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Absorbing boundary conditions for wave equation calculation

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INTRODUCTION

Because of the limitations of our seismic survey profile and computer core size, we can only solve the infinite domain wave equation on a finite domain. The computational edges we introduce generate artificial reflections. These reflections, when propagated inward, mask the true solution of the problem. So we must establish absorbing boundary conditions through which the waves propagate with as little reflection as possible.

The subject of absorbing boundary conditions has been extensively studied by many authors, e. g. , Clayton et al. (1977), Reynolds (1978) for modeling, and Clayton et al. (1980) for migration. In this paper we present a method to derive absorbing boundary conditions by directly evaluating the edge reflection coefficient. The method can be used to find absorbing boundary conditions for both modeling and migration equations. And we display numerical results for each.

Absorbing boundary conditions for modeling

The 3-D cone dispersion relation of the two-dimensional scalar wave equation is:

$$k_x^2 + k_z^2 = (\frac{\omega}{v})^2 \tag{1}$$

where k_x and k_z are the horizontal and vertical wave-numbers, respectively, and ω is the temporal frequency. Decomposition of the wavefield into leftgoing ($-x$) and rightgoing ($+x$) waves, requires equation (1) to be changed into the following form:

$$k_x = \pm\sqrt{(\frac{\omega}{v})^2 - k_z^2} \tag{2}$$

where the positive sign is used for rightgoing, the negative sign for leftgoing waves.

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Our starting point is that the total wavefield P_T at the boundary is the sum of the incident wavefield P_I from the interior and the reflected wavefield P_R from the boundary, such that $P_T = P_I + P_R$. If we let $P_T = P_I$, then $P_R = 0$, we obtain a perfect nonreflecting boundary. So we use the leftgoing (rightgoing) wave equation as a boundary condition for the left(right) edge. To get difference equations in the space-time domain, we need to rationally approximate equation (2).

Here we give two forms of rational approximation to (2):

$$A1 : k_x = ak_z + b\frac{\omega}{v}, \tag{3}$$

and

$$A2 : k_x = \frac{(\frac{\omega}{v})^2 - bk_z^2}{ck_z + d\frac{\omega}{v}}. \tag{4}$$

In the space-time domain:

$$A1 : P_x - aP_z + \frac{b}{v}P_t = 0, \tag{5}$$

and

$$A2 : \frac{a}{v^2}P_{tt} - bP_{zz} - cP_{xz} + \frac{d}{v}P_{xt} = 0. \tag{6}$$

The criterion for the absorbing boundary condition is that it is transparent to the incidence wavefield, that is, the boundary reflection coefficient is very small. Our motivation is to find a boundary condition with a reflection coefficient as small as possible.

Consider a rightgoing plane wave:

$$P_I = e^{i(k_x x + k_z z - \omega t)}. \tag{7}$$

Suppose the boundary reflection coefficient is R, then the right boundary reflection is:

$$P_R = Re^{i(-k_x x + k_z z - \omega t)}. \tag{8}$$

Locally near the boundary, the total wave field $P_T = P_I + P_R$ must satisfy both the boundary condition and the interior wave equation. Applying $A2$ to $(P_I + P_R)$, we obtain:

$$|R| = \left| \frac{bk_z^2 - a(\frac{\omega}{v})^2 + ck_x k_z + dk_x\frac{\omega}{v}}{a(\frac{\omega}{v})^2 - bk_z^2 + ck_x k_z + dk_x\frac{\omega}{v}} \right|. \tag{9}$$

From the interior two-way wave equation dispersion relation, we have:

$$\frac{vk_z}{\omega} = \sin\theta, \tag{10}$$

and

$$\frac{vk_x}{\omega} = \cos\theta. \tag{11}$$

where θ is the angle of incidence measured from the normal to the boundary. Then we have:

$$|R(\theta)| = \left| \frac{(b\sin^2\theta - a) + c\sin\theta\cos\theta + d\cos\theta}{(a - b\sin^2\theta) + c\sin\theta\cos\theta + d\cos\theta} \right|. \tag{12}$$

In the same way we obtain the reflection coefficient for A1:

$$|R(\theta)| = \left| \frac{b + a\sin\theta - \cos\theta}{b + a\sin\theta + \cos\theta} \right|. \tag{13}$$

We first give a reflection coefficient $R_0 = 0.1$, then by trial and error, we will find a, b, c, and d, so that $|R(\theta) \leq R_0|$ for as many θ as possible. We have

A1: a=-0.55, b=1

A2: a=8, b=8, c=-7.5, d=10.

In Figure 1, the reflection coefficients for A1, A2, versus C2 from Clayton et al. (1977) are plotted. The low angle reflection coefficients for A1 and A2 can be decreased, but we have to sacrifice the high angle counterparts. Figure 2 displays the finite difference modeled shot records of different absorbing boundary conditions on the model given by Reynolds (1978, p. 1103).

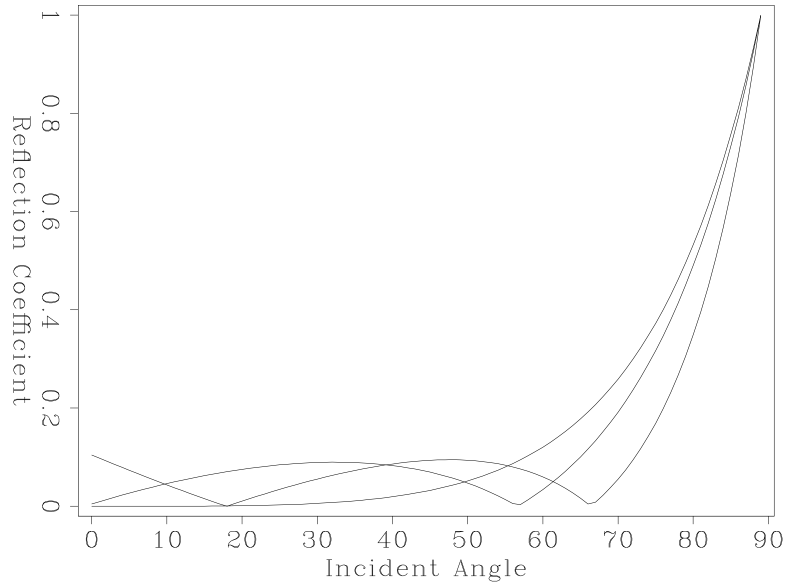

Figure 1: Graph of reflection coefficients for boundary conditions, A1, A2 versus C2 (Clayton et al. 1977). mo-fig1 [ER]

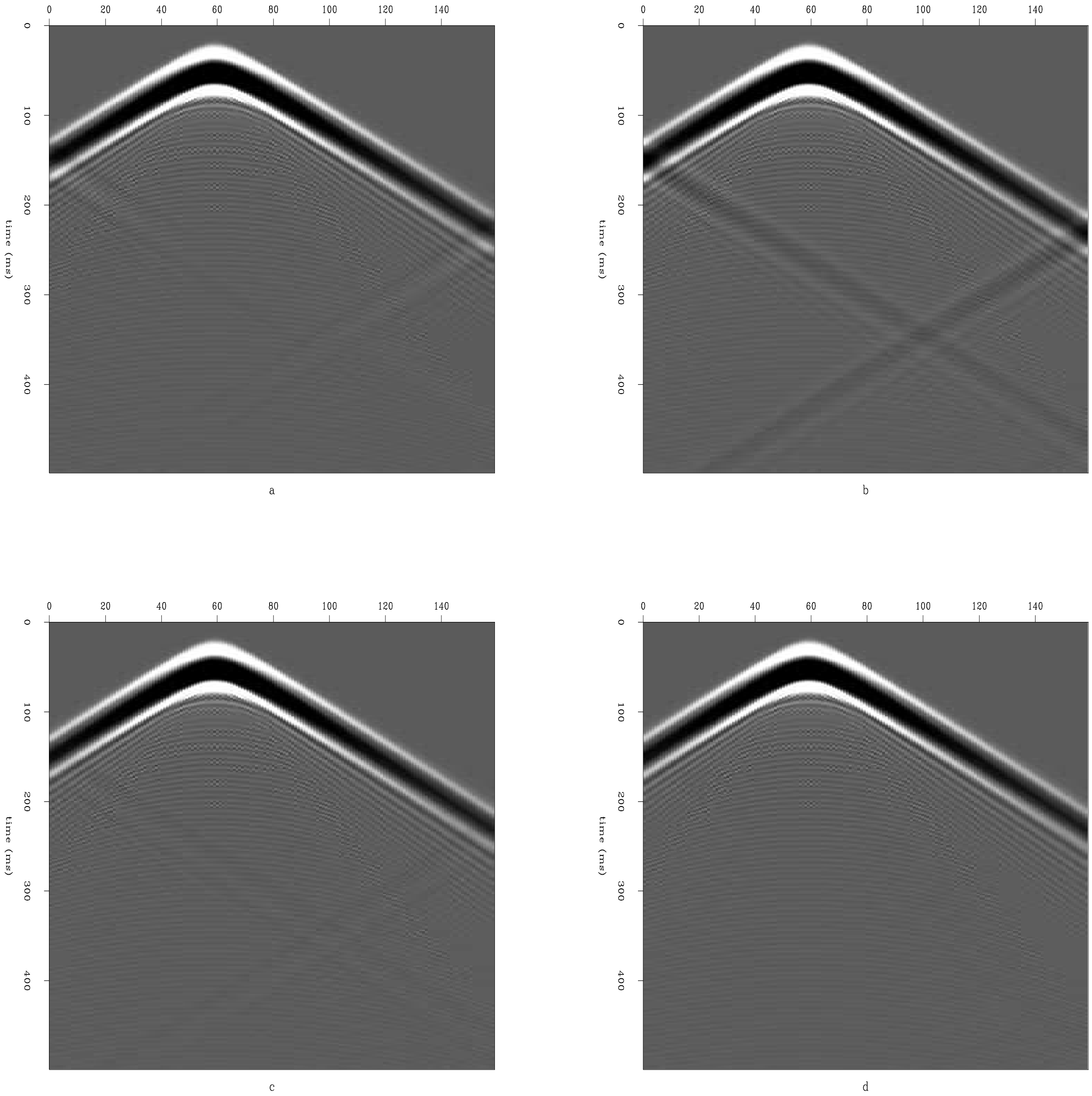

Figure 2: Modeled shot records by different absorbing boundary conditions, (a) A1, (b) A2, (c) C2 by Clayton et al. (1977), (d) Reynolds (1978) mo-fig2 [ER]

ABSORBING BOUNDARY CONDITIONS FOR LIWEQ

We develop absorbing boundary conditions for Zhiming Li's (1986) migration equation. In the space-time domain, the transforms are:

$$\begin{pmatrix} x' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} x \\ \frac{1}{\sqrt{2}}(z - vt) \\ \frac{1}{\sqrt{2}}(t + \frac{z}{v}) \end{pmatrix}, \tag{14}$$

and the inverse transforms are:

$$\begin{pmatrix} x \\ z \\ t \end{pmatrix} = \begin{pmatrix} x' \\ \frac{1}{\sqrt{2}}(z' + vt') \\ \frac{1}{\sqrt{2}}(t' - \frac{z'}{v}) \end{pmatrix}. \tag{15}$$

In the frequency-wavenumber domain, the transforms are

$$\begin{pmatrix} k_x \\ k_z \\ \frac{\omega}{v} \end{pmatrix} = \begin{pmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} k_x' \\ k_z' \\ \frac{\omega'}{v} \end{pmatrix} = \begin{pmatrix} k_x' \\ \frac{1}{\sqrt{2}}(k_z' - \frac{\omega'}{v}) \\ \frac{1}{\sqrt{2}}(k_z' + \frac{\omega'}{v}) \end{pmatrix}. \tag{16}$$

Then the original dispersion relation changes to

$$k_x'^2 - \frac{2}{v} k_z' \omega' = 0. \tag{17}$$

Because in coordinate transformation rotation, the k_x axis is fixed, so the transformation is actually a rotation with k_x as the symmetry axis. After coordinate rotation, the conic surface for $k_x' > 0$ is the same as the conic surface for $k_x > 0$, similarly for $k_x' < 0$ versus $k_x < 0$. So coordinate transformation does not change the propagation directions of the leftward and rightward traveling waves. We apply the leftgoing (or rightgoing) wave equation in the new coordinate system as a boundary condition, then we obtain the absorbing boundary conditions:

$$k_x' = \pm\sqrt{\frac{2}{v} k_z' \omega'}. \tag{18}$$

To get the difference equation in the space-time domain, we must approximate the square root in (18). The order of k_z' in the approximate equation must be no more than one, for only upcoming waves are to be present in our migration wavefield. Because under coordinate transformation only our viewing angle is changed, the object (cone) itself is not changed. We can use the approximate formulas in the last section. After coordinate rotation, A1 and A2 change into:

$$B1 : k_x' = a k_z' + b \frac{\omega'}{v} \tag{19}$$

and

$$B2: k_x^{'} = \frac{\frac{a}{v} k_z^{'} \omega^{'}}{c k_z^{'} + b \frac{\omega^{'}}{v}}. \tag{20}$$

In the space-time domain, B1 and B2 correspond to:

$$B1: P_{x'} + \frac{b}{v} P_{t'} - a P_{z'} = 0 \tag{21}$$

and

$$B2: c P_{x'z'} - \frac{b}{v} P_{x't'} + \frac{b}{v} P_{z't'} = 0. \tag{22}$$

Because of coordinate transformation, the $(x-z)$ plane changes to the $(x^{'}-z^{'})$ plane, then the rightgoing wave must be considered in thethe new $(x^{'}-z^{'})$ plane.

In the (x, z, t) domain, the rightgoing wave is

$$P_I = e^{i(kx\cos\theta - kz\sin\theta - \omega t)} \tag{23}$$

where $k = \frac{\omega}{v}$, and $0 < \theta < 90^0$. P_I could not be a downgoing wave, for in our migration wavefield there is only an upcoming wave component, no downgoing wave. After coordinate transformation,

$$P_I^{'} = e^{i[kx^{'}\cos\theta + \frac{k}{\sqrt{2}} z^{'}(1-\sin\theta) - \frac{\omega}{\sqrt{2}} t^{'}(1+\sin\theta)]}. \tag{24}$$

Let the right boundary reflection coefficient be R, then the wavefield near the boundary is:

$$P(x^{'}, z^{'}, t^{'}) = P_I^{'} + P_R^{'} = e^{i[kx^{'}\cos\theta + \frac{k}{\sqrt{2}} z^{'}(1-\sin\theta) - \frac{\omega}{\sqrt{2}} t^{'}(1+\sin\theta)]}$$
$$+ e^{i[-kx^{'}\cos\theta + \frac{k}{\sqrt{2}} z^{'}(1-\sin\theta) - \frac{\omega}{\sqrt{2}} t^{'}(1+\sin\theta)]}. \tag{25}$$

Substituting the above equation into boundary equations, we get:

$$B1: |R(\theta)| = \left| \frac{\sqrt{2}\cos\theta - b(1+\sin\theta) - a(1-\sin\theta)}{\sqrt{2}\cos\theta + b(1+\sin\theta) + a(1-\sin\theta)} \right| \tag{26}$$

and

$$B2: |R(\theta)| = \left| \frac{c(1-\sin\theta) + b(1+\sin\theta) - \frac{a}{\sqrt{2}}\cos\theta}{c(1-\sin\theta) + b(1+\sin\theta) + \frac{a}{\sqrt{2}}\cos\theta} \right|. \tag{27}$$

For a reflection coefficient $R_0 = 0.1$,we find the coefficients a, b, and c, by trial and error, so that $|R(\theta)| \le R_0$ in over as large a θ range as possible.

B1: a=1, b=0.35

B2: a=17, b=2, c=13.

In Figure 3, the reflection coefficients for B1, B2 are plotted. We apply the condition B2 to migrate three different frequency wavelets, and the results are displayed in Figure 4.

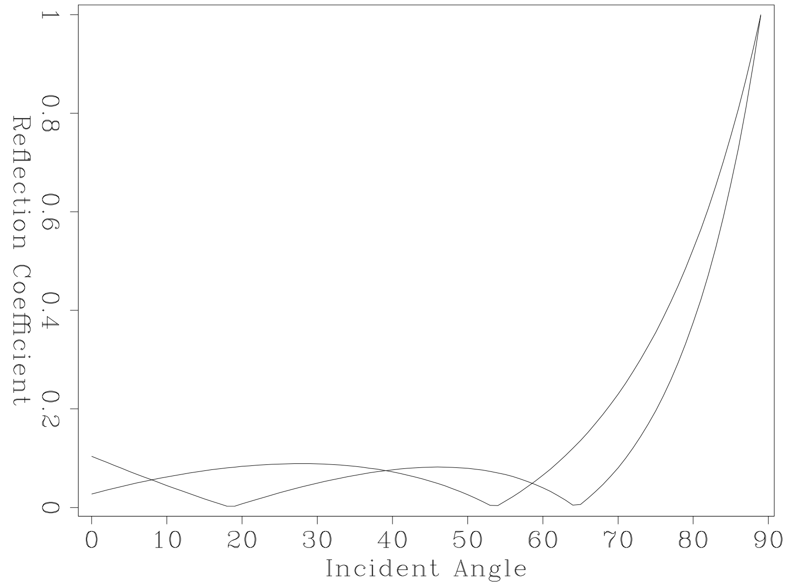

Figure 3: Graph of reflection coefficients for boundary conditions B1 and B2. mo-fig3 [ER]

CONCLUSIONS

In this note we present a direct method to derive absorbing boundary conditions for wave equation modeling and migration. For modeling, our conditions are not competitive with the methods that are already published. For migration, our conditions give good absorbing effect.

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Figure 4: Migration of three different frequency wavelets,(left) zero-value boundary, (right) absorbing boundary condition B2.

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