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Reciprocity of tensor wavefields

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ABSTRACT

If reciprocity principles are invoked in seismic data processing algorithms, one has to make sure that those processes maintain reciprocity. I outline briefly the derivation of reciprocity principles for elastic wave equations and show that discretized or approximated wave equations can lose symmetry properties and thus reciprocities. This effect is important if one is concernd about true-amplitude processes and if one needs to use reciprocity arguments. I show an example of an elastic finite-difference approximation to the wave equation that is not reciprocal, but can be made reciprocal by symmetrizing the Green's function kernel properly.

INTRODUCTION

Reciprocity principles in wave propagation problems are well known and mathematical aspects are detailed in Morse and Feshbach (1953) and with applications to seismic data in Aki and Richards (1980). Those descriptions are based on a symmetry property of the Green's funtions for the underlying wave propagation operator. Full wave equation theory is the basis for those investigations. Knopoff and Gangi (1959) and Gangi (1980) verify reciprocity principles in measurements for seismic waves on the laboratory scale. Fenati and Rocca (1984) demonstrate reciprocity in field data to a remarkable degree, even though their source and receiver geometry/type were not exactly reciprocal. All these investigations have been concerned with full dynamic reciprocity principles, not just traveltime, but full waveform reciprocity. Razavy and Lenoachca (1986) have investigated the influence of analytical and numerical approximations on reciprocity principles, which becomes important when using approximate solutions and reciprocity arguments together in a wave propagation problem. Based on these findings, I will show that my particular finite-difference approximation to the elastic wave equations maintains reciprocity and I will show two field data examples of 9 component data showing reciprocity (or lack thereof).

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WAVE EQUATIONS AND RECIPROCITY

Given a set of dynamic equations, which describe the propagation of a wave in some medium, conservation principles lead to a reciprocal relation. Consider the set of dynamic equations

$$\rho \frac{\partial^2}{\partial t^2} u_i^{(1)} = X_i^{(1)} + \frac{\partial}{\partial x_i} \sigma_{ij}^{(1)} \tag{1}$$

$$\rho \frac{\partial^2}{\partial t^2} u_i^{(2)} = X_i^{(2)} + \frac{\partial}{\partial x_i} \sigma_{ij}^{(2)} \tag{2}$$

in a volume V, where u_i are components of a displacement vector, X_i are components of the body force vector and σ_{ij} are components of the stress tensor. The raised parentheses $^{(.)}$ indicate various positions of the source. These equations describe the force balance of a medium, without specifying the particular way in which stresses might be related to displacements. It is not necessary to assume any particular constitutive relation at this point. The force equations

$$\frac{\partial}{\partial x_i} \sigma_{ij}^{(1)} = f_i^{(1)} \qquad \frac{\partial}{\partial x_i} \sigma_{ij}^{(2)} = f_i^{(2)} \tag{3}$$

describe the distribution of boundary forces f_i on the enclosing surface S_s, when the body force $X_i^{(1)}$ and $X_i^{(2)}$ are applied while

$$u_i^{(1)} = g_i^{(1)} \qquad u_i^{(2)} = g_i^{(2)} \tag{4}$$

define the distribution of displacement vectors g_i on the surface S_u when the same body forces are applied. See Figure 1 for an illustration showing the state of the medium in the two cases. The above equations are a set of equations that describe the physics of motion and boundary

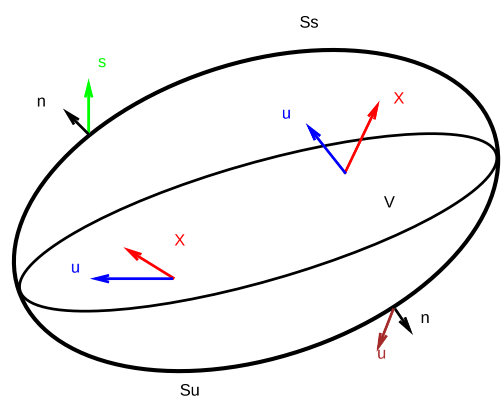

Figure 1: A medium with a certain volume V and surface S showing two different locations at which a body force is applied and the resulting displacement is measured. On the surface of the body, stresses and displacments may develop or may be specified as boundary conditions. martin1-betti [NR]

conditions of a wave propagation problem. The equations of motion (1) and (2) augmented by a constitutive relationship can be generally written in terms of a linear differential operator L

$$Lu = X \tag{5}$$

with appropriate boundary conditions (3) and (4). In further analysis I assume that initial acceleration and displacements are zero before some time and represent a causal wave propagation problem. Forming an inner product of the above equations with $u_i^{(2)}$ and $u_i^{(1)}$, respectively, leads in the time domain to a general integral relation. This relation is often referred to as Betti's reciprocal theorem (Aki and Richards, 1980). It relates the work done in each of the experiments.

$$\begin{aligned}
&\int_V \int_0^t X_i^{(1)}(x,t-\tau)u_i^{(2)}(x,\tau)d\tau dV \\
&+\int_{S_s} \int_0^t \sigma_{ij}^{(1)}(x,t-\tau)g_i^{(2)}(x,\tau)n_j d\tau dS \\
&+\int_{S_u} \int_0^t \sigma_{ij}^{(1)}(x,t-\tau)g_i^{(2)}(x,\tau)n_j d\tau dS \\
&=\int_V \int_0^t X_i^{(2)}(x,t-\tau)u_i^{(1)}(x,\tau)d\tau dV \\
&+\int_{S_s} \int_0^t \sigma_{ij}^{(2)}(x,t-\tau)g_i^{(1)}(x,\tau)n_j d\tau dS \\
&+\int_{S_u} \int_0^t \sigma_{ij}^{(2)}(x,t-\tau)g_i^{(1)}(x,\tau)n_j d\tau dS
\end{aligned}$$

It is noteworthy that the dependency of the stress field on the displacement field does not enter explicitly into this equation. In fact it is valid for a large variety of media (inhomogeneous, discontinuous, elastic, anisotropic, etc.). The above integral relation can now be used to derive special properties of the Green's function. The definition of a Green's function, is the solution of the impulse repsonse problem

$$L\ G(x,t|\xi,\tau) = -4\pi\delta(x-\xi)\delta(t-\tau) \tag{6}$$

A particular case of this Green's function would be the elastic case with causal initial conditions

$$\rho\frac{\partial^2}{\partial t^2}G_{in} = \delta_{in}\delta(x-\xi)\delta(t-\tau) + \frac{\partial}{\partial x_i}(c_{ijkl}\frac{\partial}{\partial x_l}G_{kn}) \tag{7}$$

in the Volume V, with initial conditions

$$G(x,t|\xi,\tau) = 0 = \frac{\partial}{\partial t}G(x,t|\xi,\tau). \tag{8}$$

If G satisfies homogeneous boundary conditions on S, $f_i = g_i = 0$, a relation between receiver and source positions is possible. Let $X^{(1)} = \delta_m(\xi_1,\tau_1)$ and $X^{(2)} = \delta_n(\xi_2,\tau_2)$, be impulsive forces in the m and n direction, then the displacements can be expressed as $u_i^{(1)} = G_{im}(x,t|\xi_1,\tau_1)$ and $u_i^{(2)} = G_{in}(x,t|\xi_2,\tau_2)$. Substituting those expressions in equation (6) results in a reciprocal relationship between the Green's tensor components:

$$G_{nm}(\xi_2,\tau+\tau_2|\xi_1,\tau_1) = G_{mn}(\xi_1,\tau-\tau_1|\xi_2,-\tau_2) \tag{9}$$

Choosing the reference time to be $\tau = 0$ leaves then the final reciprocal relation:

$$G_{nm}(\xi_2,\tau_2|\xi_1,\tau_1) = G_{mn}(\xi_1,-\tau_1|\xi_2,-\tau_2) \tag{10}$$

It is the spatial part of the reciprocity principle that I will use later.

Reciprocity and self-adjointness of operators

Reciprocity and self-adjointess of operators are closely related to each other. The adjoint of an operator L, defined generally as in (6), is obtained from the solution of the problem

$$L^* G^*(x,t|\xi,\tau) = -4\pi\delta(x-\xi)\delta(t-\tau) \tag{11}$$

L^* is then the adjoint operator of the operator L and G^* is the adjoint Green's function of L. The operator L is said to be self adjoint if $L = L^*$. To see how self-adjointness relates to reciprocity, use the generalization of the kernel in Green's theorem.

$$u^{(1)} \cdot L \cdot u^{(2)} - u^{(2)} \cdot L^* \cdot u^{(1)} = div \cdot P(u^{(1)}, u^{(2)}) \tag{12}$$

$u^{(1)}$ and $u^{(2)}$ are arbitrary solutions to the problem. $P(u^{(1)}, u^{(2)})$ is called the "bilinear concomittant" and is some linear combination of functions of $u^{(1)}$ and $u^{(2)}$. Assuming homogeneous boundary conditions on the surface S of the volume V, we can integrate Eqn. (12) in time and space and see that the right hand side vanishes leaving us with the relation:

$$u^{(1)} L u^{(2)} = u^{(2)} L^* u^{(1)} \tag{13}$$

The structure of equation (6) and equations (12) and (13) are very similar, in that a dot product between the dual fields X and $u^{(1)}$, and $u^{(1)}$ and $Lu^{(2)}$ are formed. If the dot product test, equation (13), for a self-adjoint operator is valid for any arbitrary solutions $u^{(1)}$ and $u^{(2)}$, then Betti's theorem is automatically satisfied. A convenient reciprocity principle for its Green's function can be derived. In contrast, however, the fact that an operator is reciprocal, does not imply self adjointness. An example of the latter would be the diffusion equation.

Reciprocity in approximations

As shown in the previous section, a reciprocity relation holds for a large variety of wave propagation problems. Reciprocity in a wave propagation problem may be defined as a symmetry property of the wavefield, due to symmetry of the Green's function. An important question raised by Ravazy and Lenoachca (1986) was whether an approximation to the original problem still preserves the original reciprocity relationships. Even if such an approximation is more accurate, it might implicitly result in a Green's kernel that is no longer symmetric and thus violates spatial reciprocity. They indeed found when investigating the scalar wave equation, that some analytical high frequency approximations and some numerical finite-difference approximations destroyed the reciprocity relationships of the original problem. It is therefore very important to verify that if reciprocity arguments are used to derive a data processing operation, the resulting algorithm and its numerical implementation should maintain reciprocity. Figure 2 shows a combination of two homogeneous elastic media in which a pair of source/receiver locations are marked. The two materials have different stiffnesses and densities. At both locations, a source with identical time history is activated and at both locations the wavefield is recorded. Source and receiver activate and register both x and z components. Figure 3 shows a plot of the four components of received wavefields. Across a row the receiver component is the same, while across the column the source component is the same. In each quadrant two seismograms are overlaid, one at location (1) the other at location (2). The diagonal plots show identical source and receiver components and the seismograms match perfectly. The off-diagonal plots clearly show nonidentical seismograms; the source component is different from the receiver component. Compare this to Figure 4, where the off-diagonal components show a perfect match. In contrast to Figure 3, reciprocal components are selected. All seismograms are now reciprocal and match perfectly. Thus the anisotropic elastic wave equation operator is symmetrically implemented using high order finite-differences on a staggered grid. Approximating the continuous wave equation has not broken the original symmetry.

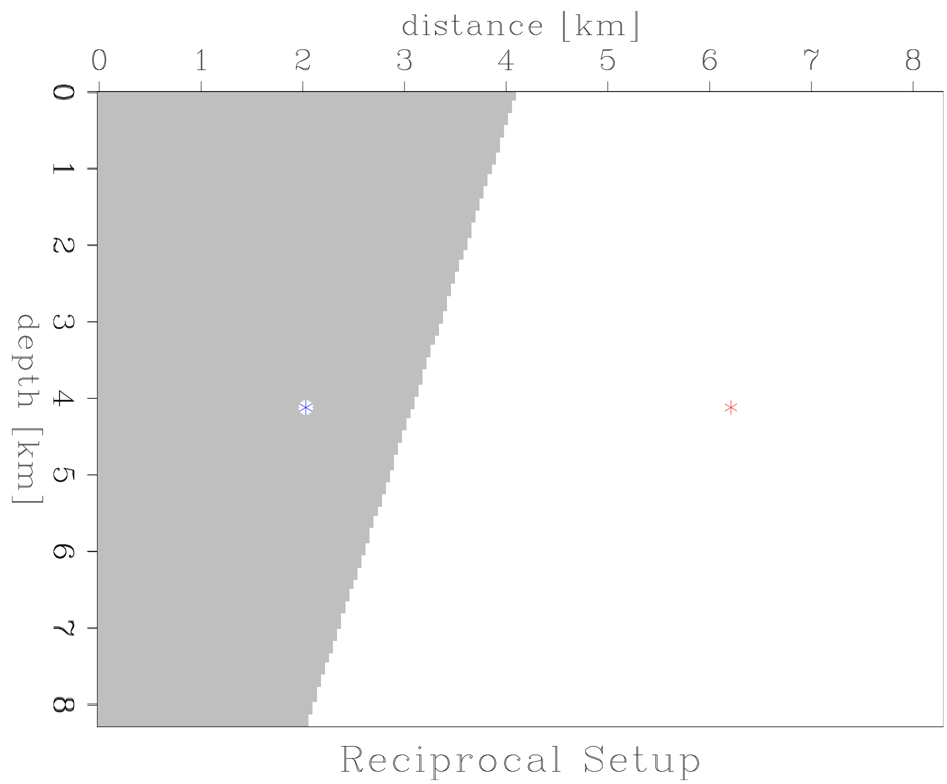

Figure 2: Test medium for a reciprocal multi component experiment. The left location (A) is hosted in a different medium that the right location (B). Both stiffnesses and densities are different at these locations. martin1-medium [ER]

Spatially bandlimited sources

The previous example was for a spatially impulsive source activated on the finite-difference grid. The spatial frequency content of the source thus generates frequencies up to the spatial Nyquist frequency. It is remarkable that even for the highest frequencies (where the finite-difference approximation becomes less accurate and dispersive) numerical reciprocity holds. However, from a physical point of view, sources on a finite-difference grid are usually not introduced as a spatial impulse but in a bandlimited manner in order to reduce the spatial

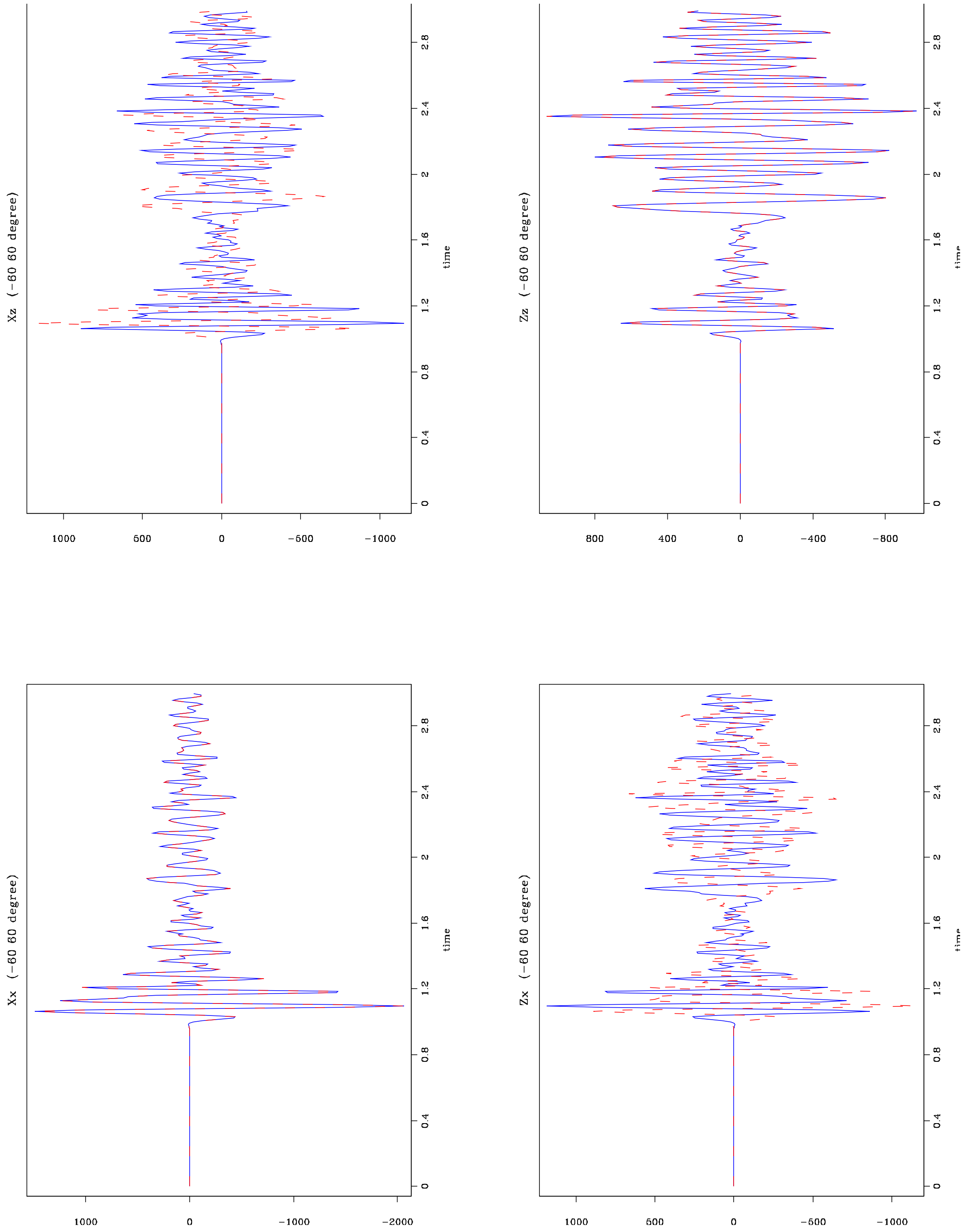

Figure 3: A spatially reciprocal experiment with non-reciprocal components is carried out in the medium shown in Figure 2. A source with an inclination of -60 degrees from the vertical is recorded into a receiver with an inclination of +60 degrees. The experiment is carried out at both locations. Shown are the direct recordings and the off-diagonal components are clearly non-reciprocal. martin1-drot.-60.60.i [ER]

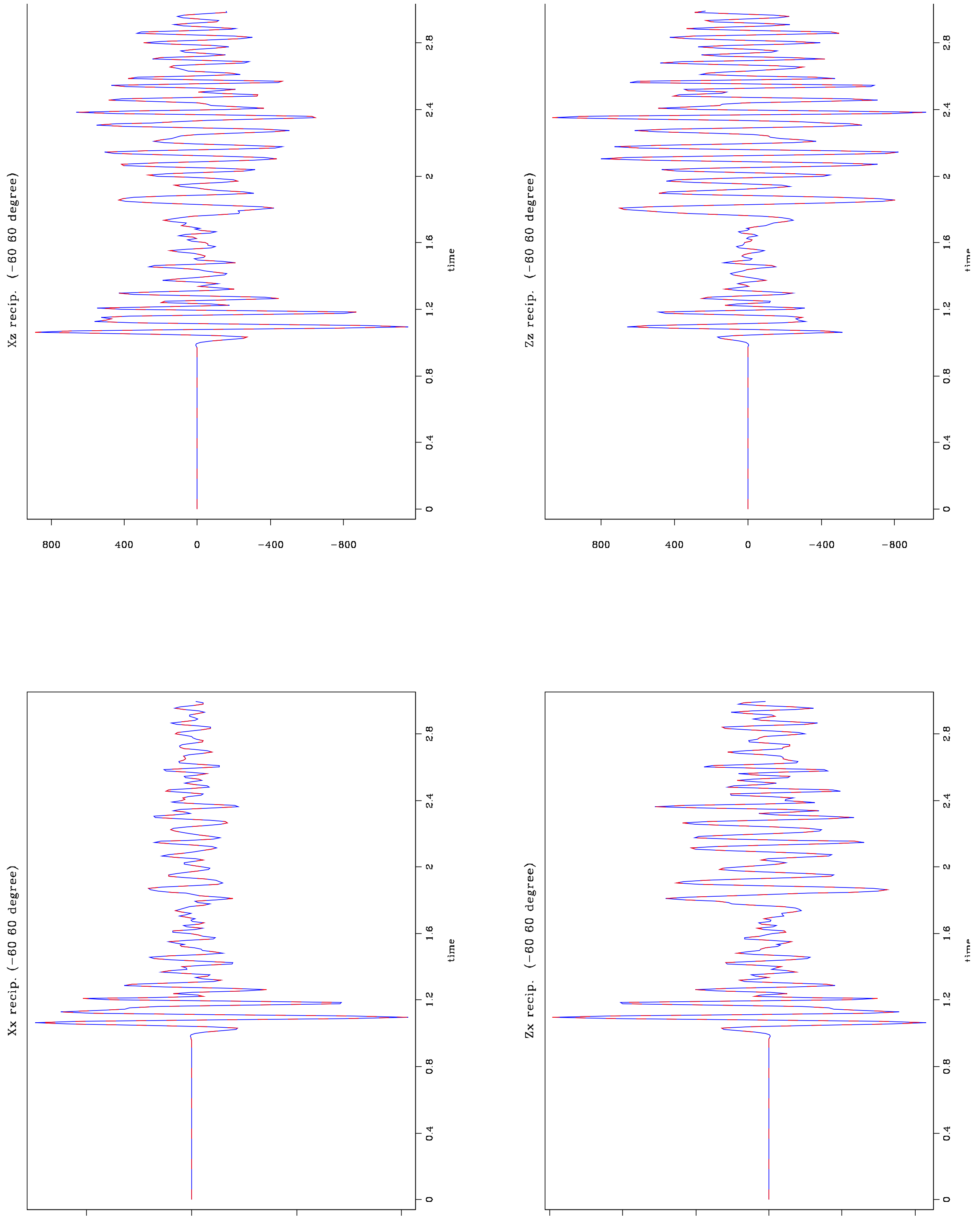

Figure 4: A reciprocal experiment is carried out in the medium shown in Figure 2. A source with an inclination of -60 degrees from the vertical is recorded into a receiver with an inclination of +60 degrees. The experiment is carried out at both locations. Shown are the reciprocal recordings and the all components do now match. This is in contrast to Figure 3. martin1-rrot.-60.60.i [ER]

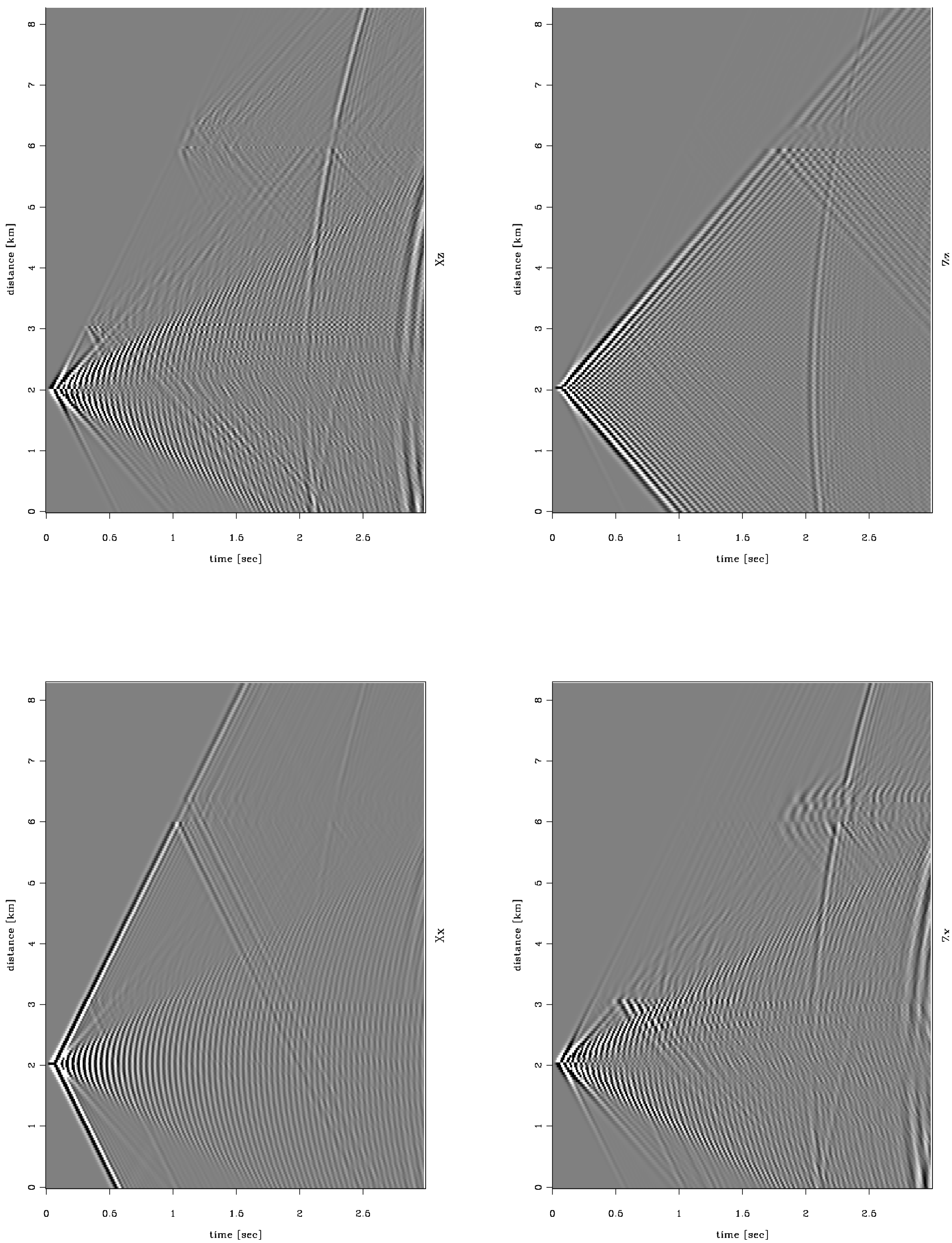

Figure 5: Four component shot gathers are modeled with finite-differences in the medium shown in Fig. 2. Fig. 3 and 4 compare reciprocal traces in these gathers. martin1-seismo.i [ER]

frequency content such that the difference operators are able to approximate spatial derivatives accurately. If reciprocity has to be maintained, then a spatially bandlimited receiver has to be used. In this way the duality of the experiment is maintained. Let P denote the staggered-grid finite-difference operator that propagates the entire wavefield, and S an operator that injects sources at certain locations in the medium; R denotes a related operator that extracts the wavefield at the receiver points. For point sources and receivers these two operators just consist of δ-functions at the source and receiver locations. For a source function f, the recorded data D are given by

$$D = \quad R \quad \cdot \quad P \quad \cdot \quad S \quad f \tag{14}$$

which in matrix form might appear schematically, like this

$$D = \begin{pmatrix} v_1 & v_2 & v_3 & & & \\ & v_1 & v_2 & v_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & & & & \\ \cdot & & & & \\ \vdots & & & \ddots & \\ \cdot & & & & \end{pmatrix} \begin{pmatrix} w_1 & & & \\ w_2 & w_1 & & \\ w_3 & w_2 & \ddots & \\ & w_3 & \ddots & w_1 \\ & & \ddots & w_2 \\ & & & w_3 \end{pmatrix} \quad f \tag{15}$$

Spatially bandlimited receivers and sources can be implemented using appropriate weight functions in the projection operators R and S. Commonly used weights are multi-dimensional Gaussian weights. f denotes the vector of impulsive sources on the gridded model, while P is the staggered-grid finite-difference modeling operator. Injection and extraction operators S and R maintain reciprocity if are transposes $R = S^t$. Figures 6 and 7 show the above example with bandlimited source, but using point receivers, hence not reciprocal. The Gaussian weight is of the general form $exp^{-\alpha[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]}$ and extends over four gridpoint halfwidth. The data traces match remarkably quite well, but deviations in the waveform are noticable. If now also the receiver is spatially bandlimited in the same way as the source, reciprocity is again restored and the waveforms match exactly. In many cases of data processing, imaging, inversion or optimization, reciprocity arguments are invoked. If such arguments are used, numerical implementations of operators should be designed to be reciprocal. I showed the staggered-grid finite-difference wave equation operator as one such example that when implemented conventionally is not quite reciprocal. However by symmeterizing the kernel, complete reciprocity can be obtained.

What does it mean for seismic data ?

Recording multicomponent seismic is most often carried out over a surface. The medium under investigation is parameterized by properties such as stiffness and density of the medium. Not knowing the medium completely, experiments are designed to give us the best information. Seismic data realistically are recorded at very sparse locations within the medium itself, never "at every point" in the medium. Thus the complete Green's tensor with complete spatial coverage is surely impossible to obtain and will always be bandlimited. For an elastic medium, the collected data will only approximate the Green's tensor, even if data are collected in a manner that spans the source and receiver component space. For an ideal experiment with

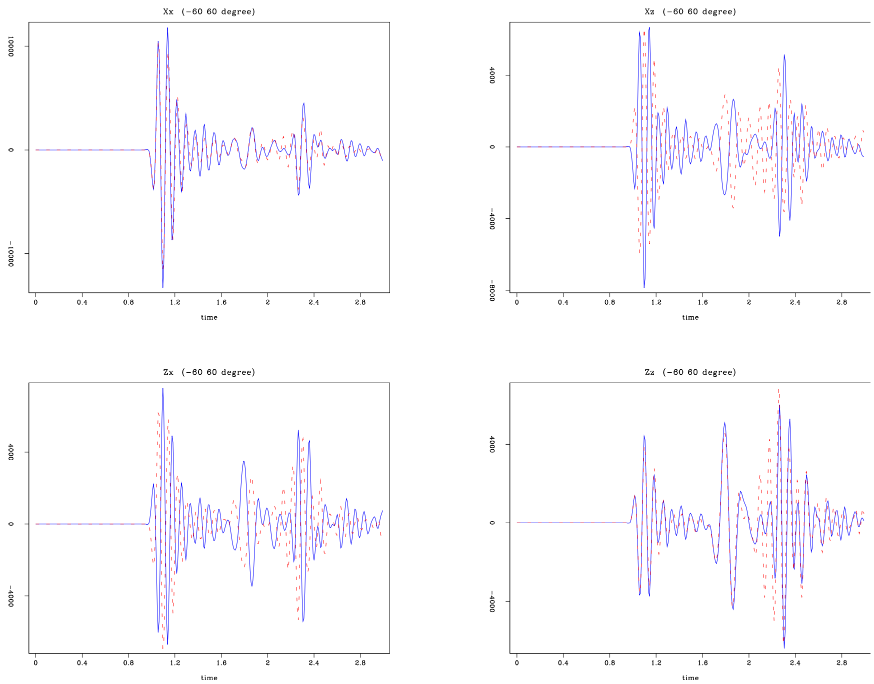

Figure 6: Spatially bandlimited source, point receiver. martin1-drot.-60.60.b [ER]

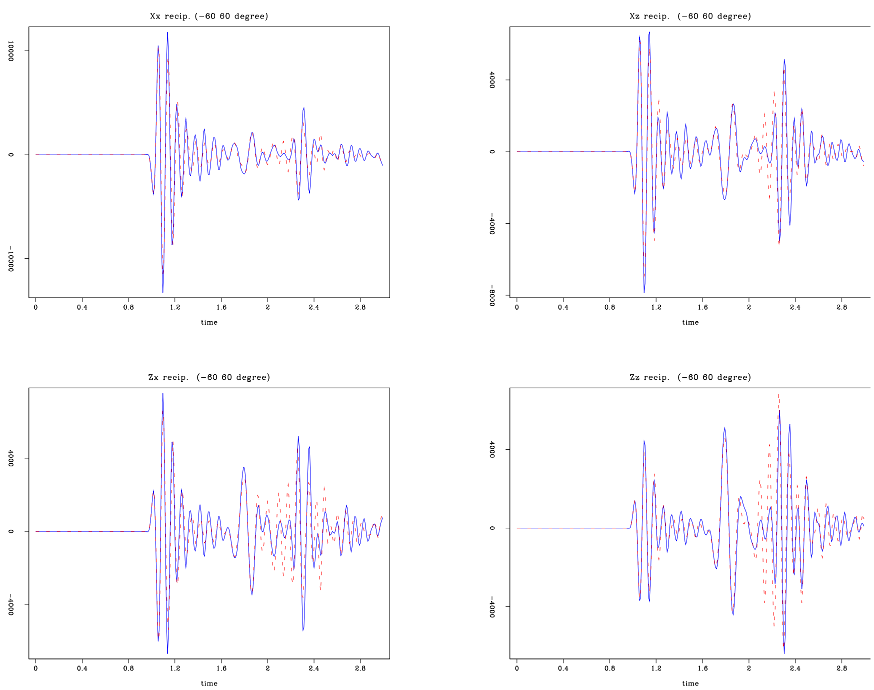

Figure 7: Spatially bandlimited source, point receiver. reciprocal traces. martin1-rrot.-60.60.b [ER]

multi-component sources and receivers, we should use at least 3 linearly independent source directions and 3 linearly independent receiver directions. Under ideal conditions then we record the full bandlimited Green's tensor G_{ij} $i, j = 1,2,3$ at a given source-receiver pair. Using fewer components might lead to ill-conditioned inversion and imaging results. Types of such incomplete experiments: p source into 3 component receivers (missing other two wave types) or vertical source into horizontal geophones. One is lucky if there are near-source wave-type conversions such that significant amounts of the "originally missing" source-wave types are generated. However, the experiment is not orthogonal, but rather a superposition of those wave type experiments.

HOW CAN IT BE USED?

Knowing that reciprocity holds for arbitrary inhomogeneous and arbitrary aniso-tropic media, we can make use of the reciprocity relationship in various ways. The most commonly used practice is to reduce data acquisition by assuming ideal recording conditions. Then, only part of the data have to be collected and the rest can be inferred by invoking the reciprocity relationship. Under this assumption one has only to collect G_{Xy}, G_{Xz} and G_{Yz} in addition to G_{Xx},G_{Yy} and G_{Zz}, because by reciprocity $G_{Yx} = G_{Xy}$, $G_{Xz} = G_{Zx}$, $G_{Yz} = G_{Zx}$. The emphasis lies here on "ideal" conditions. The other option is to claim real world conditions are never ideal and to use the data redundancy to estimate other than material parameters, namely source or receiver variability or classification of noise sources. But to justify such an approach it is very important to determine the degree to which real data typically is "not reciprocal". Thus reciprocity measurements give experimenters a much needed handle on how accurate and reproducible their data are.

FIELD DATA: ARE THEY RECIPROCAL ?

The Pembrook data (traces shown in the next Figures) has split spread geometry and thus is ideal for comparing reciprocal trace pairs. During the acquisition no particular effort has been made to ensure that the employment of sources and geophone in the field was perfectly reciprocal. In that respect this dataset presents a typical acquisition for multi-component data. Surface impact sources were used for acquiring this data set. Such sources are relatively weak compared to explosive sources. Consequently each source is not only activated once, but multiple times and the resulting traces stacked to create the final field trace. Furthermore each source is activated at an angle with respect to the vertical, so that horizontal and vertical components can be created by weighted subtraction or addition. Figures 8 through 13 show reciprocal trace pairs at two different offsets (near and far). None of the trace pairs is perfectly reciprocal, but in general the trend is the same. Most noticable are time shifts and amplitude differences for each trace pair. The differences persist over the whole length of the trace with good fits at various times. Such variations in the reciprocal match can be explained by noise sources that are not part of the reciprocal experiment, such as drill, pumping or surface noise.

These questions then remain:

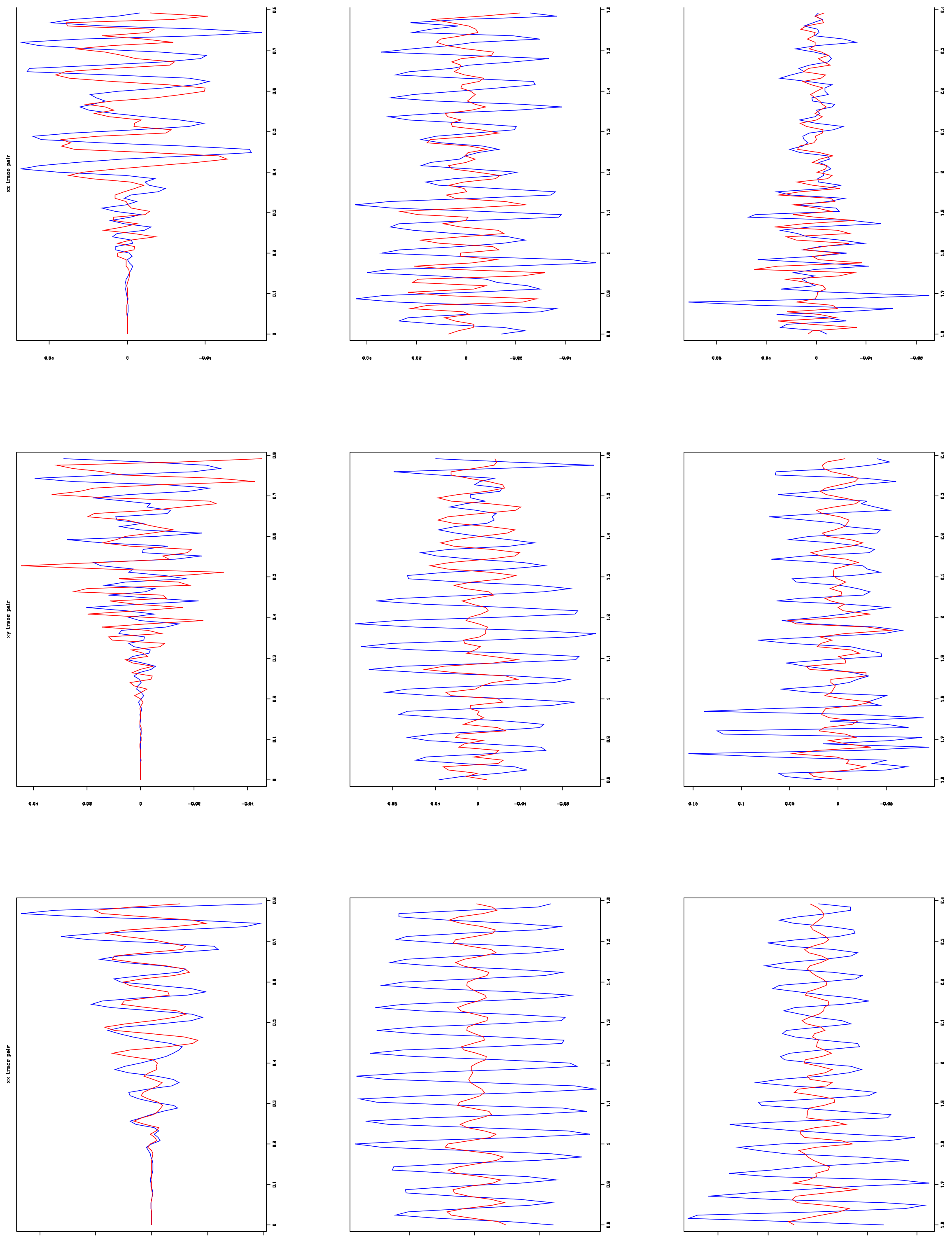

Figure 8: X source near offset reciprocal trace comparisons. Left column is x receiver, middle column is y receiver and right column is z receiver. Time progresses from top to bottom. martin1-PemXxyz10 [ER]

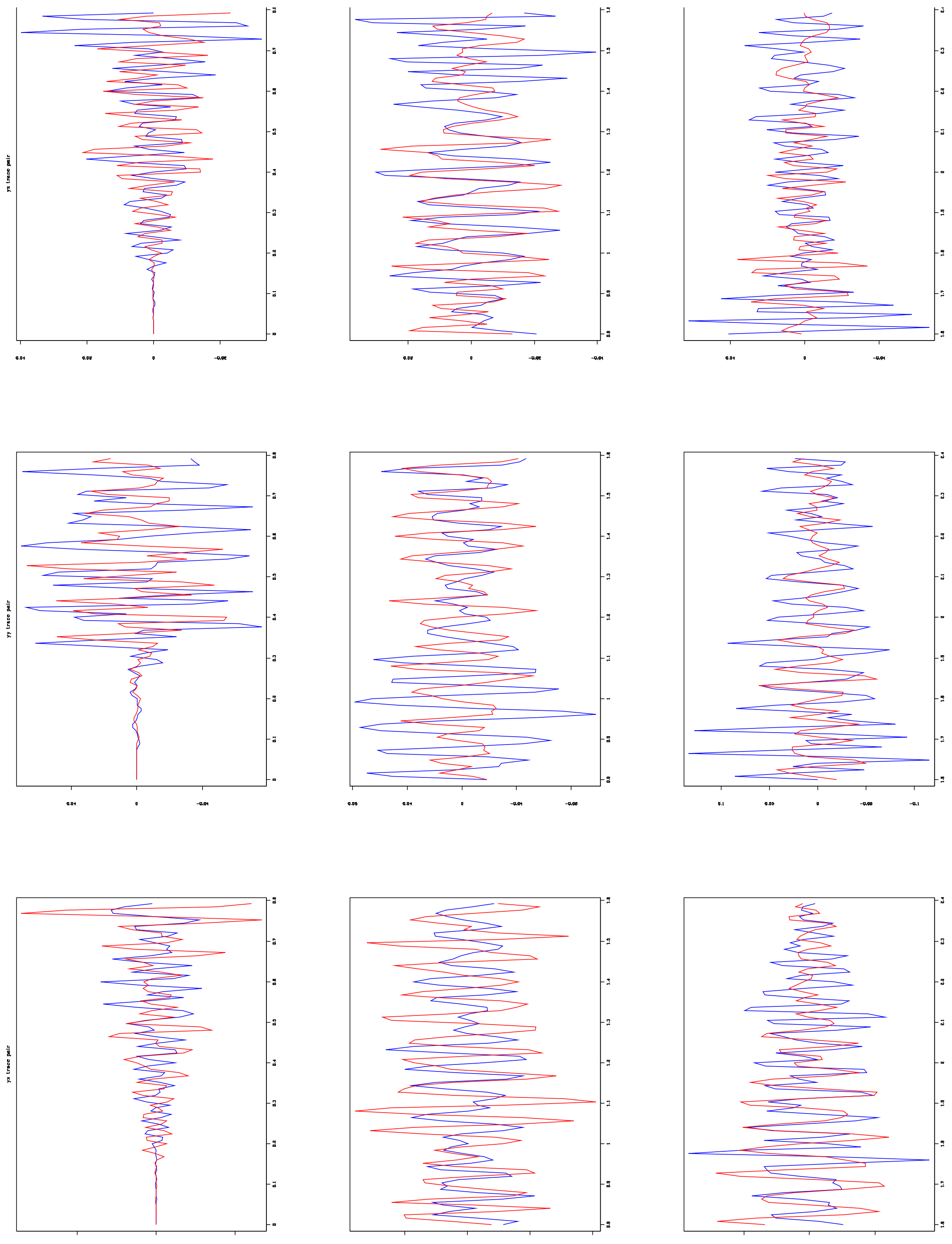

Figure 9: Y source near offset reciprocal trace comparisons.Left column is x receiver, middle column is y receiver and right column is z receiver. Time progresses from top to bottom. martin1-PemYxyz10 [ER]

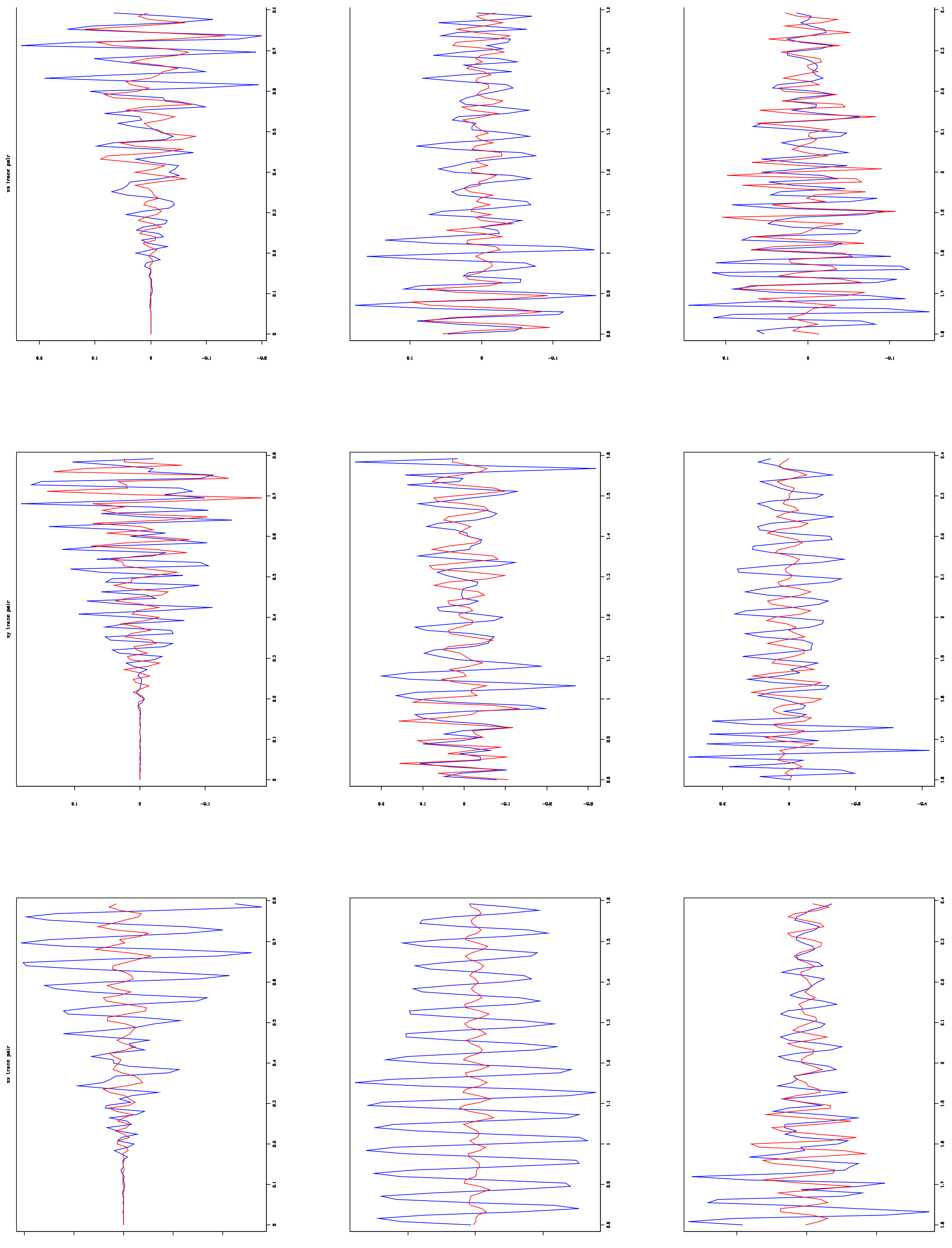

Figure 10: Z source near offset reciprocal trace comparisons.Left column is x receiver, middle column is y receiver and right column is z receiver. Time progresses from top to bottom. martin1-PemZxyz10 [ER]

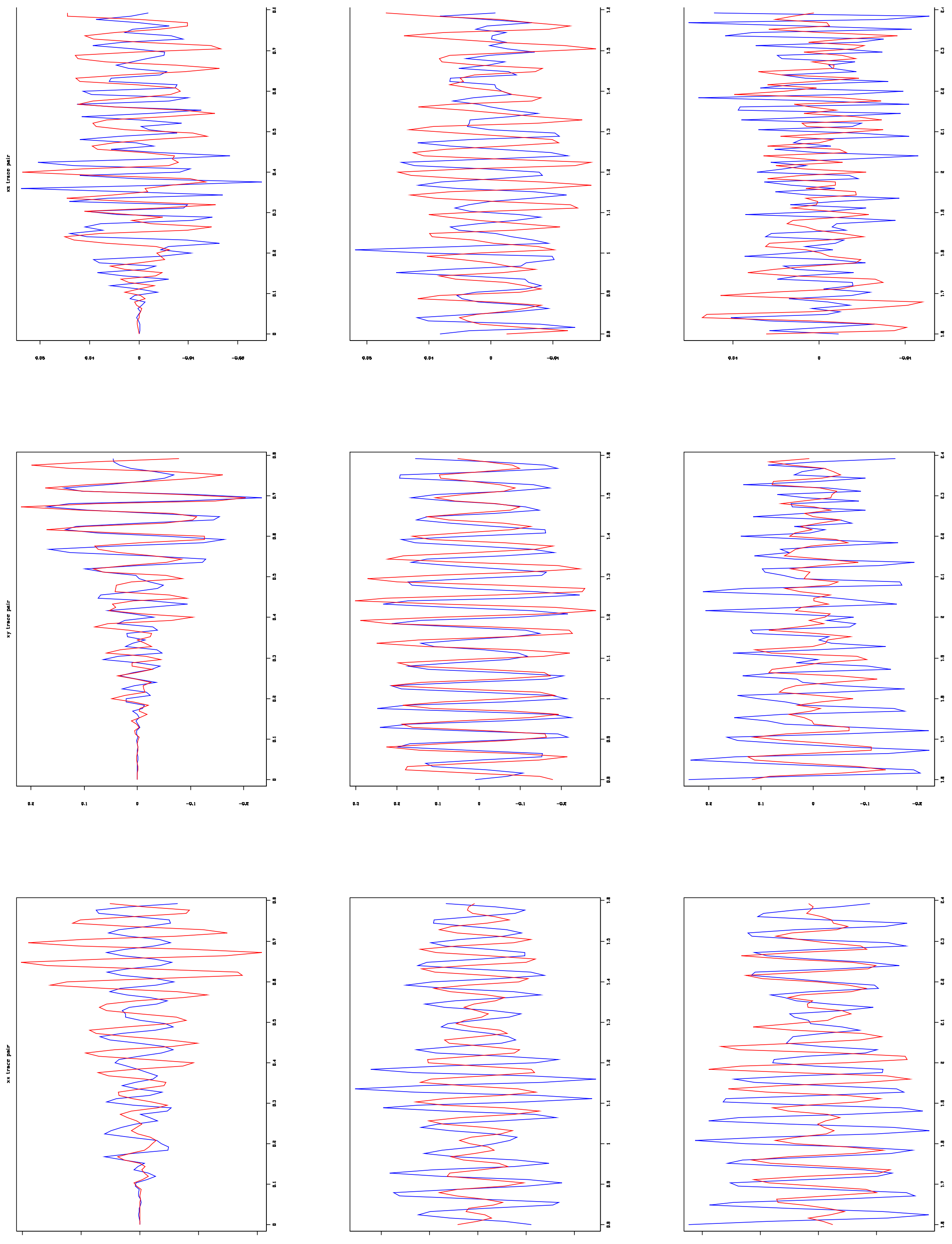

Figure 11: X source far offset reciprocal trace comparisons.Left column is x receiver, middle column is y receiver and right column is z receiver. Time progresses from top to bottom. martin1-PemXxyz20 [ER]

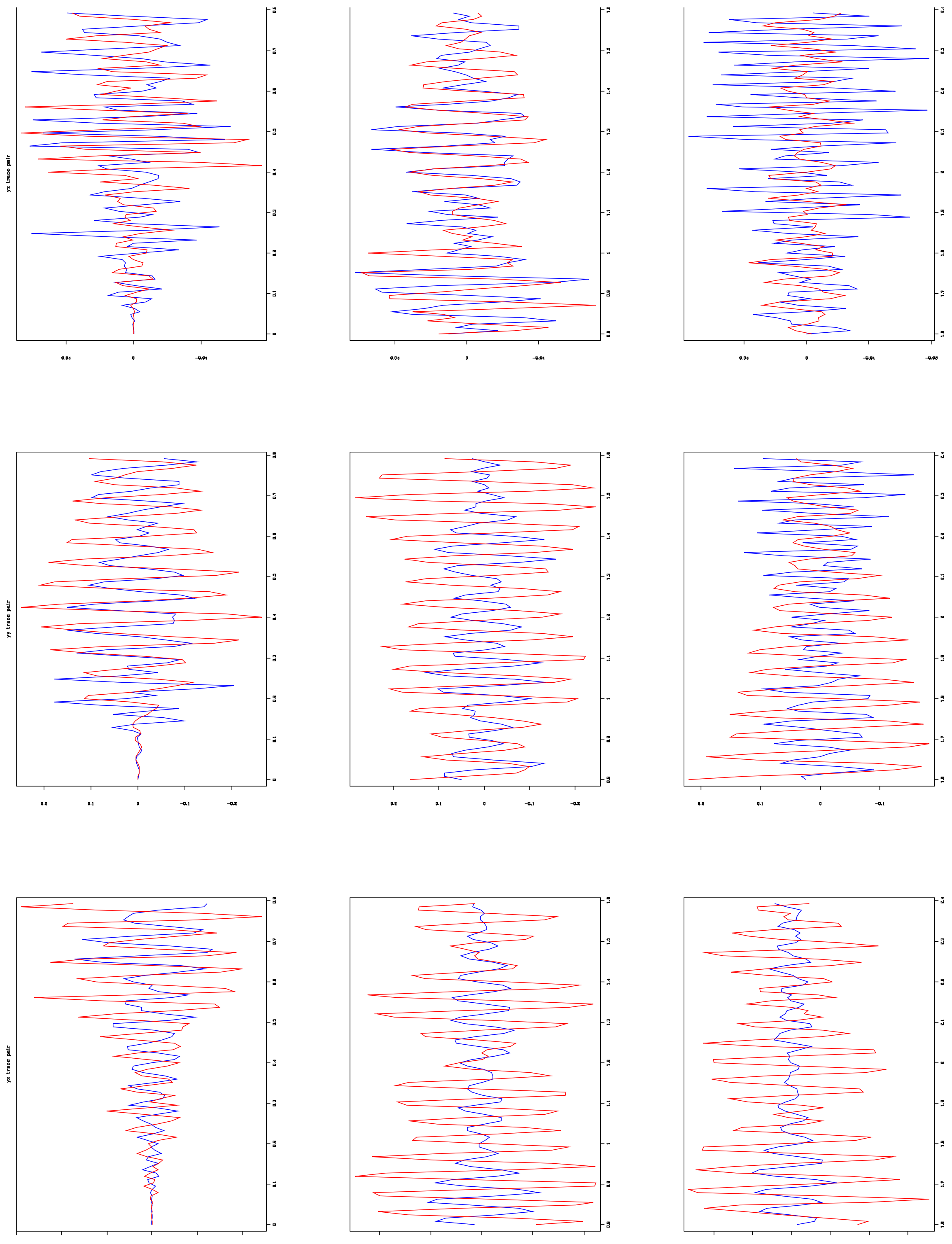

Figure 12: Y source far offset reciprocal trace comparisons.Left column is x receiver, middle column is y receiver and right column is z receiver. Time progresses from top to bottom. martin1-PemYxyz20 [ER]

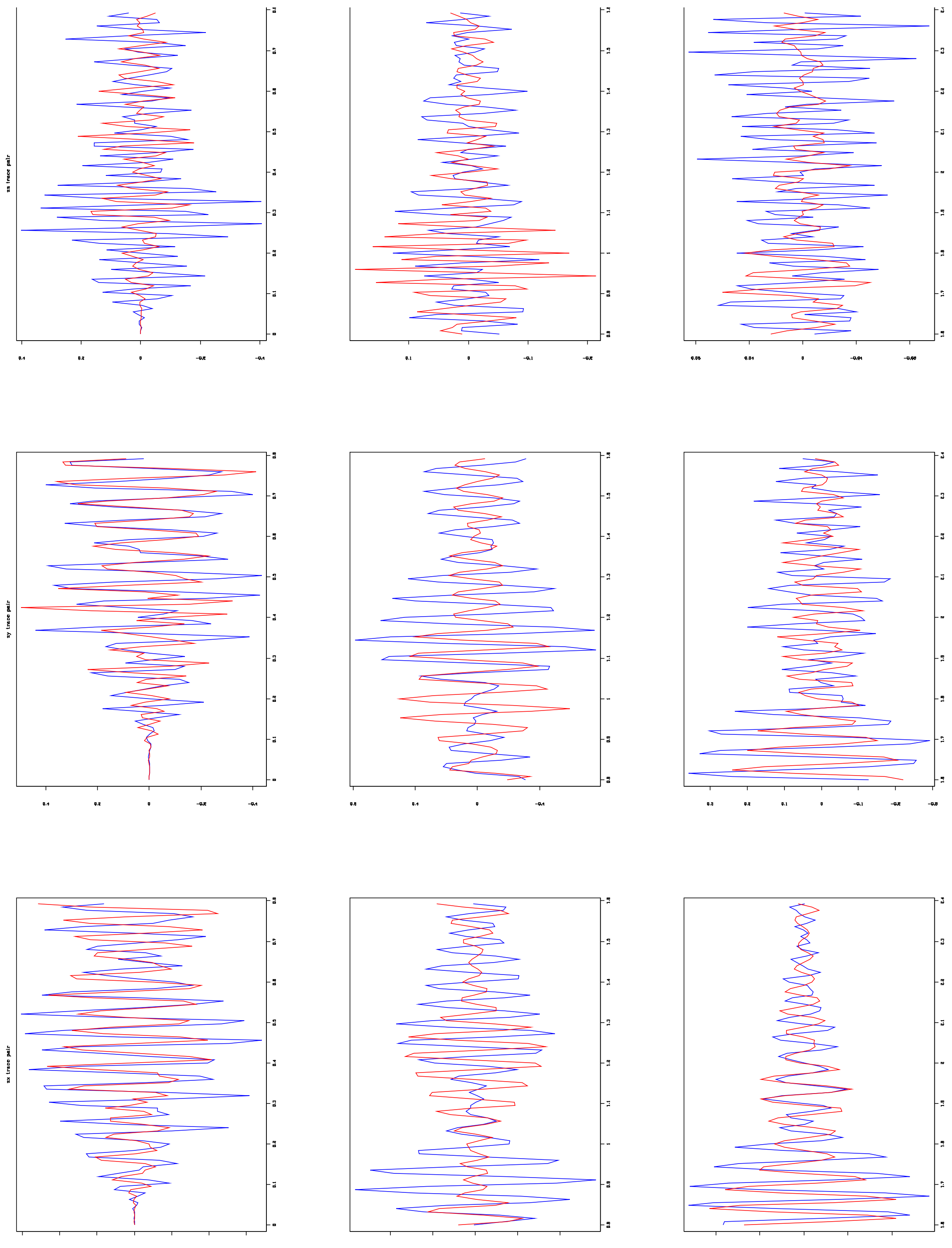

Figure 13: Z source far offset reciprocal trace comparisons.Left column is x receiver, middle column is y receiver and right column is z receiver. Time progresses from top to bottom. martin1-PemZxyz20 [ER]

- Are the differences in the reciprocal trace pairs significant ?
- To what mechanism do we attribute them ?
- Can we estimate source and receiver properties ?

The next section tries to give a reason as to why we should atrribute them to differences in the source behaviour.

BLAME IT ON THE SOURCE

Differences in reciprocal trace pairs can have many causes. One reason can be the uncertainty in source and receiver positioning, or in case of multi-component sources the misalignment of components. These two causes have nothing to do with the medium itself, but rather are purely geometrical effect. It may also be that a source is not behaving identically at each location. Nonlinearities in the source surface interaction and noise may prevent exact duplication of otherwise reciprocal data. Nonlinearity is important in this aspect, since Green's functions are for linear equations and nonlinear effects have unknown consequences for reciprocity. If we assume that noise is random with respect to the reciprocal experiment and that receivers respond reasonably isotropically, we can blame mismatches mainly on the source.

SUMMARY

I have outlined the importance of symmetrizing Green's function kernels of elastic wave propagation operators. I have shown an example of a non-reciprocal, but very accurate modeling operator. Symmetrizing the kernel achieved total reciprocity, however, for a spatially bandlimited source, it also required a spatial bandlimiting of the receivers. Numerical methods should always be developed such that they are accurate but still inherently reciprocal.

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