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# Kinematic ray tracing in anisotropic layered media: practical details

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***keywords:*** *ray tracing, phase, velocity, anisotropy*

**ABSTRACT**

I review a procedure to trace rays in layered transversely isotropic models with dipping interfaces. Group velocities are used to propagate the ray across each homogeneous layer and phase velocities are used to find out how a given ray changes its direction when impinging on an interface. The equation that relates the ray parameter of the incident ray with the angle of the emergent phase at each interface is studied in detail. Examples of ray tracing in simple models are shown.

## INTRODUCTION

Estimating velocity anisotropy tomographically from cross well traveltime data is a process that often has two degrees of nonlinearities: the anisotropy itself and the ray bending from one iteration to the next. Under certain conditions one of these nonlinearities can be neglected and the computations related to other one are simplified. For example, assuming weak anisotropy, Pratt and Chapman (1992) and Chapman and Pratt (1992) proposed to do the ray tracing in isotropic media and to use those rays to back project in the anisotropic model. Michelena et al. (1991) simplify the problem even more and present examples where the velocity contrasts are small enough such that the rays traced on the anisotropic model are straight and the only non linearity that remains is in the estimation of the anisotropy. Unfortunately, these simplifications are not valid in general and therefore, the problem needs to be solved by tracing rays appropriately in inhomogeneous anisotropic media.

Methods for dynamic ray tracing in inhomogeneous anisotropic media have been proposed by Červený (1972), Hanyga (1982), Jech and Pšenčík (1989) and Gajewski and Pšenčík (1990), among others. These methods are based on the solution of the dynamic ray tracing equations. In order to solve only the kinematic problem (what is needed to perform traveltime tomography), Byun (1982) and Byun (1984) proposes a different technique based on the application of Snell's law at each interface.

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In this paper, I review Byun's procedure for kinematic ray tracing. From the boundary conditions at each interface, I derive the equation that relates the ray parameter $p$ of the incident phase to the angle of the transmitted phase. By using this equation and the equations that relate phase and group velocities, I explain a procedure that uses alternatively both types of velocity to propagate a ray across a layer and to figure out how it changes direction when impinging upon an interface. At the end, I show examples of $P$-wave ray tracing for a cross-well geometry in different layered transversely isotropic models. In a separate paper (Michelena, 1992), I use this ray tracing procedure to do traveltime tomography in azimuthally anisotropic media.

## BOUNDARY CONDITIONS

A wave impinging on a plane interface between two media must satisfy the kinematic boundary condition

$$\mathbf{v}_1 \;=\; \mathbf{v}_1. \tag{1}$$

that says that the particle velocity $\mathbf{v}$ must be continuos at all points on the boundary between medium 1 and medium 2 (Auld, 1990).

If the wave field is described as a superposition of functions of the form $\exp(i\mathbf{kr})$, the condition (1) implies that for incident and scattered waves the component of $\mathbf{k}$ tangential to the boundary ($\mathbf{k}_{\|}$) must be the same. If we divide $\mathbf{k}$ by the frequency $\omega$ of the incident wave and define a vector $\tilde{\mathbf{k}}$ of magnitude $1/v$ (where $v$ is the phase velocity)

$$\tilde{\mathbf{k}} \;=\; \frac{\mathbf{k}}{\omega} \;=\; \frac{\mathbf{k}}{\|\mathbf{k}\|}\frac{1}{v}, \tag{2}$$

the continuity of $\mathbf{k}_{\|}$ implies the continuity of $\tilde{\mathbf{k}}_{\|}$, usually called ray parameter $\mathbf{p}$:

$$\mathbf{p_1} \;=\; \mathbf{p_2}. \tag{3}$$

Since we know that $\mathbf{p}$ is parallel to the interface, the ray parameter is usually considered as an scalar $p$ with a positive or negative sign in the front to indicate the sign of angle of the incident phase with respect to the normal.

Auld (1990) examines condition (3) by looking at the corresponding slowness surfaces (a plot of $\|\tilde{\mathbf{k}}\|$ as a function of its direction), as Figure 1 shows. In this figure the slowness surfaces for both $P$ and $S$ waves are represented for each isotropic medium. The vector $CA = AB = \mathbf{p}$ is the projection on the interface of the point in the slowness surface that corresponds to the incident $P$ wave phase. From the construction of Figure 1 it is possible to derive easily Snell's law that gives the angles of the scattered phases for both $P$ and $S$ waves:

$$\begin{aligned} p \;&=\; \frac{1}{v_{1_P}}\sin(\theta_{P_i}) \;=\; \frac{1}{v_{2_P}}\sin(\theta_{P_t}) \;=\; \frac{1}{v_{2_S}}\sin(\theta_{S_t}) \\ &=\; \frac{1}{v_{1_S}}\sin(\theta_{S_r}) \;=\; \frac{1}{v_{1_P}}\sin(\theta_{P_r}). \end{aligned} \tag{4}$$

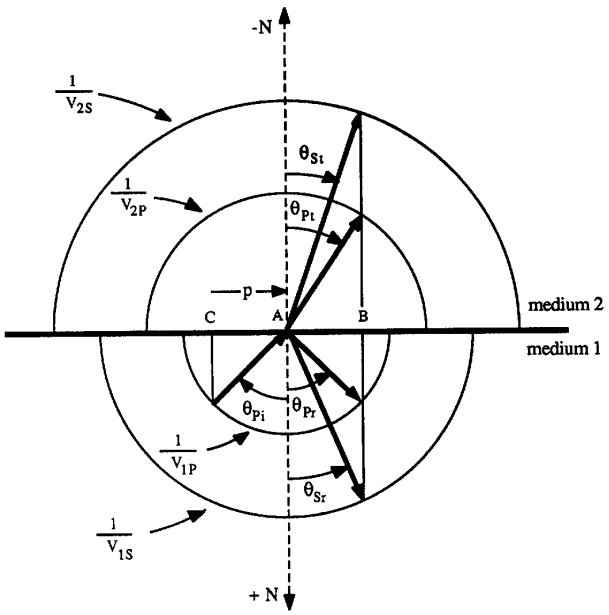


Figure 1: Plane wave scattering at a plane boundary between two isotropic media. From this construction Snell's law can be easily derived. reinaldo2-slow-surf [NR]

Snell's law tells us how a given phase changes its direction when it crosses the interface between two media. It also tells us how rays change direction when crossing an interface, but the medium must be isotropic for this to be true, like in Figure 1. In isotropic media, since rays and waves travel in the same direction with the same velocity, boundary conditions valid for waves are also used to predict the behavior of rays. In anisotropic media, however, this simplification is no longer valid because in general rays and waves travel with different velocities in different directions. This is shown in Figure 2.

## INITIAL VALUE RAY TRACING

When the medium is horizontally layered and isotropic, tracing a ray across the different interfaces is easy. From the medium velocities and the direction of the ray at the source ($p$), successive applications of (4) tells us how a ray changes its direction as it travels. The only angles that matter are those of the incident and refracted rays. If the layers are dipping the problem is slightly more complicated because $p$ has the be calculated every time the ray reaches a new interface and we have to deal with three instead of two angles at each boundary.

When the medium is anisotropic, we have to deal with seven angles at each interface to figure out how a ray changes its direction. All these angles (positive counterclockwise) are shown in Figure 3. The continuity of the slowness across the interface

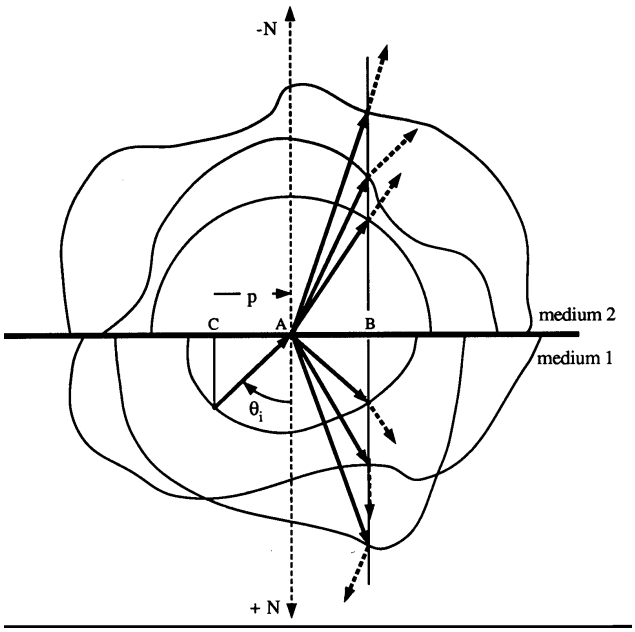


Figure 2: Plane wave scattering at a plane boundary between two anisotropic media. The slowness surfaces are separated and nonspherical. Ray directions (dashed arrows) and phase directions (continuos arrows) are no longer the same.(Modified from Aki and Richards, 1980.) reinaldo2-slow-surf-ani [NR]

(equation 3) is expressed as follows:

$$p = \frac{\sin(\theta_i + \gamma_1)}{v_1(\theta_i)} = \frac{\sin(\theta_t + \gamma_2)}{v_2(\theta_t)} \tag{5}$$

where $v_1(\theta_i)$ and $v_2(\theta_t)$ are the phase velocities in the corresponding media evaluated at the incident and transmitted phase angles respectively. The sums $(\theta_i + \gamma_1)$ and $(\theta_t + \gamma_2)$ have to be done considering the proper signs of the angles. From equation (5) we notice that when $p = 0$ (normal incidence) the angle of the transmitted phase does not depend on the phase velocity and is $\theta_t = -\gamma_2$. Any other relation derived from (5) must reproduce this simple result.

Given the function $v(\theta)$ for each medium, equation (5) tells us how the incident phase $\tilde{\mathbf{k}}$ changes its direction when crossing the interface between two media. However, it does not say what happens to the direction and velocity of the associated ray. These two quantities can be calculated from the equations that relate phase and ray velocities (Byun, 1984):

$$v(\theta) = V(\phi)\cos(\phi - \theta) \tag{6a}$$

$$\tan(\phi - \theta) = \frac{1}{v(\theta)}\frac{dv(\theta)}{d\theta} \tag{6b}$$

$$V^2(\phi) = v^2(\theta) + \left(\frac{dv(\theta)}{d\theta}\right)^2 \tag{6c}$$

where $V(\phi)$ is the ray velocity along the ray angle $\phi$.

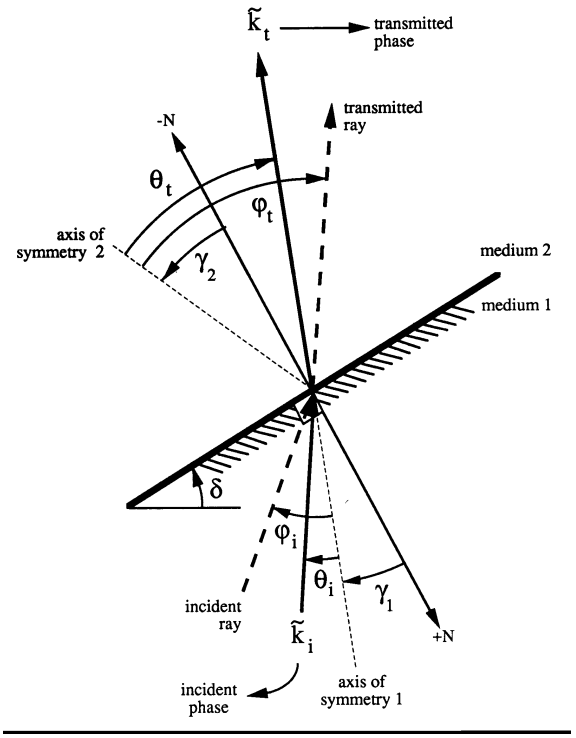


Figure 3: Incident ($i$) and transmitted ($t$) group and phase angles of a ray impinging on a plane interface between two transversely isotropic media. reinaldo2-all-angles [NR]

Equations (5) and (6) are the basic relations to trace a ray in a medium where $v(\theta)$ and $\gamma$ are given for each layer. Byun (1984) proposes the following procedure to do that: (1) Assign a value to $\theta = \theta_i$ at the source position. (2) Repeat the next steps for each layer: (a) Evaluate $v(\theta)$ and $\frac{dv(\theta)}{d\theta}$ at the corresponding $\theta$. (b) Find $\phi_t$ and $V(\phi_t)$ form equation (6). (c) Trace a ray along the angle $\phi_t$ until it reaches the next interface. (d) Find $p$ of the incident wave using equation (5). (e) Solve equation (5) for $\theta_t$. (3) Go to step 2-a.

The previous ray tracing procedure is valid for any phase velocity function $v(\theta)$. What changes from one choice of the velocity function to another is how to solve step 2-e (to find $\theta_t$ as a function of $p$ using equation (5)). The next section focuses on this step when the medium is transversely isotropic.

## SOLVING FOR THE PHASE ANGLE

### *P* and *SV* waves

In a transversely isotropic medium, the phase velocity $v(\theta)$ for $P$ and $SV$ waves is expressed as follows (Auld, 1990):

$$2\rho v^2(\theta) = c_{11}\sin^2(\theta) + c_{33}\cos^2(\theta) + c_{44} \tag{7}$$

$$\pm\sqrt{[(c_{11}-c_{44})\sin^2(\theta)+(c_{44}-c_{33})\cos^2(\theta)]^2+(c_{13}+c_{44})^2\sin^2(2\theta)},$$

where $c_{ij}$ are four of the five elastic constants that describe an homogeneous transversely isotropic medium. The sing "+" in front of the square root corresponds to $P$ waves and the "−" to $SV$ waves.

An equation for the transmitted phase angle $\theta_t$ as a function of the incident $p$ can be obtained after substituting equation (7) into equation (5), resulting the following fourth order polynomial in $\tan(\theta_t)$:

$$a_0+a_1\tan(\theta_t)+a_2\tan^2(\theta_t)+a_3\tan^3(\theta_t)+a_4\tan^4(\theta_t) \;=\; 0, \tag{8}$$

where the coefficients $a_i$ are given in Appendix A. A similar expression obtained by Byun (1984) is no correct because $\theta_t = -\gamma_2$ is not a root of the polynomial when $p = 0$, as equation (5) predicts.

As shown in Appendix A, the coefficients $a_i$ are the same for both $P$ and $SV$ waves (the $\pm$ in front of the square root in equation (7) is lost in the derivation of (8)) and they are independent on the sign of $p$. In principle, by solving this equation at each interface, it should be possible to find the angle of the transmitted phase by computing the arctangent of one of its roots. However, the procedure is not straight forward. In one hand, we have to identify the solutions that correspond to $p$ or $-p$ and $P$ or $SV$ waves. On the other hand, finding an angle from its tangent is not a one to one transformation because $\tan(\theta) = \tan(\theta + n\pi)$ and therefore, infinite angles satisfy equation (8). Besides these problems, we have to deal usually with the difficulties of solving equation (8) numerically.

Let's examine first the numerical problems. It is well known that most algorithms used to find roots of polynomials fail in the presence of multiple (or close to multiple) roots. This is the case of equation (8) when $p = 0$. From equation (5), we obtained that in this case the transmitted phase angle is $\theta_t = -\gamma_2$ and therefore, it is not necessary to solve equation (8) when $p = 0$. Problems may also arise when $p \approx 0$ because the roots are close of being multiple. The way I found around this problem was based on the assumption that the slowness surface is an smooth function of $\theta$ and then, when $p \approx 0$, $v_2(\theta_t) \approx v_2(-\gamma)$. If this is true, $\theta_{t_0} = -\gamma_2$ can be used as "starting" point to find $\theta_t$ by successive applications of equation (5) as follows:

$$\theta_{t_{i+1}} \;=\; \sin^{-1}(pv_2(\theta_{t_i})) - \gamma_2. \tag{9}$$

Typically, only one iteration is required.

Figure 4 shows how to solve equation (8) graphically. From the construction with the slowness surfaces, we see that the infinite roots of equation (8) that I mention before correspond to only eight different points in the slowness surfaces for both $P$ and $SV$ waves. In this figure I assume that the incident phase (not shown) is crossing the interface from medium 1 to medium 2. For simplicity, I assume that both media are isotropic. The slowness surfaces in medium 2 are represented one half by continuous circles and the other half by dotted circles. The slowness surfaces in medium 1 are

not shown. For a given $\pm p$ value at the interface, this figure shows the position of all the possible points in the $P$ and $SV$ slowness surfaces whose angle measured from the symmetry axis satisfy equation (8). These points are obtained by simple application of the kinematic boundary condition (3). The points *A, B, C,* and *D* are the solutions for negative $p$ and the points *H*, *G, F* and *E* are obtained by adding $\pm\pi$ to the previous ones and they are the solutions for positive $p$. If $|p| > \frac{1}{v_p}$ two of the roots of (8) are complex and the and there are no transmitted $P$ waves. If $|p| > \frac{1}{v_s}$ the four foots are complex and there are no transmitted $SV$ waves either. I will assume that when the transmitted ray travels parallel to the interface, it doesn't go back to the previous medium, which is a simplification of real the behavior of head waves.

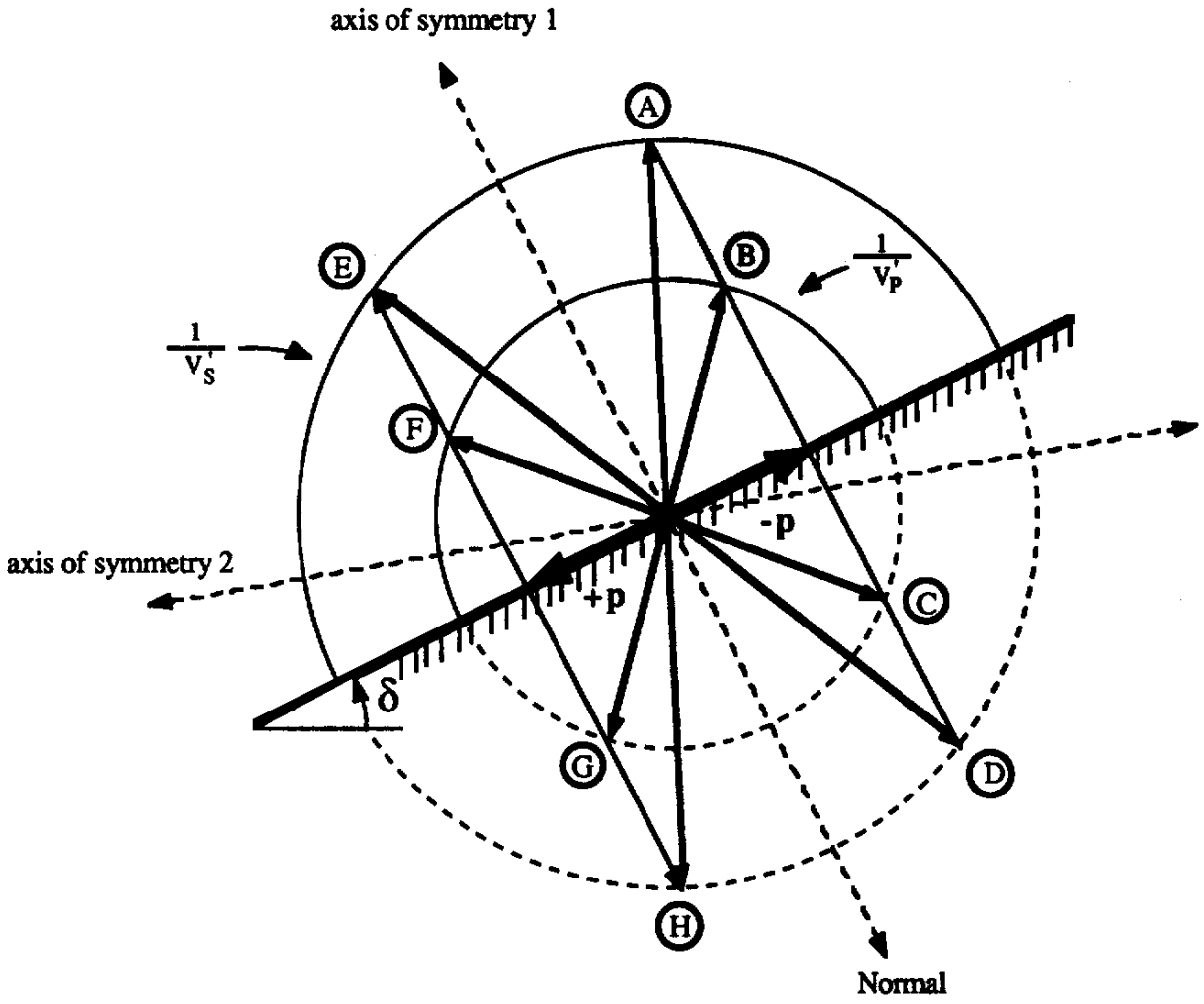


Figure 4: Graphical solution for the angle of the transmitted phase as a function of the incident $p$. The the tangent of the angles at $A$, $B$, $C$, $D$, $E$, $F$, $G$ and $H$ are roots of equation (8). Among these eight angles, only one reproduces the incident $p$ for the corresponding wave type in the expected direction. When solving equation (8) numerically, only those angles at $\pm 90$ degrees from the axis of symmetry are obtained. The axes of symmetry 1 and 2 are two of the possible axes of symmetry of the isotropic medium where the transmitted phase travels. reinaldo2-eight-roots [NR]

When the axis of symmetry and the normal to the interface are parallel (axis of symmetry 1 in Figure 4), the arctangent of the roots of equation (8) give the angles of the points $A$, $B$, $E$ and $F$. This is because the FORTRAN function ATAN gives its results in the interval $(-\pi/2, +\pi/2)$ and this is exactly the domain of the phase angle measured from the axis of symmetry. If we take 2 as the axis of symmetry (which is true here because the medium is isotropic), the roots of equation (8) give the points $E$, $F$, $G$ and $H$. None of this points is the correct one when, for example, a $P$ wave impinges on the interface with negative $p$. The point $B$ is the solution in this case.

The previous example tells us that when the axis of symmetry and the normal to the interface are not parallel, the roots of equation (8) may not give the expected direction for the refracted phase. For this reason, it is necessary to examine all the eight angles and pick the one that verifies equation (5) for the desired wave type (with this, the number of possible angles is reduced from eight to two) and produces a ray pointing towards the expected direction (which further reduces the number of angles from two to one). This will be explained in more detail in the next section, for *SH* waves.

### *SH* waves

*SH* waves in a transversely isotropic medium have elliptical slowness and wave surfaces. An elliptical slowness surface may not be a good *global* (ie., for all angles) approximation for a $P$ or $SV$ slowness surface in a transversely isotropic medium, but it is a good *paraxial* approximation (Dellinger and Muir, 1991). Hence the importance of elliptical anisotropy in data fitting, specially when the measurements have limited aperture, as explain by Dellinger Muir (1991) and Michelena (1992).

Byun (1982) shows that when the slowness surfaces are elliptical the calculation at each interface of the refracted ray angle $\phi_t$ is simpler, because the phase angle $\theta_t$ does not need to be explicitly computed. Byun's expression for the ray parameter as a function of the ray is angle at each interface is

$$p = \frac{v_{\|}^{-2}\sin(\gamma_2) + v_{\perp}^{-2}\cos(\gamma_2)\tan(\phi_t)}{\sqrt{v_{\|}^{-2} + v_{\perp}^{-2}\tan^2(\gamma_2)}}. \tag{10}$$

where $v_{\|}$ and $v_{\perp}$ are the *SH* wave velocities in the direction parallel and perpendicular to the axis of symmetry. After some algebra it is possible to solve for the ray angle $\phi_t$ resulting

$$\tan(\phi_t) = \left(\frac{v_{\perp}}{v_{\|}}\right) \frac{sin(\gamma_2)\cos(\gamma_2) \pm \sqrt{v_{\|}^2\cos^2(\gamma_2) + v_{\perp}^2\sin^2(\gamma_2) - (pv_{\|}v_{\perp})^2}}{p^2v_{\perp}^2 - \cos^2(\gamma)}. \tag{11}$$

Since equation (11) is independent of the sign of $p$, it gives the solutions of (10) for both positive and negative $p$. The correct $\phi_t$ is the one that gives the expected sign of $p$ and correspond to a ray in the desired direction.

If we take the extra step of solving explicitly for the phase angle at each interface, we have to look among the four different points in the slowness surface whose angles satisfy equation (5) and keep the one that produces a ray pointing in the desired direction. The "desired direction" depends on the type of problem where the ray tracer is used. Figure 5 shows how to pick the roots that correspond to transmitted rays, the ones needed in transmission traveltime tomography. In this figure, the horizontal distance from the center of the each ellipse to the vertical line is equal to the incident $p$. As we see, to obtain transmitted rays (dashed arrows with black head), it is enough to pick the root that produces a ray pointing towards medium 2. In some cases the transmitted ray can travel parallel to the interface with the phase direction (continuous arrows) parallel to it or not (Figures 5-f and 5-e respectively). The emergent phase angle may be greater than 90 degrees for the transmitted ray, as shown in Figure 5-g. The ray pointing towards medium 2 may also go backwards, as Figure 5-h shows. When the vertical line doesn't intersects the ellipse, there are not transmitted but only reflected rays (angle of incidence greater than the critical angle). However, in order to keep track more easily of the rays that hit the interface at post-critical angles, I will assume that those rays continue traveling along the interface. The inversion procedure I describe in a separate paper (Michelena, 1992) doesn't use these rays.

### Model of heterogeneities

The models where I'm going to trace rays consist of homogeneous transversely isotropic layers with dipping interfaces which are not allowed to cross in the area of interest. Each layer is defined by the intersects and dips of the boundaries, the five elastic constants that define a transversely isotropic medium and the orientation of the axis of symmetry with respect to the vertical. If the anisotropy is elliptical, only the velocities parallel and perpendicular to the axis of symmetry are required, as explained in Michelena (1992).

## TWO POINT RAY TRACING

In transmission traveltime tomography, we usually need to link every source with the corresponding receiver positions. The way how I do this with the initial value ray tracing just described is as follows:

- (a) An sparse fan of rays is traced from the given source position. The aperture of this fan is greater than the aperture of the receiver positions. The end points of these rays ($z_{end}$) and the shooting angles ($\alpha_{init}$) are saved.
- (b) The angle $\alpha_{init}$ that corresponds to a given receiver location is obtained (by using splines interpolation) from the relation $\alpha_{init}$ vs. $z_{end}$. This relation is a function almost everywhere. In some places it can be be multivalued because the

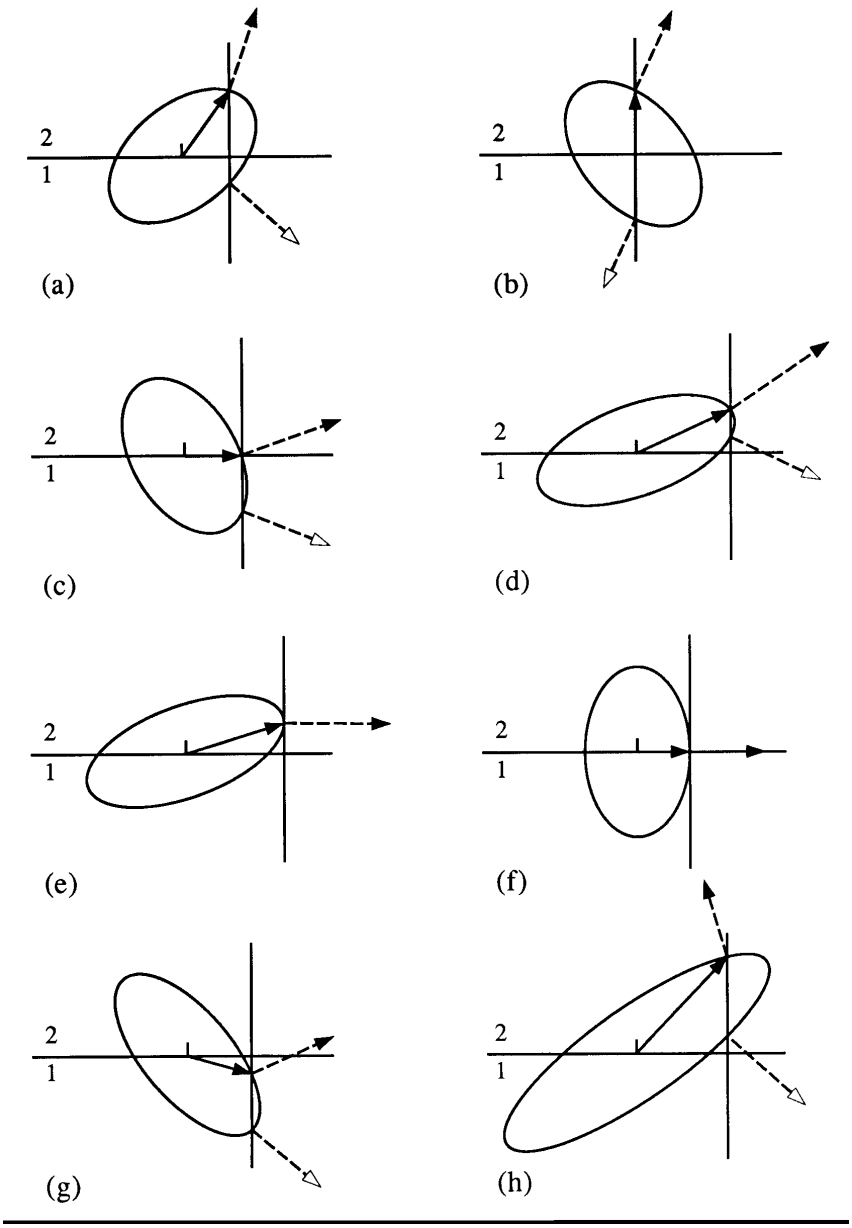


Figure 5: Roots of equation (5) (scattered phase angles) for negative $p$ when the slowness surface is elliptical. Only the slowness surface in medium 2 is shown. In each case, the incident $p$ is measured from the center of the ellipse to the vertical line. Continuous arrows show the scattered phase directions that correspond to transmitted rays (dashed arrows with black head). Dashed arrows with white head are ray directions with the same $p$ but pointing towards medium 1. (a) Most common case. (b) Normal incidence ($p = 0$). (c) Phase direction parallel to the interface, ray not. (d) Two phase directions less than 90 degrees. (e) Ray direction parallel to the interface, phase not. (f). Both ray and phase directions parallel to the interface. (g) Two phase directions greater than 90 degrees. (h) One ray goes backwards. reinaldo2-which-root [NR]

same receiver location can correspond to different shooting angles (all of them greater than the critical angle). The points were the function is multivalued must be removed before the interpolation. Because of the way the medium is described, $\alpha_{init}$ vs. $z_{end}$ always increases or always decreases (except where multivalued).

- (c) A ray is traced along the newly found $\alpha_{init}$ and the $z_{end}$ that results is compared to the expected receiver position. If the difference is greater than a predetermined value, the function $\alpha_{init}$ vs. $z_{end}$ is updated with the latest values of $\alpha_{init}$ and $z_{end}$ and the procedure goes again to step (b). If the difference is smaller than a predetermined value, the ray has been captured.

## EXAMPLES

In the following examples, I will show fans of $P$ wave ray paths in different models for a cross-well configuration (Appendix B shows the elastic constants used for each model). The fans are traced at constant intervals in the phase angle which only correspond to constant intervals in the ray angle when the medium is isotropic. This is shown in Figure 6a, where rays have been traced in the simplest model: homogeneous isotropic. When the model is transversely isotropic with vertical axis of symmetry, the density of rays changes with angle, as Figure 6b shows. Figure 6c shows the rays paths in a homogeneous transversely isotropic model with axis of symmetry tilted with respect to the vertical ($\gamma = 30$). If two point ray tracing for the same receiver positions is performed in the corresponding homogeneous models, we won't see any difference among the three ray path diagrams because the ray angles depend only on the relative positions of source and receivers. The differences among the three cases will be in the traveltimes.

Models like the ones shown in Figure 7 may arise after different combinations of deposition, erosion and rotation of the layers involved. The areas of the model that are finely layered can be modeled as homogeneous anisotropic layers with axis of symmetry perpendicular to the layering. The only difference between the two models in Figure 7 is in the axis of symmetry the third layer. Figure 8 show the rays traced through these models for a source at 3500 ft.

## CONCLUSIONS

I have reviewed the basic steps to trace rays in layered transversely isotropic models. Special emphasis was made on the calculation of the angle of the transmitted phase as a function of the ray parameter of the incident phase. When the angle of incidence is small, I use a procedure to estimate the angle of the transmitted phase different from when the angle of incidence is large. In the later case the transmitted phase angles are calculated from the roots of a four order polynomial in the tangent of the

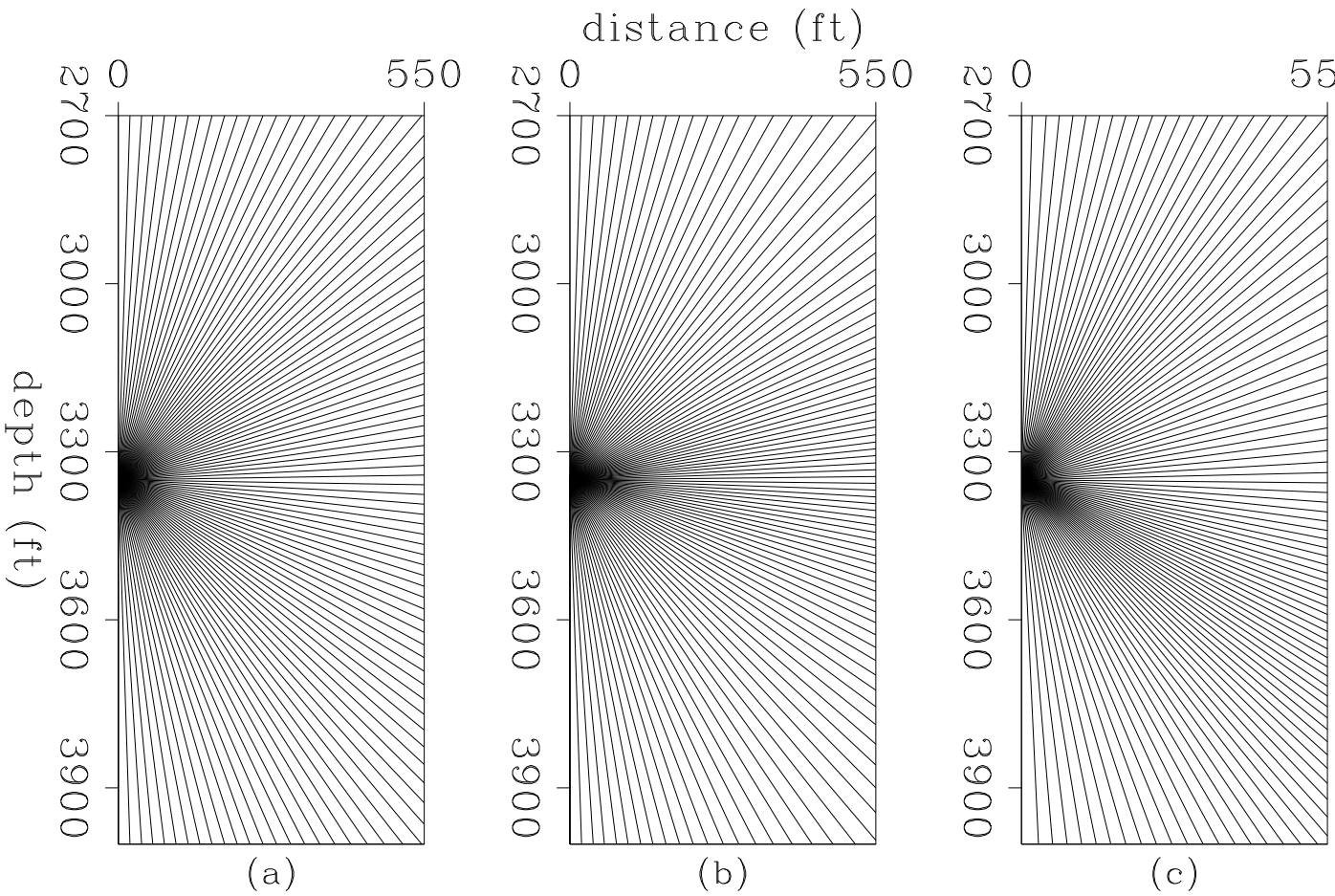


Figure 6: Ray paths in different homogeneous models for a source located at 3350 ft. (a) Isotropic. (b) Transversely isotropic with vertical axis of symmetry. (c) Transversely isotropic with $\gamma = -30$. Note how the density of rays changes with angles in the anisotropic models. reinaldo2-rays-homo [NR]

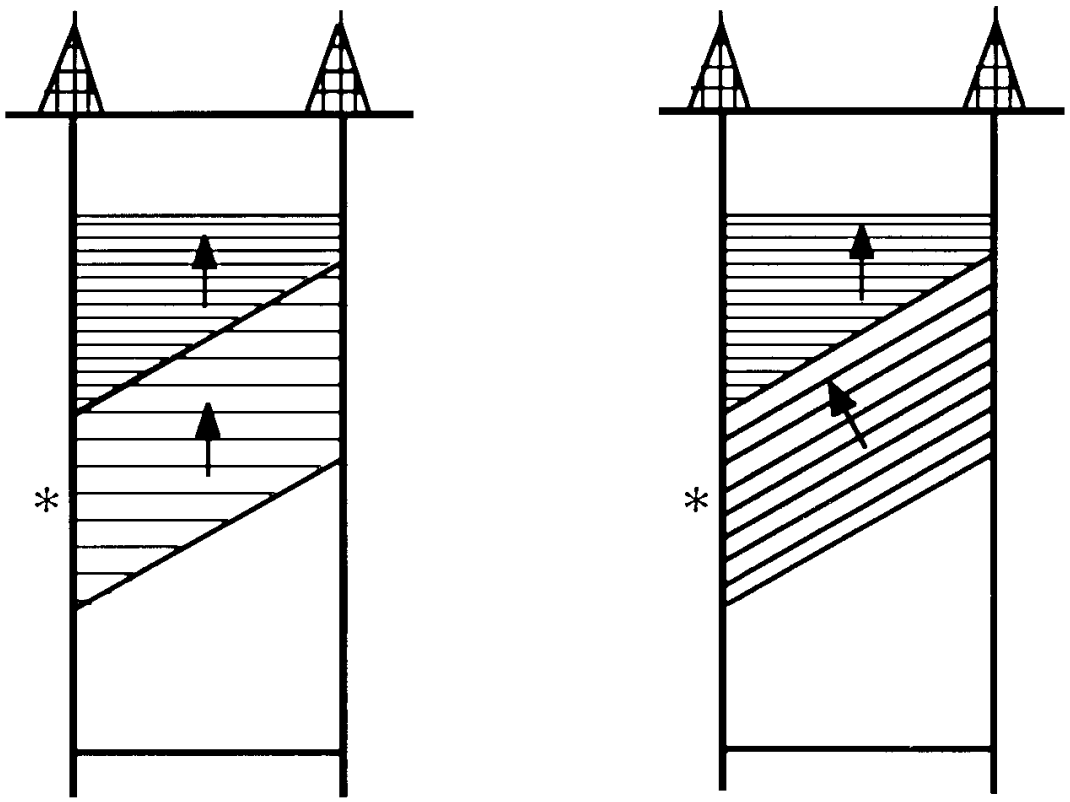

Figure 7: Anisotropic layered models I used to test the ray tracing. Two of the four homogeneous blocks are anisotropic with orientation of the axes of symmetry indicated by the arrows. The axis of symmetry is perpendicular to direction of fine layering. In both models, the layers 1 and 4 (from top to bottom) are isotropic. The third block of thin layers in the model at the left ($\gamma = 0$) can be the result of deposition in an eroded boundary whereas the same block in the model at the right ($\gamma = 30$) can be the result of tilting after deposition over a plane boundary. The position of the source is indicated by "$_*$". reinaldo2-models [NR]

angle (for $P$ and $SV$ waves) and analytically for $SH$ waves. The appropriate root must reproduce the $p$ of the incident phase for the given wave type, producing a ray in the desired direction.

This algorithm is used in a separate paper (Michelena, 1992) to do anisotropic traveltime tomography in layered media with strong velocity contrasts.

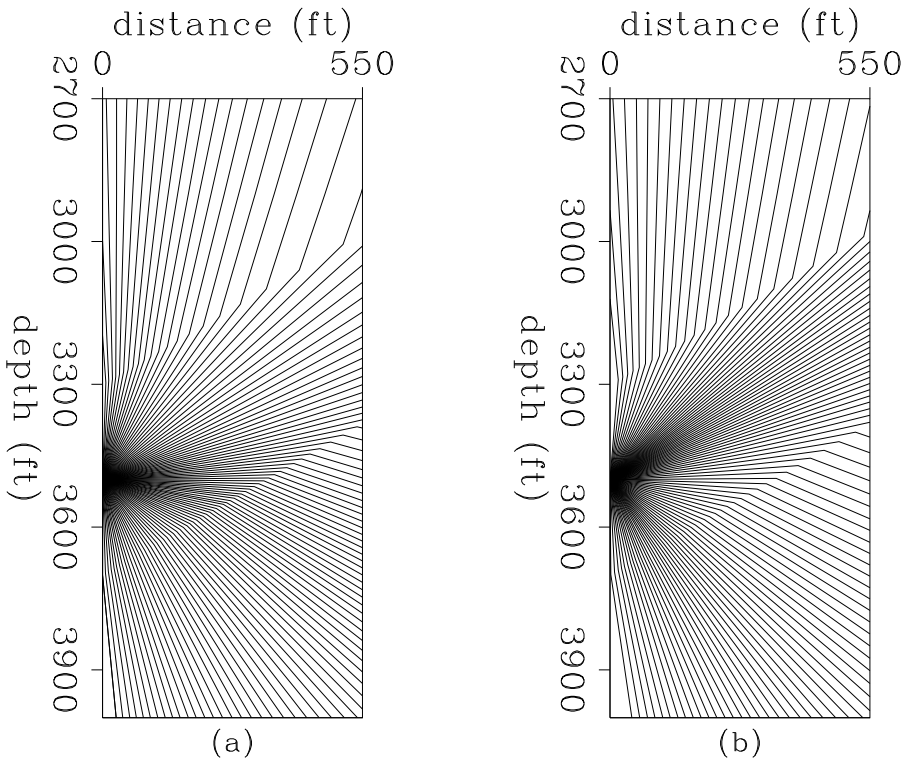


Figure 8: Ray paths in the models of Figure 7 for a source located at 3500 ft. (a) Axis of symmetry vertical. (b) Axis of symmetry tilted. reinaldo2-rays-hete [NR]

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## APPENDIX A

In this appendix, I show the explicit form of the coefficients $a_i$ in equation (8).

$$\begin{aligned} a_0 &= c^4 p^4 - T_0^2 \\ a_1 &= -2T_0 T_1 \end{aligned} \tag{12}$$

$$
\begin{aligned}
a_2 &= 2b^2c^2p^4 - 4d^2p^4 - T_1^2 - 2T_2T_0 \\
a_3 &= -2T_2T_1 \\
a_4 &= p^4b^4 - T_2^2 \\
T_0 &= 2\sin^2(\gamma_2) - p^2(c^2 + 2a^2) \\
T_1 &= 2\sin(2\gamma_2) \\
T_2 &= 2\cos^2(\gamma_2) - p^2(b^2 + 2a^2) \\
a^2 &= c_{44}/\rho \\
b^2 &= (c_{11} - c_{44})/\rho \\
c^2 &= (c_{33} - c_{44})/\rho \\
d^2 &= [(c_{11} - c_{44})(c_{33} - c_{44}) - (c_{13} + c_{44})^2]/\rho^2
\end{aligned}
$$

where $c_{ij}$ are four of the five elastic constants that describe an homogeneous transversely isotropic medium, $\gamma_2$ is the orientation of the axis of symmetry measured from the normal to the interface, $p$ is the ray parameter and $\rho$ is the density.

## APPENDIX B

The elastic constants of the models used in this paper to trace the rays for $P$ waves are the following:

| | Layer 1 | Layer 2 | Layer 3 | Layer 4 |
|---|---|---|---|---|
| Figure 4 (center and right) | $c_{11} = 9894$<br>$c_{33} = 9005$<br>$c_{44} = 4949$<br>$c_{13} = 5655$ | | | |
| Figure 5 | $c_{11} = 9005$<br>$c_{33} = 9005$<br>$c_{44} = 4949$<br>$c_{13} = 5655$ | $c_{11} = 9894$<br>$c_{33} = 9005$<br>$c_{44} = 4949$<br>$c_{13} = 5655$ | $c_{11} = 12545$<br>$c_{33} = 10845$<br>$c_{44} = 5968$<br>$c_{13} = 6805$ | $c_{11} = 9005$<br>$c_{33} = 9005$<br>$c_{44} = 4949$<br>$c_{13} = 5655$ |

In all cases, $\rho = 1$. $c_{ij}/\rho$ has units of velocity squared $(ft/s)^2$.