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Short Note

On the validity of a variable velocity acoustic wave equation

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INTRODUCTION

I consider the validity conditions for the variable velocity acoustic wave equation (1). This equation is sometimes used in geophysical applications of seismic wave propagation, imaging and inversion, without regard to the specific assumptions underlying its derivation or the validity of its use as a practical approximation. To gain some insight into these validity conditions, I carry out the necessary mathematical analysis to further illuminate the subject. To apply my results to seismic data, one has to assume that the Earth is a linear isotropic non-viscous fluid, which it clearly is not. I therefore hope to extend this analysis in a future work using the elastodynamic wave equation. In the meantime, my results may be useful in as much as treating seismic waves as uncoupled acoustic (scalar) waves has been useful to the geophysical community in the past.

VALIDITY CONDITION RESULTS

In this section I will state the validity condition results of the mathematical analysis that follows in the main body of the paper. A valid acoustic wave equation governing fluid particle motion for a pressure wavefield $P(\underline{\mathbf{x}}, t)$ is

$$\{\partial_{tt} - v^2(\underline{\mathbf{x}})\nabla^2\}P(\underline{\mathbf{x}}, t) = 0\,. \tag{1}$$

1. It is perfectly rigorous to use (1) in a linear isotropic non-viscous fluid with a constant mass density ρ_o and a spatially variable bulk modulus $\lambda(\underline{\mathbf{x}})$ such that

$$v_1^2(\underline{\mathbf{x}}) = \lambda(\underline{\mathbf{x}})/\rho_o\,. \tag{2}$$

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2. In "reasonable" background models where $\rho(\underline{\mathbf{x}})$ varies smoothly with respect to pressure wavelength, **1** can be extended to $\rho_o = \rho(\underline{\mathbf{x}})$ such that

$$v_2^2(\underline{\mathbf{x}}) = \lambda(\underline{\mathbf{x}})/\rho(\underline{\mathbf{x}})\,, \tag{3}$$

when the following validity condition is satisfied:

$$\frac{|\partial_i \rho|}{|\rho|} \ll \frac{|\partial_i \ddot{u}_i|}{|\ddot{u}_i|}\,. \tag{4}$$

Condition (4) is invalid near regions of discontinuities in $\rho(\underline{\mathbf{x}})$ (e.g., reflecting boundaries), but is valid in regions of smooth-varying spatial properties (e.g., a smoothed migration model).

MATHEMATICAL ANALYSIS

Conservation of linear momentum

Conservation of linear momentum $\rho\dot{u}_i$ can be expressed as a global balance of surface tractions T_i and volumetric body forces f_i:

$$\int_V \rho \ddot{u}_i \, dV = \int_S T_i \, dS + \int_V f_i \, dV\,, \tag{5}$$

integrated over the bounding surface S and volume V. Note that in (5) I have already invoked conservation of mass in a quiescent fluid by assuming that the material derivative $D_t\rho$ vanishes identically. Since (5) is valid for an arbitrary volume, it can be re-expressed in local form:

$$\rho \ddot{u}_i = \partial_j \sigma_{ij} + f_i\,. \tag{6}$$

The traction vector T_i has been replaced with the gradient of the stress tensor, $\partial_j \sigma_{ij}$. Assuming a non-viscous fluid, which supports no shear stress, a linear isotropic stress-strain relation can be written as

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk}\,. \tag{7}$$

Since pressure is identified with the normal compressional stress components, pressure and strain displacement can be related as

$$P = -\lambda \epsilon_{kk} = -\lambda \partial_k u_k\,, \tag{8}$$

where the standard strain-displacement definition has been invoked:

$$\epsilon_{ij} = 1/2(\partial_i u_j + \partial_j u_i)\,. \tag{9}$$

Substituting (7) into (6) and assuming body forces f_i (gravity, hydrostatic pressure, etc.) to be negligible or non-deviatoric:

$$\rho \ddot{u}_i = \partial_j \lambda \delta_{ij} \epsilon_{kk} + 0\,, \tag{10}$$

and using (8) results in

$$\rho \ddot{u}_i = \partial_i \lambda \partial_k u_k\,. \tag{11}$$

Taking the divergence of both sides of (11) to ensure compressional motion only,

$$\partial_i \rho \ddot{u}_i = \partial_i \partial_i \lambda \partial_k u_k\,, \tag{12}$$

and expanding gradient terms on the left results in

$$\rho \partial_i \ddot{u}_i + \ddot{u}_i \partial_i \rho = \partial_i \partial_i \lambda \partial_k u_k\,. \tag{13}$$

Up until this point, the mass density and bulk modulus terms have been unrestricted with respect to spatial variability, i.e., $\rho = \rho(\underline{\mathbf{x}})$, and $\lambda = \lambda(\underline{\mathbf{x}})$.

Case 1: constant mass density

Assume the mass density is constant: $\rho(\underline{\mathbf{x}}) = \rho_o$. Then the general result (13) can be evaluated as

$$\rho_o \partial_i \ddot{u}_i = \partial_i \partial_i \lambda \partial_k u_k\,. \tag{14}$$

Recalling the pressure-displacement relation (8), equation (14) can be rewritten as

$$\rho_o \partial_{tt} P = \lambda \partial_i \partial_i P\,, \tag{15}$$

or,

$$\{\partial_{tt} - v_1^2(\underline{\mathbf{x}}) \nabla^2\} P(\underline{\mathbf{x}}, t) = 0\,, \tag{16}$$

where

$$v_1^2 = \lambda(\underline{\mathbf{x}})/\rho_o \tag{17}$$

is the propagation phase velocity of the pressure waves.

Hence, for a linear isotropic non-viscous fluid medium with constant mass density and spatially variable bulk modulus, the variable velocity acoustic wave equation (1) is rigorously valid for use in wave propagation, imaging and inversion applications.

Case 2: spatially smooth mass density variation

Assume that the mass density $\rho(\underline{\mathbf{x}})$ is smooth in some sense on the scale of the dominant pressure wavelengths of concern. This leads us to examine the relative magnitude of the two terms on the left hand side of (13):

$$|\rho \partial_i \ddot{u}_i| \quad \text{and} \quad |\ddot{u}_i \partial_i \rho| \,. \tag{18}$$

It is convenient to normalize the two terms by $|\rho \ddot{u}_i|$ and scale by the Fresnel length L_f, since spatial changes in properties are only meaningful with respect to the spatial wavelengths and frequency components in the propagating wavefield:

$$R_1 = \frac{|\partial_i \ddot{u}_i|}{|\ddot{u}_i|} L_f \quad \text{and} \quad R_2 = \frac{|\partial_i \rho|}{|\rho|} L_f \,. \tag{19}$$

Examining these two terms, it is apparent that (1) is valid when

$$\frac{|\partial_i \rho|}{|\rho|} \ll \frac{|\partial_i \ddot{u}_i|}{|\ddot{u}_i|} \tag{20}$$

and

$$v_2^2(\underline{\mathbf{x}}) = \lambda(\underline{\mathbf{x}}) / \rho(\underline{\mathbf{x}}) \,. \tag{21}$$

As a matter of practical interest, consider the following example. Assume a propagating wavefield in a region with a phase velocity v_2 of 3 km/s, an average density $\tilde{\rho}$ of 2.0 g/cc, and a dominant frequency content of 30 Hz. The dominant spatial wavelength will be on the order of 100 m, and we will assume that the wave energy will be coherent over a quarter-wavelength Fresnel zone of $L_f = 25$ m. A reasonable "background" density gradient characteristic of the Gulf of Mexico would be about 0.2 g/cc/km. Using these values, and approximating

$$R_2 \approx \frac{|\Delta \rho|}{L_f} \cdot \frac{L_f}{|\tilde{\rho}|} = \frac{|\Delta \rho|}{|\tilde{\rho}|} \,, \tag{22}$$

where $\tilde{\rho}$ is an average over the Fresnel zone and $\Delta\rho$ is the total density contrast over the Fresnel zone, I evaluate the density gradient term R_2 to be $\approx 0.25\%$. Similarly, for the displacement gradient term,

$$R_1 \approx \frac{|\Delta \ddot{u}_i|}{|\tilde{\ddot{u}}_i|} \approx \frac{|\ddot{u}_{max}|}{1/2 |\ddot{u}_{max}|} \,, \tag{23}$$

or, $R_1 \approx 200\%$. For this realistic exploration example, $R_2 \ll R_1$ by about three orders of magnitude! In fact, scaling this result implies that a density gradient of up to 16 g/cc/km over a 25 m Fresnel zone, or a total density contrast of 0.4 g/cc/L_f, will result in an error of less than 10% if the variable velocity acoustic equation (1) is used:

$$\nabla\rho \leq 16 \text{ g/cc/km} \quad \text{implies} \quad R_2/R_1 \leq 10\% \,. \tag{24}$$

On the other hand, this result also implies that very large contrasts in density and very high frequency data may be sufficient to invalidate the use of (1). The validity condition (20) is likely to be violated at strong geologic discontinuities such as shale/carbonate boundaries or at basement volcanic unconformities given sufficiently high frequency data.

CONCLUSIONS

The variable velocity acoustic wave equation (1) is rigorously valid for use in wave propagation and imaging/inversion applications for the case of a linear isotropic non-viscous fluid medium with constant mass density and spatially variable bulk modulus. However, for reasonably smooth contrasts in density and standard frequency-content data, the validity condition (20) is sufficiently satisfied for the use of (1) in realistic exploration geophysics applications. I hope to extend this analysis in a future work to define the validity conditions for approximating elastic wave propagation with a variable velocity scalar wave equation.

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