncertainty we shall denote the symbol v for each operator whose action depends on the choice of velocity. (This symbol may mean a scalar value or a function $v(\mathbf{r}), v(t_0)$ and so on). In particular, we shall write $NMO(v_{st}), \mathbf{P}^x_{w_0}(v_m)$ and $\mathbf{P}^s_{w_r}(v_m)$ (v_m-migration velocity, v_{st}-stacking velocity) . If $v_m \neq v_{ex}$, then operators $\mathbf{P}^x_{w_0}(v_m)\mathbf{D}^k_{(-)}$ and $\mathbf{P}^s_{w_r}(c_m)\mathbf{D}^k_{(-)}$ do not change the order of discontinuity. So the only reason for the inequality $q^{'} \neq q$ lies in action of operators $\sum_s$ and $\sum_h NMO(v_{st})$.

Prestack migration

If $v_m = v(\mathbf{r})$ (when $v(\mathbf{r})$ is true velocity), then for each s the position of reflectors is correct. It is not difficult to show that in this situation the operator $\sum_s$ supplies the order of touching $r = \infty$ then, with accordance to equation (140),

$$q_I^{'} = q.$$

If $v_m \neq v(\mathbf{r})$ then each reflector's image will be distorted and, in a general case, we shall have $r = 1$, so

$$q \leq q_I \leq q + \frac{1}{2}.$$

Post-stack migration

The operator $\sum_h \cdot NMO(v_{st})$ supplies the order $r = \infty$ only in the situation when mid-point travel-time curve is a hyperbola (as stacking lines). This is possible only for the simplest model of planar reflectors, and for the stacking velocity

$$v_{st} = \frac{v}{\cos\phi}.$$

In this case $q_{\rm II}^{'} = q$.

Let's examine a more complicated media. In any situation there is an optimal value v_{opt} of stacking velocity that supplies the order of touching $r \geq 3$. Let us put

$$v_{st} = \frac{1}{\sqrt{vt_0t^{''}}} \tag{141}$$

where

$$t_0 = t_X(0),\ t^{''} = \frac{d^2t_X}{dh^2},$$

$t_X(h)$ - common-mid-point travel-time function.

One can easily check that condition (141) supplies the order of touching $r = 2$. But both stacking hyperbola or common-mid-point travel-time curve are even functions (with respect to h) so third derivatives of both functions at $h = 0$ are equal to 0. It means that $r = 3$ and

$$q_{II}^{'} = q + \frac{1}{4}$$

We shall give some examples of optimal stacking velocities:

- if $v(x,z)$ is constant and reflector is curved, then

$$v_{opt} = \frac{v}{\cos\phi}$$

where ϕ is a slope of the reflector at the base of the normal ray from the point X.

- If $v(x,z) = v(z)$ and $\phi = 0$, then

$$v_{opt} = v_{rms} = \left(2t_0^{-1} \int_0^H v(z)dz \right)^{\frac{1}{2}}$$

(H is the reflector's depth).

- If $v(x,z) = v(t)$ and a reflector is curved, then

$$v_{opt} = \left(\frac{\int_0^H \frac{v(z)dt}{[1-p^2v^2(z)]^{\frac{3}{2}}}}{\int_0^H \frac{dz}{v(z)\sqrt{1-p^2v^2(z)}}} \right)^{\frac{1}{2}}$$

where $p = v^{-1}(H)\sin\phi$.

In practice, the optimal value of extreme velocity can be achieved as a result of velocity spectra analysis.

So, we have

$$q \leq q_{II}^{'} \leq q + \frac{1}{4}.$$

Accurate optimization of migration velocity for prestack migration is also possible. Let us consider the example of a flat horizontal reflector in vertically inhomogeneous velocity $v = v(z)$. It can be shown that at $v_m = v_{rms}$ the second derivative of the function $z = h(x)$ which describes the shape of the reflector's image equals to 0 (under $x = s$). The function $h(x)$ is symmetrical with respect to the point $x = s$ and it means that again $r = 3$ and $q_I^{'} = q + \frac{1}{4}$.

It is interesting to note that the operator $\sum_s \mathbf{P}_{w_s}^r(v_{rms})\mathbf{D}_{(-)}^k$ gives the good order $q + \frac{1}{4}$ but wrong depth $H^{'} = \frac{1}{2}t_0 v_{rms} \neq H = \frac{1}{2}t_0\bar{v}$ ($\bar{v}$ is an average velocity). On the other hand, the operator $\sum_s \mathbf{P}_{w_s}^r(\bar{v})\mathbf{D}_{(-)}^k$ gives the order $q + \frac{1}{2}$ but the correct depth $H^{'} = H$.

To perform post-stack migration for this model we can use the operator $\mathbf{P}_{w_0}(v)\mathbf{D}_{(-)}^k \sum_h NMO(v_{rms})$ which simultaneously gives $q^{'} = q + \frac{1}{4}$ and $H^{'} = H$.

When there is no symmetry in the medium, optimal migration velocity in prestack migration guarantees only $r = 2$. So in general,

$$q \leq q_I^{'} \leq q + \frac{1}{3}$$

It is necessary to remember that $q'_{II} = q$ is possible only for the simplest model of medium. Prestack migration can be used in more complicated situations, provided there is enough knowledge of the velocity model. But when the velocity model is not well known then post-stack velocity is more robust. It is necessary to add that estimation of optimal stacking velocities needs less computer time then estimation of optimal migration velocities. But, at the same time, optimal stacking velocities enable us to obtain the correct velocity model. So, in complicated situations, it is necessary to use both schemes in a definite order.

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