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Applying finite-difference method to ray tracing

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ABSTRACT

For a given slowness model, the traveltime field can be calculated, on a regular grid, with finite-difference methods. The contour lines of this field define the trajectories of wavefronts. From the orthogonal relations between wavefronts and rays, one can calculate another field whose contour lines define the trajectories of rays. Two-point ray tracing can then be accomplished by finding the contour line of a given value. The method is limited to rays traveling in one direction and associated with first arrivals. The algorithm can efficiently trace rays to destinations on a regular grid, and thus can be applied to tomographic inversion and time-to-depth conversion of migrated images

INTRODUCTION

Ray tracing plays an important role in many methods in seismic data processing. Conventional ray tracing method is based on the ray tracing system derived from the eikonal equation by the method of characteristics (Nolet, 1987; Červený, 1987). The result of this method gives the coordinates of a ray as a function of the arclength along the ray. The destination of the ray can not be controlled.

Some methods of seismic tomography, migration and velocity-inversion require tracing rays for given pairs of points. This is much more difficult than tracing rays for given initial conditions. Langan et al. (1985) solve this problem by using an iterative method. In each iteration, the initial conditions are updated according to the difference between the given destination and the actual destination of the last ray tracing. Obviously, this method is inefficient. Van Trier (1988) presents a ray tracing method that is vectorizable. He uses this method to trace many rays at once, and then interpolates the result to approximate two-point ray tracing. For complicated velocity models, this method may fail to trace rays to shadow zones.

Recently, several finite-difference schemes are developed to calculate the traveltimes for an arbitrary velocity model. Among them, Vidale's method (Vidale, 1988)

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and Van Trier's method (Van Trier, 1989) draw most attention. Because these methods give the traveltime field on a regular grid, one can plot the wavefronts by contouring this field. It is known that rays and wavefronts are orthogonal. Can one calculate another field whose contours define the trajectories of rays? The answer is yes.

From the orthogonal relation between the gradient directions of rays and wavefronts, I derived an partial differential equation. The solution of this equation is a function that has constant values along each ray. Therefore, two-point ray tracing can be efficiently done by tracing the contour lines of the function. This method handles rays traveling in one direction and corresponding to first arrivals. These properties may be disadvantages of the method in some applications. However, for others, such as seismic tomography and time-to-depth conversion of migrated sections, they are exactly what are needed.

In this paper, I first derive the orthogonal equation in the general case. Then I describe, in detail, how two-point ray tracing is done for point-source and line-source. Finally, I show two examples of the applications of the method.

ORTHOGONAL RELATION

The eikonal equation in two dimensions relates the two gradient components of the traveltime field:

$$\tau_x^2 + \tau_z^2 = s^2, \tag{1}$$

where s is a two-dimensional slowness model and τ is the traveltime field. Subscripts x and z denote partial derivatives with respect to x and z, respectively. Using a finite-difference method, one can solve this equation for τ. The contour lines of τ define the trajectories of wavefronts. It is well known that, in an isotropic medium, the trajectories of rays are always perpendicular to the trajectories of wavefronts and that the gradient directions of a field are always perpendicular to the contour lines of the field. Therefore, the gradient of τ is orthogonal to the gradient of function p whose contour lines are the trajectories of rays:

$$\mathrm{grad}[\tau] \cdot \mathrm{grad}[p] = 0.$$

If we assume that the traveltime field τ is known, this equation leads to an linear, first-order partial differential equation for p

$$\tau_x p_x + \tau_z p_z = 0. \tag{2}$$

In the following sections, I will explain how to solve this equation for two specific cases.

POINT SOURCE

In two dimensions, a point source initiates waves with circular wavefronts. In this case, the partial differential equation (2) should be solved in the polar coordinates.

Orthogonal equation

Let us assume the location of the point source is x_0 on the surface. With the coordinate transformation:

$$\left\{ \begin{array}{lcl} x & = & x_0 + r\sin\theta \\ z & = & r\cos\theta, \end{array} \right. \tag{3}$$

we can derive the orthogonal equation in the polar coordinates:

$$\tau_r p_r + \frac{1}{r^2}\tau_\theta p_\theta = 0, \tag{4}$$

where $\tau = \tau(r, \theta, x_0)$ and $p = p(r, \theta, x_0)$. The function p has a physical meaning of surface horizontal slowness:

$$p = \frac{\partial \tau}{\partial x_0} = -s_0 \sin\theta_0, \tag{5}$$

where s_0 is the slowness at the source location, θ_0 is the angle between incident ray at the surface and the vertical. Along each ray, the surface horizontal slowness p maintains to be constant. Therefore, we can trace a ray by following the trajectory of a contour line of p.

Initial and boundary conditions

Assuming the slowness around the source is constant s_0, we can find following initial conditions:

$$\left\{ \begin{array}{lcl} p(0, \theta, x_0) & = & -s_0 \sin\theta \\ p_r(0, \theta, x_0) & = & 0 \\ p_\theta(0, \theta, x_0) & = & -s_0 \cos\theta. \end{array} \right. \tag{6}$$

The boundary conditions on two sides of the model are simple. One can assume outgoing rays and use one-sided finite-differences.

Example

I choose Van Trier's method to solve the eikonal equation in the polar coordinates because this method gives the traveltime derivatives that appear in the orthogonal equation. If Vidale's method is used, these derivatives have to be numerically computed from the traveltime field. Once the traveltime derivatives are known, equation (4) becomes a first-order, linear partial-differential equation and is readily solved using the standard finite-difference techniques. After p has been computed, the coordinates of the ray that connects the source location to any given point can be found by tracing the contour line starting from the given point. It is guaranteed that all contour lines of field p will end at the source location.

I test the method on a synthetic slowness model. The results are shown in Figure **??**. The raster images of the synthetic slowness model are overlain with the wavefronts and rays. The wavefronts are plotted at a constant time-interval. The rays are traced to the destinations distributed uniformly along the boundaries of the model. The ray-bending effects are clearly seen.

LINE SOURCE

A line source initiates plane waves. The incident angle of the plane waves is defined to be the angle between the direction of wave propagation and the vertical. We can rotate the coordinates of the slowness model so that the plane waves travel along one of the axes in the new coordinates.

Orthogonal equation

Let us suppose (x, z) are the coordinates before the rotation and (x', z') after the rotation. One can find the relations between these two coordinates:

$$\left\{ \begin{array}{lcl} x & = & x' \cos\theta_0 - z' \sin\theta_0 \\ z & = & z' \cos\theta_0 + x' \sin\theta_0, \end{array} \right. \tag{7}$$

where θ_0 is the incident angle of the plane waves. It is easy to show that both the eikonal equation and the orthogonal equation are invariant under the transformation defined in equation (7). Therefore, we can use equations (1) and (2) after replacing (x, z) by (x', z'). The new definition of the function p is as follows:

$$p(x, z) = \frac{\partial \tau}{\partial \theta} = -s_0 x_0, \tag{8}$$

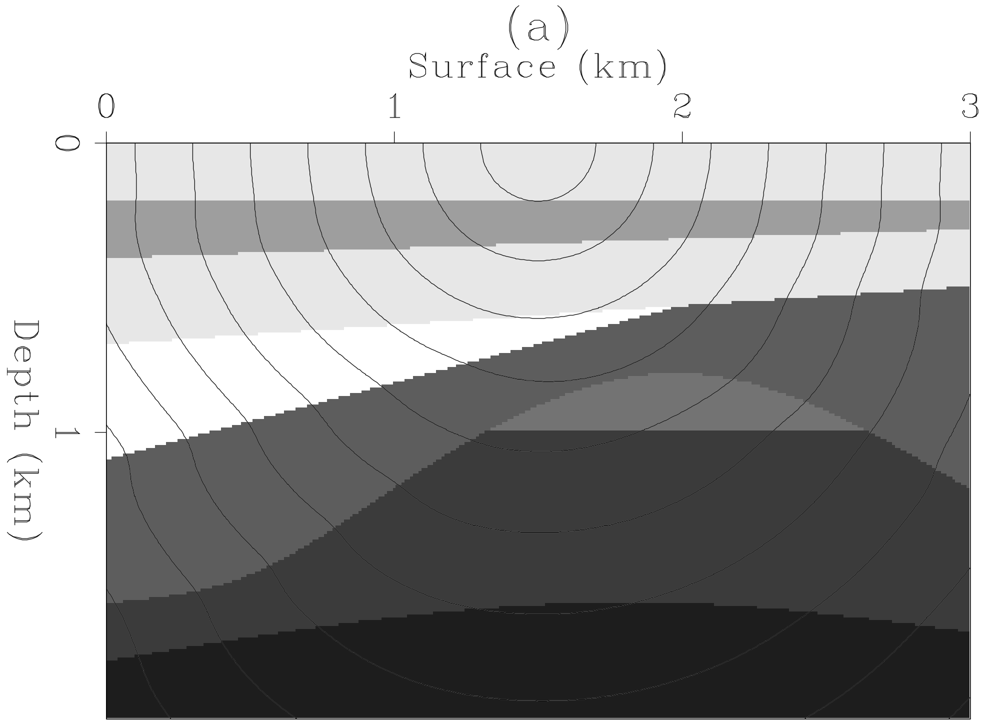
where s_0 is the slowness at the surface, and x_0 is the surface location of a incident ray. I do not know the meaning of p defined here. I guess it is a horizontal traveltime. Along each ray, p maintains to be constant. Therefore, we can trace a ray by following the trajectory of a contour line of p.

Initial and boundary condition

Assuming the slowness along the surface is constant s_0, we can find following initial conditions:

$$\left\{ \begin{array}{lcl} p(x, z) & = & -s_0 x \\ p_x(x, z) & = & -s_0 \\ p_z(x, z) & = & 0. \end{array} \right. \tag{9}$$

The boundary conditions are similar to those described in the last section.



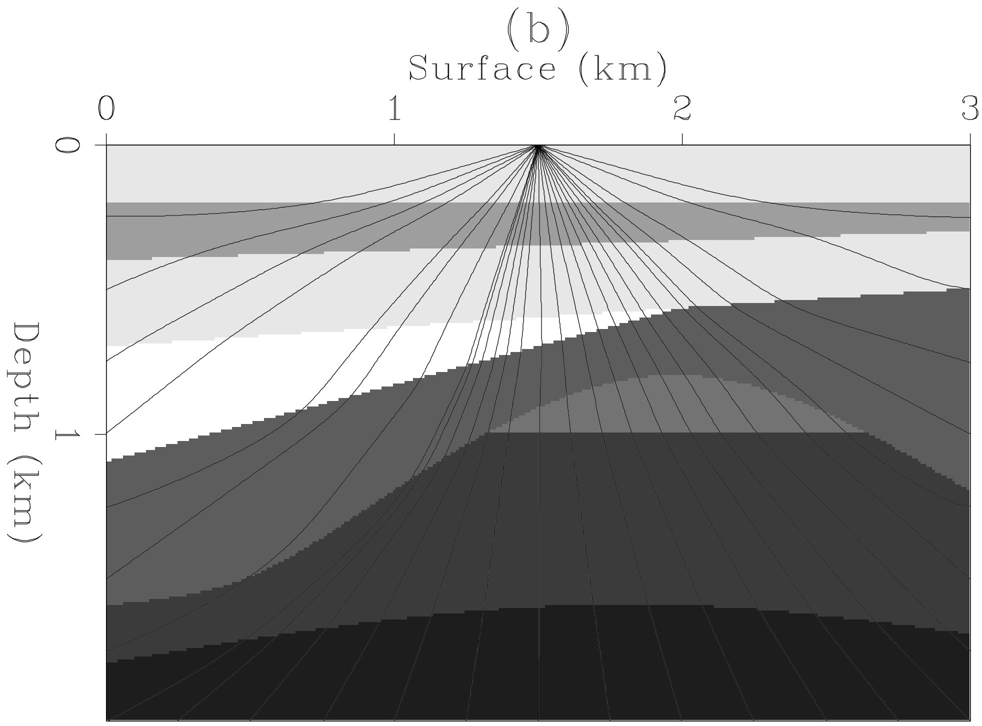


Figure 1: Ray tracing for a point source: (a) wavefronts; (b) rays. The background shows the slowness model. High intensities denote high velocities or low slowness. lin3-point [NR]

Example

In this example, I assume a line source generates plane waves with a incident angle of 30 degrees. I use the algorithm to trace the rays in a synthetic model. The results are displayed in Figure **??**. Again, the raster images of the synthetic slowness model are overlain with the wavefronts and rays. The wavefronts are plotted at a constant time-interval. The rays are traced with the starting points distributed uniformly along the surface. The incident angles of the rays are equal to 30 degrees.

APPLICATIONS

This section shows two applications of the fast two-point ray tracing algorithm.

Seismic tomography

Cross-well tomography uses the traveltimes of first arrivals to estimate the velocity model between two wells. If the velocity anomalies are large, ray-bending effects must be considered. In this case, one needs to find the trajectories of rays between each pair of sources and receivers. In Figure **??**, I simulate this situation by tracing rays through a smooth slowness model. The rays start from three sources on the left and end to the uniformly spaced receivers on the right. The velocity variations cause the rays to bend accordingly.

Time to depth conversion

One can use image-ray corrections to convert a migrated time-section to a depth-section (Larner et al., 1981). The image-rays are rays that have vertical incident angles at the surface. Usually, the trajectories of these rays are calculated by ray tracing. The conventional one-point ray tracing method can set the starting points of the rays. However, it can not control the destinations of the image-rays. Therefore, the converted depth-image has to be interpolated. For complicated velocity models, rays may not penetrate shadow zones; the interpolation is thus cumbersome. Using the two-point ray tracing described in this paper, one can trace image-rays to any imaging points. The interpolation is done in the input time-section that is uniformly sampled. In Figure **??**, I show the image-rays calculated with the fast two-point ray tracing algorithm. The destinations of these rays are uniformly distributed along the bottom of the slowness model.

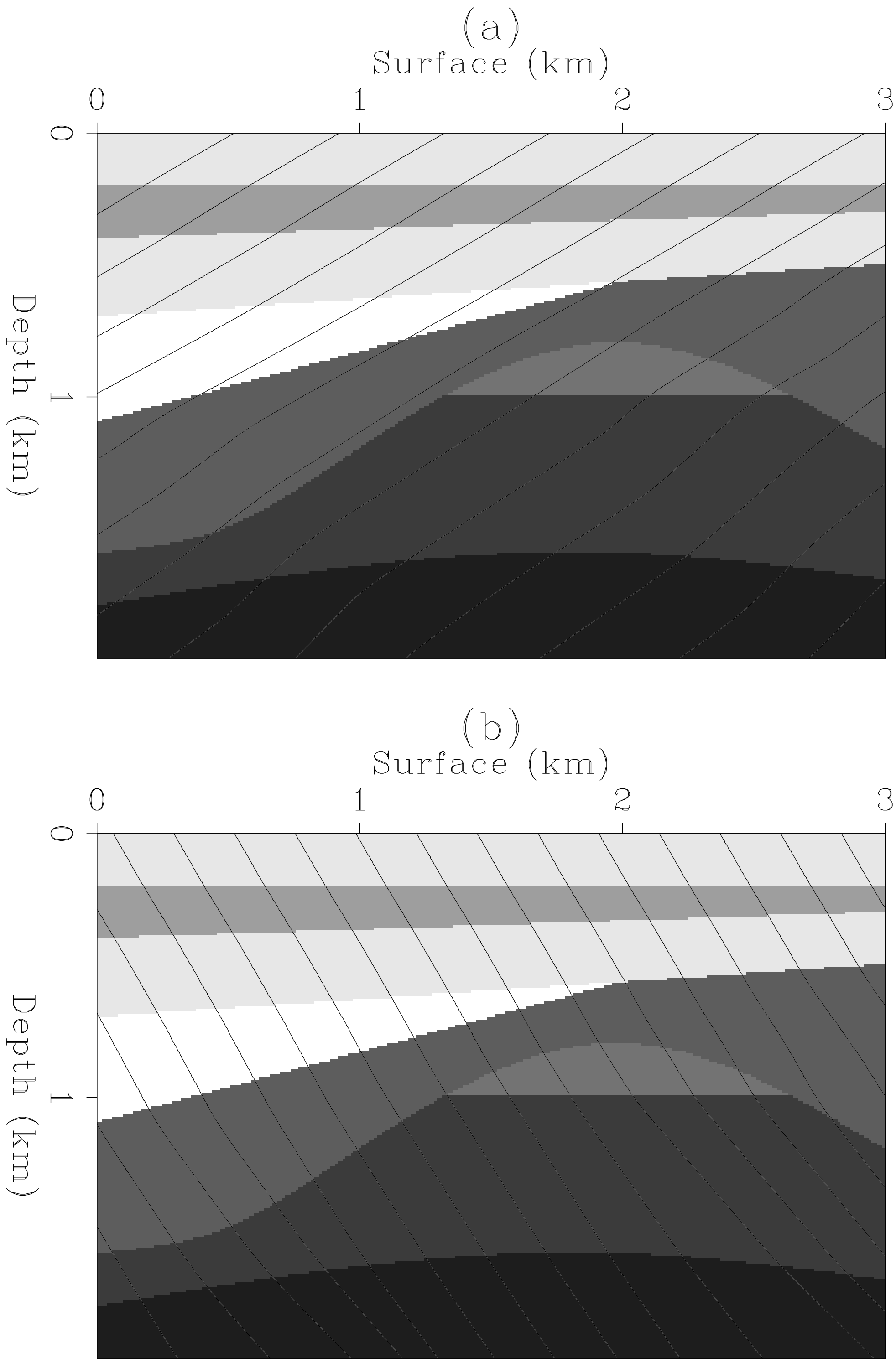


Figure 2: Ray tracing for a line source: (a) wavefronts; (b) rays. The background shows the slowness model. High intensities denote high velocities or low slowness. lin3-line [NR]

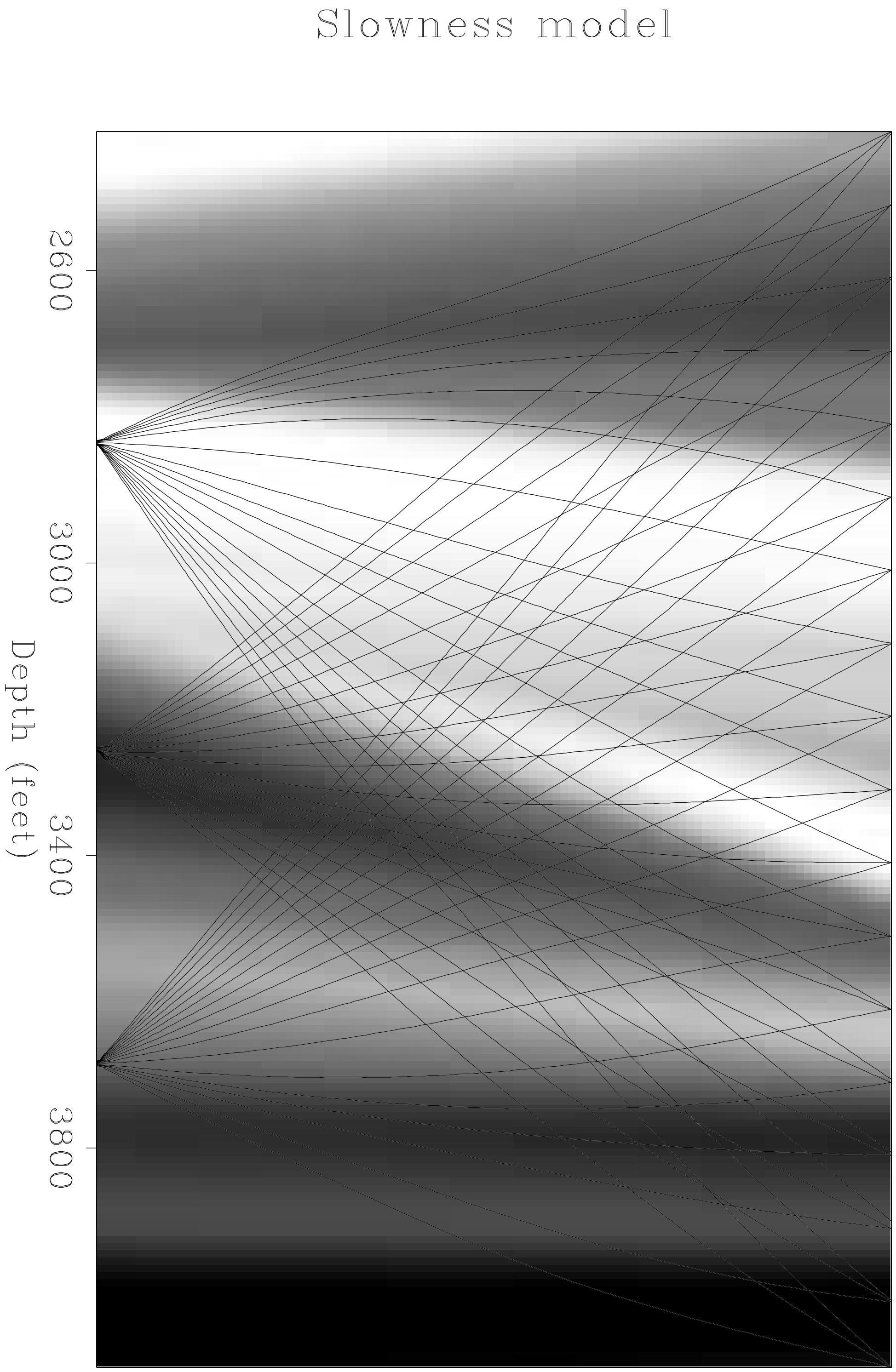


Figure 3: Ray tracing from three source on the left to the uniformly spaced receivers on the right. The background shows the slowness model. High intensities denote high velocities or low slowness. lin3-tomotrc [NR]

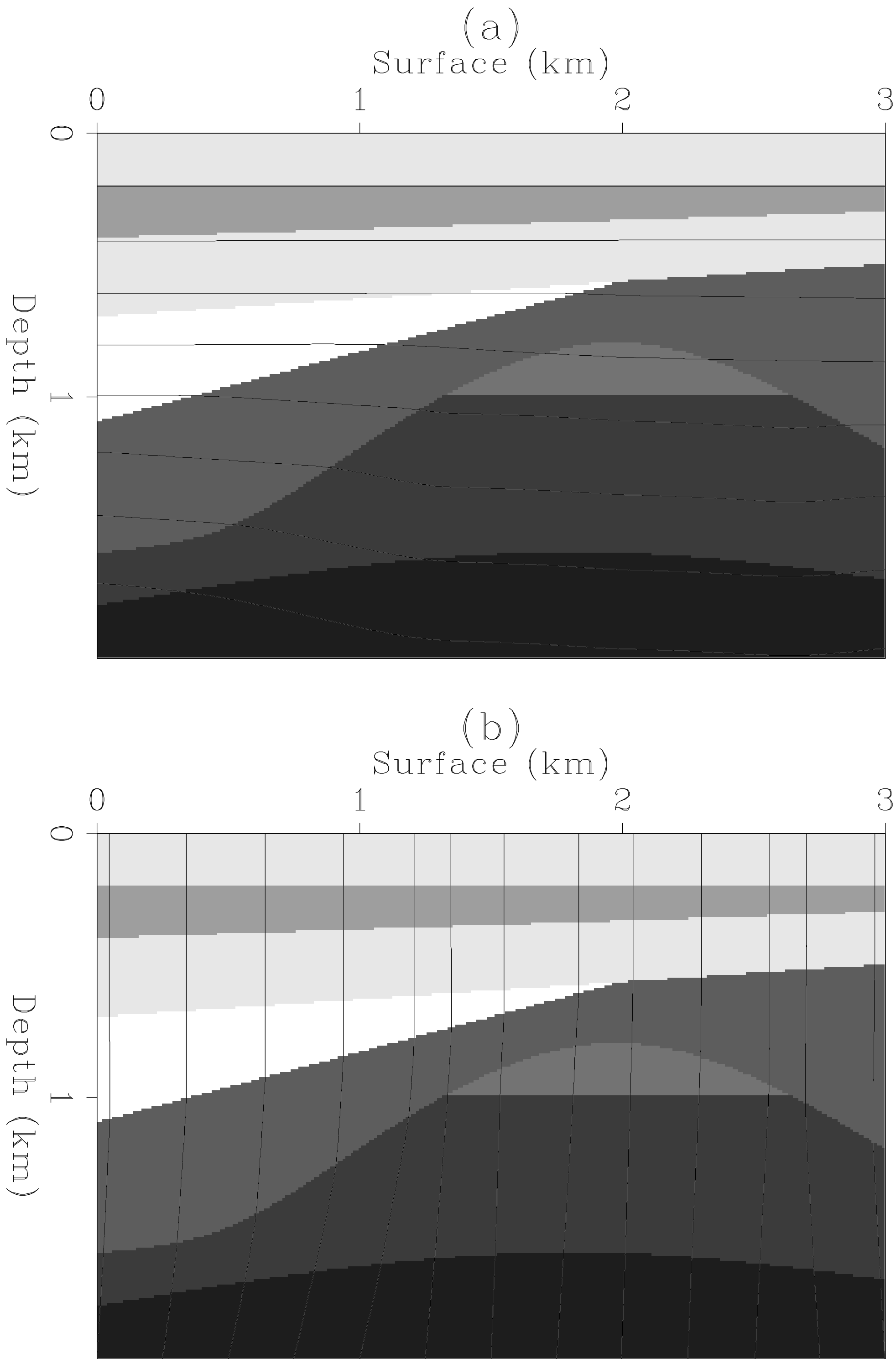


Figure 4: Image-rays with uniformly spaced destinations. The background shows the slowness model. High intensities denote high velocities or low slowness. lin3-ttod [NR]

CONCLUSIONS

I have presented an efficient algorithm for two-point ray tracing. The method can trace rays to destinations on a regular grid. The computations are vectorable. the method can be applied to the traveltime tomography and time-to-depth conversion of migrated time-sections.

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