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A general theory for equivalent media

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ABSTRACT

Conservation of deformation energy, volume and mass lead to a general formulation for finding the static homogeneous equivalent of an arbitrary heterogeneous anisotropic medium. The homogeneous equivalent has identical integral properties and on a large scale, shows the same average behavior in displacements and stresses. The medium is not restricted to a special geometry.

INTRODUCTION

Equivalent Medium theory tries to predict the gross behavior of a heterogeneous medium. While there exist theories for calculating equivalent homogeneous counterparts for special geometries, most of them consider isotropic solids or isotropic solid-fluid media. Recently Schoenberg and Muir (1989) introduced an elegant averaging scheme for anisotropic horizontal layers. That scheme demands layers with thicknesses much smaller than the length of the elastic wave which is propagating through it. It is a static approach, but results show that in the long-wavelength limit reasonable results can still be obtained. A more general averaging scheme would have two objectives: allow arbitrary geometry and lift the static assumption. In this paper I show a way to describe a heterogeneous anisotropic medium and relate it to an equivalent homogeneous counterpart by requiring integral equivalence. The way of defining equivalence is of uttermost importance. In the following I assume equivalence as outlined for the Schoenberg & Muir scheme by Nichols and Karrenbach (1990). Generalizing that notion, a heterogeneous medium and its homogeneous counterpart exhibit the same mass, the same changes in volume and the same deformation energy. If a stress is applied to the exterior surface of a heterogeneous body, the homogeneous equivalent shows identical change in deformation energy within and the same average displacements and forces on its boundaries.

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EQUIVALENT QUANTITIES

Three basic conservation laws govern the behavior of an inhomogeneous anisotropic medium and its equivalent homogeneous counterpart. An arbitrary region V in three dimensions is shown in Figure **??**. The region is bounded by surface S. The quantities we want to preserve in our equivalent medium are: total energy change, total volume change and total mass.

The volume, shown in Figure **??**, changes its shape in response to the application of some external force on its exterior surface. The total energy during this process is the sum of the elastically stored energy in volume V, the total surface energy on boundary S, the kinetic energy and the energy provided by sources within the medium:

$$E_{\text{tot}} = \oint_V \epsilon(r) : \sigma(r) dV + \oint_S -u(r) \cdot F(r) dS + E_{\text{kin}} + E_{\text{source}} \tag{1}$$

where u the displacement vector and F is the vector of force applied to the exterior surface S. The contribution of energy in response to a force acting on the exterior of the medium is calculated by the dot product of displacement and force integrated over the whole exterior surface. The material has a stiffness that may depend on spatial variables. The strain $\epsilon(r)$ in the medium is proportional to the stress $\sigma(r)$ present. So part of the energy is elastically stored in the medium. It can be calculated by taking the double dot product of stress and strain and integrating it over the whole volume. In the static case we neglect kinetic energy E_{kin}, which is equivalent to waiting until the medium reaches a state of equilibrium. Omitting the E_{source} term implies having no elastic sources within the medium. The total volume change δV in response to an application of an external force is

$$\delta V = \oint_V d\delta V \tag{2}$$

The total mass m of the medium after deformation is

$$m = \oint_V \rho \, dV \tag{3}$$

The equivalent homogeneous medium should have the following desired properties. There should be no change in the total mass. The total volume change should be identical to the homogeneous medium. When external forces are applied, the change in the sum of surface and deformation energy should be the same for the homogeneous and heterogeneous medium. On interior boundaries we want *welded* contacts; internal interfaces are not allowed to slip. This specification is important, since it governs the later choice of boundary decomposition.

INTERIOR BOUNDARY CONDITIONS

The medium can be totally heterogeneous internally on any scale. The shapes of these heterogeneities define material discontinuity surfaces on which we specify interior boundary conditions. On each point on these interior boundaries, the normal stress components σ_N and the tangential strain components ϵ_T are continuous. We can define a local transformation of stress and strain and stiffness matrix at such an interior boundary into normal and tangential components. In order to get this decomposition for any arbitrary surface, we rotate the coordinate system in order to align the local surface normal with the z axis of the coordinate frame. The determination of the tangential stress and strain components is then identical to the one given in Schoenberg & Muir (1989) for the layered case.

Using this local mapping we can cast the constitutive relation into a form that indicates the boundary conditions in effect. Thereby we create the hybrid matrix (consisting of stiffnesses and compliances) $\mathbf{X}$. The left hand side consists of quantities $(\sigma_T,\ \epsilon_N)$ which are not preserved when crossing the interior boundary. These quantities however are a linear combination of quantities that are preserved $(\sigma_N,\ \epsilon_T)$. We end up with the constitutive relation

$$\begin{pmatrix} \sigma_T \\ \epsilon_N \end{pmatrix} = \begin{pmatrix} \mathbf{X}_{TT} & \mathbf{X}_{TN} \\ \mathbf{X}_{NT} & \mathbf{X}_{NN} \end{pmatrix} \begin{pmatrix} \epsilon_T \\ \sigma_N \end{pmatrix} \tag{4}$$

The quantity $< \mathbf{X} >$ will be a function of the position on the discontinuity surface and thus implicitly depend on the normal vector at that point. Thus the mapping of the stiffness at those points into the group matrix $\mathbf{X}$ as described before, will be a function of spatial coordinates.

The mapping is not defined when the normal onto the surface is not defined. This requires that the discontinuity surfaces cannot have points with infinite curvature. Thus the discontinuity surface has itself to be continuous on the scale where we carry out the mapping and summation. With these restrictions in mind, the *static* total energy H in the medium is given by

$$H = \oint_V \begin{pmatrix} \epsilon_T & \sigma_N \end{pmatrix} \begin{pmatrix} \mathbf{X}_{TT} & \mathbf{X}_{TN} \\ \mathbf{X}_{NT} & \mathbf{X}_{NN} \end{pmatrix} \begin{pmatrix} \epsilon_T \\ \sigma_N \end{pmatrix} dV + \oint_S -u \cdot F dS \tag{5}$$

EVALUATION OF VOLUME INTEGRALS

Above we have formally defined the equations which govern the averaging process (2,3 and 5). Instead of calculating the volume integral, the problem can be transformed into an evaluation of surface integrals. Greens Theorem is valid for tensors of any order. In its most general form it is written as

$$\oint_V \nabla \cdot A \ dV = \oint_S A \cdot n dS \tag{6}$$

A can be a vector, then $\nabla \cdot$ is the divergence operator and the theorem is termed "Gauss' Theorem"; or A can be a tensor, with the appropriate definition for $\nabla \cdot$.

Field discontinuities

If a volume contains an arbitrary surface Σ across which A is discontinuous, then application of Greens Theorem to the subvolumes, Figure **??** turns a volume integral in a sum of surface integrals.

$$\oint_V \nabla \cdot A \; dV = \oint_S A \cdot n \; dS + \oint_\Sigma (A_1 \cdot n_1 + A_2 \cdot n_2) dS \tag{7}$$

n_1 and n_2 are the unit normals when approaching the discontinuity surface either from side 1 or the side 2. A_1 and A_2 are forces on opposite sides of the surface. The first term is the integral over the exterior surface, while the second term accounts for effects created by interior discontinuity surfaces. Applying the Greens Theorem (6) to the energy integral 5, which we want to calculate, we get:

$$\oint_V \sigma : \epsilon \; dV = - \oint_{S_{tot}} u \cdot \sigma \cdot n \; dS + \oint_V u \cdot (\nabla \cdot \sigma) \; dV \tag{8}$$

where $S_{\text{tot}} = S + \Sigma$ is the sum of exterior and interior surfaces. In the static case the divergence of the stress $\nabla \cdot \sigma$ at any point in the medium is zero. If it were not zero we would have a net force acting on some part of the medium; then the medium would not be in equilibrium and would accelerate according to Newton's law ($\nabla \cdot \sigma = \frac{\partial^2 u}{\partial t^2} - F_{\text{source}}$). This static argument is equivalent to saying: after an external force is applied to the medium, we wait for a long time until the medium is in equilibrium (or there is no energy any more in form of elastic waves traveling though the medium).

Energy at a discontinuity surface

As before, we are free to rewrite the above integral relationships in a form which incorporates the boundary conditions occurring at a material discontinuity surface:

$$H = \oint_S -u \cdot F ds + \oint_\Sigma \{(u_N \cdot \sigma_N)_1 + (u_N \cdot \sigma_N)_2\} ds + \oint_\Sigma \{(u_T \cdot \sigma_T)_1 + (u_T \cdot \sigma_T)_2\} ds \tag{9}$$

$\oint_\Sigma$ stands for an integral over all points in the medium which exhibit material and thus stress and strain discontinuities. u_N and u_T are displacements at the discontinuity interface, generated by the normal and tangential strains in effect on an infinitesimal surface element. For convenience we look at a small piece of such a discontinuity surface. The symbol $< \quad >$ means the addition of the two opposite surface effects. At a small surface region Δs, we obtain then

$$h = < u_N \cdot \sigma_N > \Delta s + < u_T \cdot \sigma_T > \Delta s \tag{10}$$

Since the boundary conditions on welded interfaces require that σ_N and u (ϵ_T) are continuous we can write for the change in surface energy:

$$dh = \sigma_N \cdot < \epsilon_N > \; \Delta s + \epsilon_T \cdot < \sigma_T > \; \Delta s \tag{11}$$

We note that only the discontinuous components are in $< \quad >$. Expressing these discontinuous components in terms of continuous ones using the constitutive equation

$$dh = \sigma_N < \mathbf{X}_{NT} \cdot \epsilon_T + \mathbf{X}_{NN} \cdot \sigma_N > \ \Delta s + \epsilon_T < \mathbf{X}_{TT} \cdot \epsilon_T + \mathbf{X}_{TN} \cdot \sigma_N > \ \Delta s \tag{12}$$

Separating out the continuous quantities and rewriting it as a matrix equation gives:

$$dh = \begin{pmatrix} \epsilon_T & \sigma_N \end{pmatrix} < \begin{pmatrix} \mathbf{X}_{TT} & \mathbf{X}_{TN} \\ \mathbf{X}_{NT} & \mathbf{X}_{NN} \end{pmatrix} > \begin{pmatrix} \epsilon_T \\ \sigma_N \end{pmatrix} \ dl \ \Delta s \tag{13}$$

This takes care of **one** boundary point. Choosing an infinitesimal boundary region ds we can integrate the above expression over all discontinuity surfaces within the volume

$$dH = \oint_\Sigma \begin{pmatrix} \epsilon_T & \sigma_N \end{pmatrix} < \begin{pmatrix} \mathbf{X}_{TT} & \mathbf{X}_{TN} \\ \mathbf{X}_{NT} & \mathbf{X}_{NN} \end{pmatrix} > \begin{pmatrix} \epsilon_T \\ \sigma_N \end{pmatrix} \ ds \tag{14}$$

Consequently we obtain the elastic energy stored by deforming the medium. Note that this expression is an exact formula for calculating the energy; however all the quantities including the stresses and strains at each point inside the medium must be known. If we are willing to sacrifice exactness, we can approximate H by assuming an average stress and strain in the medium. Then the composite medium property depends only on properties of its composites by:

$$H = \begin{pmatrix} \epsilon_T & \sigma_N \end{pmatrix} \oint_\Sigma < \begin{pmatrix} \mathbf{X}_{TT} & \mathbf{X}_{TN} \\ \mathbf{X}_{NT} & \mathbf{X}_{NN} \end{pmatrix} > \ dl \ ds \begin{pmatrix} \epsilon_T \\ \sigma_N \end{pmatrix} \tag{15}$$

It is remarkable that in the layered geometry (14) and (15) are identical and thus exact. For an arbitrary heterogeneous medium the scale of heterogeneity determines the accuracy of approximation (15) to (14). The stress and strain field in the medium is not distorted much by small scale heterogeneities. Thus on an average much larger than the scale of the heterogeneities the stress and strain field is assumed to be constant, so that we can apply relation (15).

THE EQUIVALENT HOMOGENEOUS MEDIUM

We can define then an average medium, which has the same integral properties as the original heterogeneous medium:

$$\begin{aligned} V^{equiv} &= \int_V dV \\ \rho^{equiv} \ V^{equiv} &= \int_V \rho(r) \ dV \end{aligned}$$

$$\begin{pmatrix} {\epsilon_T}^{equiv} & {\sigma_N}^{equiv} \end{pmatrix} \begin{pmatrix} {\mathbf{X}_{TT}}^{equiv} & {\mathbf{X}_{TN}}^{equiv} \\ {\mathbf{X}_{NT}}^{equiv} & {\mathbf{X}_{NN}}^{equiv} \end{pmatrix} \begin{pmatrix} {\epsilon_T}^{equiv} \\ {\sigma_N}^{equiv} \end{pmatrix} \oint_V dV =$$

$$\begin{pmatrix} \epsilon_T & \sigma_N \end{pmatrix} \oint_V < \begin{pmatrix} \mathbf{X}(r)_{TT} & \mathbf{X}(r)_{TN} \\ \mathbf{X}(r)_{NT} & \mathbf{X}(r)_{NN} \end{pmatrix} > \begin{pmatrix} \epsilon_T \\ \sigma_N \end{pmatrix} \ dV \tag{16}$$

For the homogeneous equivalent we know that it is energetically equivalent but in the interior stress distribution different from the heterogeneous medium. They share the same integral properties, and the stress and strain will be on the average equivalent. Taking the heterogeneous medium and applying a static force to its exterior boundaries results in a static stress and strain distribution within the body. This means stress and strain are functions of the spatial coordinates only. The stress and strain values are uniquely determined by the stiffness and by the external forces applied to the exterior surface. If we apply the same external forces to the homogeneous medium we will in general not end up with the same displacements as observed in the heterogeneous case, but on the average over a region much larger than the scale of the heterogeneity we will have the same result.

CONCLUSIONS

Conservation of deformation energy, volume and mass lead to a general formulation for finding the static homogeneous equivalent of an arbitrary heterogeneous medium. While applying exterior surface forces and displacements the stress and stain components at points on interior discontinuity surfaces have to be continuous according to the specified boundary conditions. For a homogeneous equivalent medium forces and displacements on the exterior boundaries will be averages of the ones observed on the heterogenous medium. The heterogeneous and homogeneous equivalent have identical integral properties and on a large scale, show the same average behavior in displacements and stresses.

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