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A Generalized Phase-Shift Method

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INTRODUCTION

A basic part of seismic migration is downward continuation of surface data into the subsurface. Gazdag (1978) introduced the phase-shift method which yields complete accuracy for laterally uniform structures. Kosloff and Kessler (1987) showed how the phase-shift method can be generalized for an arbitrary velocity structure in the space-frequency domain. The generalized phase-shift method by Kosloff and Kessler is instructive to understand how the phase-shift method works and can be implemented in both space-frequency domain and wavenumber-frequency domain.

THE GENERALIZED PHASE-SHIFT METHOD

The generalized phase-shift method is based on the solution of the temporally transformed acoustic wave equation

$$\frac{\partial^2 P}{\partial z^2} = -\frac{\omega^2}{c^2} P - \frac{\partial^2 P}{\partial x^2}, \tag{1}$$

where $P(x, z, \omega)$ denotes the temporal transform of the pressure field and $c(x, z)$ is velocity field. As suggested by Kosloff and Baysal(1983), it is convenient to recast equation (1) as a set of two first-order coupled equations given by

$$\frac{\partial}{\partial z}\begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{\omega^2}{c^2} - \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}\begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}. \tag{2}$$

The downward continuation in the migration consists of the solution of equation (2) for each frequency at all depths under the initial conditions of the values of P and $\partial P/\partial z$ at the surface $z = 0$ (Kosloff and Baysal, 1983). Equation (2) can be expressed in a more compact form

$$\frac{\partial}{\partial z}\begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix} = [A]\begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}, \tag{3}$$

where $[P, \partial P/\partial z]^T$ denotes a column vector of length $2N_x$ containing first the N_x pressures $P(idx, z, \omega)$ and then the N_x pressure derivatives $\partial/\partial z[P(idx, z, \omega)]$, for $i =$

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$0, ..., N_{x-1}$. As with the ordinary phase-shift method, the solution here is propagated in depth increments. Within each increment z to $z + dz$, the velocity is assumed to be invariant in the vertical direction although it may vary horizontally. The solution of equation (3) can then be written as

$$\begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_{z+dz} = \exp[Adz] \begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_{z} . \tag{4}$$

The solution (4) embodies a phase-shift of the eigenvector coefficients of A. It would therefore seem that a matrix diagonalization would have to be performed before each propagation.

Horizontally Uniform Structures

The $2N_x$ by $2N_x$ matrix A can be partioned according to

$$[A] = \begin{pmatrix} 0 & I_{N_x} \\ A_{21} & 0 \end{pmatrix} \tag{5}$$

where I_{N_x} denotes the N_x by N_x identity matrix and the N_x by N_x submatrix A_{21} is given by

$$[A_{21}] = \begin{pmatrix} -\frac{\omega^2}{c^2} - \frac{\partial^2}{\partial x^2} & 0 & \dots & 0 \\ 0 & -\frac{\omega^2}{c^2} - \frac{\partial^2}{\partial x^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\frac{\omega^2}{c^2} - \frac{\partial^2}{\partial x^2} \end{pmatrix} \tag{6}$$

The horizontal discretization requires an approximation for the derivative term in equation (6). As with the ordinary phase-shift method (Gazdag, 1978), this derivative can be calculated by Fourier method

$$\frac{\partial^2 P}{\partial x^2} = -k_x^2 P \tag{7}$$

Since multiplication in the space domain is equivalent to convolution in the wavenumber domain, the multiplication of the sloth (squared slowness) becomes convolution. For horizontally uniform structures, however, the Fourier transform of the velocity function is a delta function. Therefore, the submatrix A_{21} has the form

$$[A_{21}] = \begin{pmatrix} -\frac{\omega^2}{c^2} + k_x^2 & 0 & \dots & 0 \\ 0 & -\frac{\omega^2}{c^2} + k_x^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\frac{\omega^2}{c^2} + k_x^2 \end{pmatrix} \tag{8}$$

The diagonal elements of this matrix are the eigenvalues of A_{21} and a matrix in which the columns consist of the eigenvectors form an identity matrix. By letting the eigenvalues of submatrix A_{21} as

$$\lambda_i = -\frac{\omega^2}{c^2} + k_{x_i}^2,$$

the eigenvalues of matrix A become $\pm\sqrt{\lambda_i}$, for $i = 1, ..., N_x - 1$. The corresponding eigenvectors are given by

$$\begin{pmatrix} I_0 \\ \sqrt{\lambda_0} I_0 \end{pmatrix}, \begin{pmatrix} I_1 \\ \sqrt{\lambda_1} I_1 \end{pmatrix}, \ldots, \begin{pmatrix} I_{N_x-1} \\ \sqrt{\lambda_{N_x-1}} I_{N_x-1} \end{pmatrix}, \begin{pmatrix} I_0 \\ -\sqrt{\lambda_0} I_0 \end{pmatrix}, \ldots, \begin{pmatrix} I_{N_x-1} \\ -\sqrt{\lambda_{N_x-1}} I_{N_x-1} \end{pmatrix}.$$

where I_i denotes i-th column of identity matrix. With these eigenvectors we define a $2N_x$ by $2N_x$ matrix Q in which the columns consist of the eigenvectors of A. We then have the relation

$$\begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_{z+dz} = \exp[Adz] \begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_z = Qe^{\Lambda dz} Q^{-1} \begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_z \tag{9}$$

where Λ is the diagonal matrix

$$\Lambda = \begin{pmatrix} \sqrt{\lambda_0} & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \sqrt{\lambda_1} & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \sqrt{\lambda_{N_x-1}} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & -\sqrt{\lambda_0} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & -\sqrt{\lambda_{N_x-1}} \end{pmatrix} \tag{10}$$

and

$$Q^{-1} = \frac{1}{2} \begin{pmatrix} I_0^T & \ldots & I_0^T/\sqrt{\lambda_0} \\ I_1^T & \ldots & I_1^T/\sqrt{\lambda_1} \\ \vdots & \vdots & \vdots \\ I_{N_x-1}^T & \ldots & I_{N_x-1}^T/\sqrt{\lambda_{N_x-1}} \\ I_0^T & \ldots & -I_0^T/\sqrt{\lambda_0} \\ \vdots & \vdots & \vdots \\ I_{N_x-1}^T & \ldots & -I_{N_x-1}^T/\sqrt{\lambda_{N_x-1}} \end{pmatrix} \tag{11}$$

When both sides of equation (9) are multiplied by Q^{-1} from the left we obtain

$$Q^{-1} \begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_{z+dz} = \exp[\Lambda dz] Q^{-1} \begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_z \tag{12}$$

The equation (12) is a matrix representation for a set of equations

$$(P_i + \frac{1}{\sqrt{\lambda_i}} \frac{\partial P_i}{\partial z})_{z+dz} = e^{\sqrt{\lambda_i} dz} (P_i + \frac{1}{\sqrt{\lambda_i}} \frac{\partial P_i}{\partial z})_z \tag{13}$$

and

$$(P_i - \frac{1}{\sqrt{\lambda_i}} \frac{\partial P_i}{\partial z})_{z+dz} = e^{-\sqrt{\lambda_i} dz} (P_i - \frac{1}{\sqrt{\lambda_i}} \frac{\partial P_i}{\partial z})_z \tag{14}$$

The solutions of equations (13) and (14) represent upgoing and downgoing waves respectively. When an eigenvalue $\pm\sqrt{\lambda_i}$ is real, the component is propagated by a multiplication by a real exponential, and results in an evanescent component which should be removed for numerical stability. When an eigenvalue $\pm\sqrt{\lambda_i}$ is imaginary, the corresponding coefficient is propagated by phase shift.

Arbitrary velocity structures

When the velocity structure varies in the lateral direction, the Fourier transform of velocity is no longer a delta function and the submatrix A_{21} has the form

$$[A_{21}] = \begin{pmatrix} -\omega^2 s_0 + k_x^2 & -\omega^2 s_1 & \dots & -\omega^2 s_{Nx-1} \\ -\omega^2 s_1 & -\omega^2 s_0 + k_x^2 & \dots & -\omega^2 s_{N_x-2} \\ \vdots & \vdots & \ddots & \vdots \\ -\omega^2 s_{N_x-1} & -\omega^2 s_{N_x-2} & \dots & -\omega^2 s_0 + k_x^2 \end{pmatrix} \tag{15}$$

where

$$s_n(z) = \frac{1}{L}\int_{\frac{-L}{2}}^{\frac{L}{2}} c^{-2}(x,z) e^{-in\Delta kx} dx$$

Another derivation of submatrix A_{21} is possible in space frequency domain (Kosloff and Kessler, 1987). Whether the derivative in equation (6) is calculated by the Fourier method as with the ordinary phase-shift method (Gazdag, 1978), or by finite differences with periodic horizontal boundary conditions, it can be written as the cyclic convolution

$$\frac{\partial^2 P}{\partial x^2}(idx, z, w) = \sum_{j=0}^{N_x} P(jdx, z, w) W_{i-j} \tag{16}$$

where W_i denotes a convolution operator. For example, for second-order finite differences, $W_0 = -\frac{2}{\Delta x^2}$ and $W_{-1} = W_1 = \frac{1}{\Delta x^2}$, and $W_i = 0$ for $|i| > 1$. For Fourier second-derivative operator,

$$W_0 = -(\frac{2\pi}{N_x dx})^2 \frac{L(L+1)}{3}$$

and

$$W_n = \frac{1}{2}(\frac{2\pi}{N_x dx})^2 (-1)^n \frac{\cos(\frac{\pi n}{N_x})}{\sin^2(\frac{\pi n}{N_x})}$$

with $N_x = 2L + 1$(Kosloff and Kessler, 1987). With the spatial discretization and specification of the second-derivative approximation, the elements of the N_x by N_x submatrix A_{21} are given by

$$[A_{21}] = \begin{pmatrix} -\frac{\omega^2}{c_1^2} - W_0 & -W_1 & \dots & -W_{N_{x-1}} \\ -W_1 & -\frac{\omega^2}{c_2^2} - W_0 & \dots & -W_{N_x-2} \\ \vdots & \vdots & \ddots & \vdots \\ -W_{N_{x-1}} & -W_{N_{x-2}} & \dots & -\frac{\omega^2}{c_{N_x}^2} - W_0 \end{pmatrix} \tag{17}$$

where $c_i = c(idx, z)$

Tal-Ezer Method

When the velocity structure varies arbitrarily in the lateral direction, the eigenvalues and eigenvectors of matrix A can no longer be obtained by inspection. It would therefore seem that a matrix diagonalization would have to be performed before each propagation to equation (11). However, recent work by Tal-Ezer(1986) indicates how the calculation of matrix exponential can be done without having to resort to expensive matrix diagonalizations. The solution is based on a Chebychev expansion for the function e^x given by

$$e^x = \sum_{k=0}^{\infty} C_k J_k(R) T_k\left(\frac{x}{R}\right), |x| < R \tag{18}$$

where x is imaginary and $C_0 = 1$ and $C_k = 2$ for $k > 0$. The polynomials $T_k(x)$ satisfy the recurrence relations

$$T_0(x) = 1,$$

$$T_1(x) = x,$$

and

$$T_{k+1}(x) = T_{k-1}(x) + 2xT_k(x)$$

(Tal-Ezer, 1986). By analogy with equation (18), the exponent in equation (4) is expanded according to

$$\begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_{z+dz} = \exp[Adz] \begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_z = \sum_{k=0}^{\infty} C_k J_k(R) T_k(\frac{Adz}{R}) \begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_z \tag{19}$$

This expansion is valid when the eigenvalues of $[Adz]$ are purely imaginary and R must be chosen large enough to span the range of the eigenvalues of $[Adz]$. It was shown in Tal-Ezer(1986) that for $k > R$, the series expansion converges exponentially. The number of terms required in the sum in equation (19) will therefore always be finite. Equation (19) serves as the basis for implementing the generalized phase-shift migration. First the range R of the eigenvalues of $[Adz]$ need to be estimated. Based on the case of laterally uniform velocity, Kosloff showed the estimate $r = wdz/c_{min}$ with c_{min} denoting the lowest velocity in the strip $(z, z + dz)$, is sufficient for stable results. The Bessel functions $J_k(R)$ are computed next. The solution

$$\begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_{z+dz}$$

is then calculated recursively according to following: The first two values of T_0, T_1 needed to initialize the recursion are given by

$$T_0 \begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_z = \begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_z$$

and

$$T_1 \begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_z = [\frac{Adz}{R}] \begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_z$$

The next polynomial is generated by the recursion formula

$$T_{k+1}\begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_z = T_{k-1}\begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_z + 2[\frac{Adz}{R}]T_k\begin{pmatrix} P \\ \frac{\partial P}{\partial z} \end{pmatrix}_z$$

These steps are repeated until a sufficient number of terms have been calculated in the sum(19). Then the solution is carried out in the next lower level. When $P(x,z,\omega)$ has been calculated completely, the final migrated section is cumulated by

$$P_{mig}(x,z) = \sum_{\omega} P(x,z,\omega)$$

(Kosloff and Baysal, 1983).

GENERATION OF SURFACE VALUES OF P AND $\partial P/\partial Z$

The initiation of migration based on equation (4) requires the specification of both P and $\partial P/\partial z$ on the surface. Since only one of these fields is recorded in practice, the remaining field must be generated from mathematical assumptions. In a region in which the acoustic velocity c is constant, the acoustic wave equation can be doubly transformed in x and t to give

$$\frac{\partial^2 P}{\partial z^2} = -(\frac{\omega^2}{c^2} - k_x^2)P(k_x,z,\omega) \tag{20}$$

where P is the two dimensionally transformed pressure and k_x is the horizontal wavenumber. The solution to equation (20) are given by

$$P(k_x,z,\omega) = \exp\left(i\sqrt{\frac{\omega^2}{c^2} - k_x^2}\right) P(k_x,z,\omega) \tag{21}$$

the solution (21) includes only upgoing waves under the convention that z increases with depth. The doubly transformed pressure gradients can be obtained from equation (21) by differentiation

$$\frac{\partial P}{\partial z} = i\sqrt{\frac{\omega^2}{c^2} - k_x^2}\exp\left(i\sqrt{\frac{\omega^2}{c^2} - k_x^2}\right) P(k_x,0,\omega) = i\sqrt{\frac{\omega^2}{c^2} - k_x^2}P(k_x,z,\omega) \tag{22}$$

The generated vertical pressure gradient on the surface are obtained from equation (22) by setting $z = 0$ and by an inverse Fourier transformation with respect to x.

ELIMINATION OF EVANESCENT ENERGY

In the ideal solution, evanescent components corresponding to the real eigenvalues would be eliminated, whereas components corresponding to the imaginary eigenvalues would be unaltered. Unfortunately, in the general case this requires the separation of evanescent components through a costly matrix diagonalization. A method for eliminating the evanescent components without the matrix diagonalization and without severely affecting the nonevanescent components is therefore required. A scheme proposed in Kosloff and Kessler (1987) included use of a spatially variant, high-cut filter to P and $\partial P/\partial z$ after each step of the propagation of the solution. The high-cut wavenumber k_{cut} at each point of application was based on the criterion $k_{cut} = \omega / c_{max}$, where c_{max} denotes the highest velocity in a taper region surrounding the point of application.

REFERENCES

Gazdag, J., 1978, Wave equation migration with the phase-shift method: Geophysics, **43**, 1342-1351.

Kosloff, D., and Baysal, E., 1983, Migration with the full acoustic wave equation: Geophysics, **48**, 677-678.

Kosloff, D., and Kessler, D., 1987, Accurate depth migration by a generalized phase-shift method: Geophysics, **52**, 1074-1084.

Tal-Ezer, H., 1986, Spectral methods in time for hyperbolic problem: Soc. Industr. Appl. Math., J. Numer. Anal., **23**, 11-20.