Stanford Exploration Project, Report 80, May 15, 2001, pages 1–424

Chapter 1: From prestack migration to migration to zero offset

Alexander M. Popovici[1]

ABSTRACT

As a condition for further generalization of the migration to zero-offset in variable velocity media, I develop the theory for 2-D migration to zero offset (MZO) in constant velocity media, starting from prestack migration in midpoint-offset coordinates. At the end of this chapter I arrive at an integral formulation for the MZO operator, analytically derived from the double square root (DSR) prestack migration equation. The integral formulation for the MZO is very similar in form to the DSR equation, suggesting a generalization to variable velocity media via a phase-shift algorithm. Further chapters deal with offset separation and the depth variable $v(z)$ and laterally variable $v(x,z)$ velocity media.

Introducing the Double Square Root Equation

The theory for the double square root (DSR) equation is discussed in detail in the first chapter of Yilmaz's (1979) thesis. Without going into mathematical detail I will sketch the path of the basic theory for obtaining the DSR migration equation in offset and midpoint coordinates starting from the wave equation. Readers familiar with the DSR equation can skip directly to the next section. The scalar wave equation in a 2-D medium of constant density can be written as

$$\frac{\partial^2 p}{\partial z^2} + \frac{\partial^2 p}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2}, \tag{1}$$

where $p = p(t,x,z)$ is the pressure field and $v = v(x,z)$ is the earth velocity. The pressure field $p(t,x,z)$ is a finite function and can be therefore expressed as a double Fourier series

$$p(t,x,z) = \sum_{k_x}\sum_{\omega} P(\omega,k_x,z)e^{i(k_x x-\omega t)}. \tag{2}$$

Substituting equation (2) in equation (1) we obtain

$$\sum_{k_x}\sum_{\omega}[\frac{\partial^2 P(\omega,k_x,z)}{\partial z^2} - k_x^2 P(\omega,k_x,z) + \frac{\omega^2}{v^2(x,z)}P(\omega,k_x,z)]e^{i(k_x x-\omega t)} = 0. \tag{3}$$

Equation (3) should hold for any values of k_x and ω. This is possible only if each term inside the square parenthesis is zero. This reasoning is similar to the condition that if a polynomial

[1]**email:** not available

is zero for any values of x, the coefficients of the polynomial are zero. Therefore we have

$$\frac{\partial^2 P(\omega,k_x,z)}{\partial z^2} = (k_x^2 - \frac{\omega^2}{v^2(x,z)}) P(\omega,k_x,z) \tag{4}$$

valid for all values of k_x and ω. The problem is that in this form, the x-coordinate in the pressure field is Fourier transformed and there is no direct correspondence between a point (x,z) in the medium, the velocity $v(x,z)$, and the corresponding value of $p(t,x,z)$ at that location. For a constant velocity we write

$$k_z = \left[\frac{\omega^2}{v^2} - k_x^2 \right]^{\frac{1}{2}} \tag{5}$$

where k_z is constant for two given values of k_x and ω. Equation (5) is the well known **dispersion relation**. Equation (4) becomes an ordinary differential equation

$$\frac{\partial^2 P}{\partial z^2} = -k_z^2 P. \tag{6}$$

For a constant k_z equation (6) has the analytic solution

$$P = P_1\, e^{ik_z z} + P_2\, e^{-ik_z z} \tag{7}$$

To find the solution to equation (7) we would need to have two initial or boundary conditions. We only have the pressure field at $z = 0$ as a boundary condition, but we can still solve the problem if we decide to resolve only the upgoing waves, in other words to use the exploding reflectors principle. If we know the pressure field (or wavefield) at a certain depth we can propagate it forward in time or backward in time. We can also propagate it up in depth (along the z-axis) or down. To understand how we determine the propagation direction we have to analyze the values and sign of k_z. The function

$$e^{i(k_z z + k_x x - \omega t)}$$

represents a plane wave. If we ignore $k_x x$, which determines the lateral variation, we can introduce a function which we call $phase(z,t)$ defined as

$$phase(z,t) = k_z z - \omega t.$$

The phase is constant along a plane wave, and we write

$$k_z z = \omega t + const$$

for the phase of a particular plane wave. The plane wave is moving **downward** when k_z has the same sign with ω because z increases with t in order to keep the phase constant. So for the **upward** moving waves we need to have opposite signs of k_z and ω (z is decreasing when t is increasing). We have now figured out that in order to have only **upgoing** waves we have to look at the sign of ω and assign to k_z the opposite sign. Therefore equation (7) becomes:

$$P = \begin{cases} P_1\, e^{ik_z z} & ; \quad \omega \le 0; \\ P_2\, e^{-ik_z z} & ; \quad \omega \ge 0, \end{cases} \tag{8}$$

which can be written in a compact form as:

$$P = P_0 e^{-i\,\mathrm{sign}(\omega) k_z z}, \tag{9}$$

where

$$\begin{cases} P_0 = P_1 \ for\ \omega \le 0; \\ P_0 = P_2 \ for\ \omega \ge 0. \end{cases}$$

Setting $z = 0$ in equation (9) we identify P_0 as the data recorded at the surface:

$$P_0 = P(\omega, k_x, z = 0).$$

In this form we can use the data recorded at the surface $P(\omega, k_x, z = 0)$ to propagate the wavefield to any depth level

$$P(\omega, k_x, z) = P(\omega, k_x, z = 0) e^{i k_z z}. \tag{10}$$

The object of zero-offset migration is to estimate $P(t = 0, k_x, z)$ from $P(t, k_x, z = 0)$. Knowing the wavefield at any depth z_0 we can find the wavefield at any other depth $z_0 + z$. For positive values of z we have to propagate the wavefield back in time (toward $t = 0$) because we know that the wavefield travels upward. If the known wavefield is at depth z_0 and we want to find the wavefield at depth $z_0 - z$, then we are propagating the wavefield forward in time. This is the direction we use for modeling. However for depth varying velocity $v(z)$ we have k_z approximately constant only for small depth intervals (Δz) where we can consider the velocity constant. Therefore equation (10) becomes

$$P(k_x, z_0 + \Delta z, \omega) = P(k_x, z_0, \omega) e^{i k_z \Delta z} \tag{11}$$

and can be used to downward or upward extrapolate the wave field for a small depth interval. There are several restrictions on the values of k_z. Equation (6) has the solution (7) only for real values of k_z which imposes the condition

$$\frac{\omega^2}{v^2} - k_x^2 \ge 0.$$

The solution represented in equation (10) is for a single Fourier transform component of the wavefield. The general solution in time-space coordinates is obtained by summing all the Fourier coefficients obtained from equation (10)

$$p(t, x, z) = \sum_{k_x} \sum_{\omega} P(\omega, k_x, z_0) e^{i k_z z} e^{i(k_x x - \omega t)}. \tag{12}$$

In the case of a seismic experiment with many shots and receivers we can downward continue separately the shots and the receivers to any depth level. The total phase shift to the same depth level z becomes the phase shift of the shots plus the phase shift of the receivers.

$$k_z(\omega, k_g, k_s) z = -\mathrm{sign}(\omega) \left[\sqrt{\frac{\omega^2}{v^2} - k_g^2} + \sqrt{\frac{\omega^2}{v^2} - k_s^2} \right] z \tag{13}$$

where k_s and k_g are the shot and receiver wavenumbers. It is assumed here that the shots and geophones are on a flat surface at zero depth $z = 0$. We can change the system of coordinates from shot and receiver to midpoint and offset using the simple relations:

$$\begin{aligned} y &= \frac{x_g + x_s}{2} \\ h &= \frac{x_g - x_s}{2}, \end{aligned}$$

where y and h are respectively the midpoint and offset coordinates, while x_s and x_g are the shot and geophone surface coordinates. Note that the variable h represents half the total distance between the source and geophone. The total phase shift in the new wavenumber coordinates becomes

$$k_z(\omega, k_y, k_h)z = -\text{sign}(\omega)\left[\sqrt{\frac{\omega^2}{v^2} - (\frac{k_y + k_h}{2})^2} + \sqrt{\frac{\omega^2}{v^2} - (\frac{k_y - k_h}{2})^2}\right] z \tag{14}$$

where k_y and k_h are the midpoint and offset wavenumbers and z represents the depth level to which the wavefield was extrapolated. This formulation allows a wavefield organized in midpoint-offset coordinates to be downward continued to a certain depth level, and it forms the basis for the prestack migration in midpoint and offset coordinates shown in equation (15).

Isolating the Zero-offset migration

The basic concept for analytically deriving the MZO from prestack migration is to separate the latter into two processes:

- Migration to zero offset.
- Zero-offset migration.

Once the zero-offset migration is extracted out of the prestack migration operator, it is assumed that what is left is in fact an operator which transforms the common-offset data into zero-offset data, hence the name of the operator: migration to zero offset. I define the migration to zero offset as the operation that converts a common-offset section into a zero-offset section. For a constant velocity medium this is equivalent to the sequence of normal moveout (NMO) followed by dip moveout (DMO). We start with the constant velocity **prestack migration** in offset-midpoint coordinates (Yilmaz, 1979) formulated as:

$$p(t = 0, k_y, h = 0, z) = \int d\omega \int dk_h \; e^{ik_z(\omega, k_y, k_h)z} p(\omega, k_y, k_h, z = 0) \tag{15}$$

where $p(\omega, k_y, k_h, z = 0)$ is the 3-D Fourier transform of the field $p(t, y, h, z = 0)$ recorded at the surface, using Claerbout's (1985) sign convention:

$$p(\omega, k_y, k_h, z = 0) = \int dt \; e^{i\omega t} \int dy \; e^{-ik_y y} \int dh \; e^{-ik_h h} p(t, y, h, z = 0).$$

The phase $k_z(\omega, k_y, k_h)$ is defined in the dispersion relation as

$$k_z(\omega, k_y, k_h) \equiv -\text{sign}(\omega)\left[\sqrt{\frac{\omega^2}{v^2} - \frac{1}{4}(k_y + k_h)^2} + \sqrt{\frac{\omega^2}{v^2} - \frac{1}{4}(k_y - k_h)^2}\right]. \tag{16}$$

The two integrals in ω and k_h in equation (15) represent the imaging condition for zero offset and zero time ($h = 0, t = 0$). The constant velocity **zero-offset migration** (Gazdag, 1978) can be formulated as:

$$p(t = 0, k_y, z) = \int d\omega_0 \, e^{i k_z(\omega_0, k_y) z} \, p(\omega_0, k_y, z = 0) \tag{17}$$

where $p(\omega_0, k_y, z = 0)$ is the 2-D Fourier transform of the field $p(t, y, z = 0)$. The phase $k_z(\omega_0, k_y)$ is defined in the dispersion relation as

$$k_z(\omega_0, k_y) \equiv -2 \, \text{sign}(\omega_0)\sqrt{\frac{\omega_0^2}{v^2} - \frac{k_y^2}{4}}. \tag{18}$$

In order to convert equation (15) into a form similar to equation (17), I use a change of variables from ω to ω_0 such that after integrating over the variable k_h, equation (15) will be transformed into the form:

$$p(t = 0, k_y, h = 0, z) = \int d\omega_0 \, e^{i k_z(\omega_0, k_y) z} \, p_0(\omega_0, k_y, z = 0),$$

where $p_0(\omega_0, k_y, z = 0)$ represents the zero-offset data field. The rationale for casting the prestack migration equation in this form is to identify the operations needed to obtain the zero-offset field from the common-offset field. The assumption is that the output of zero-offset migration and prestack-migration is the same image. By dissecting the prestack-migration and separating the zero-offset migration operator, we isolate the migration to zero-offset (MZO) operator. Using Hale's (1983) derivation, a new variable ω_0 is introduced in order to isolate the zero-offset migration operator. The expression for the new variable ω_0 is found by equating the dispersion relation for prestack migration to the dispersion relation of zero offset migration,

$$-\frac{2\omega_0}{v}\sqrt{1 - \frac{v^2 k_y^2}{4\omega_0^2}} = -\frac{\omega}{v}\left[\sqrt{1 - \frac{v^2}{4\omega^2}(k_y + k_h)^2} + \sqrt{1 - \frac{v^2}{4\omega^2}(k_y - k_h)^2}\right]$$

and squaring the two equations twice. This algebra is demonstrated in detail in Hale's thesis (1983), Appendix 3.A, and therefore I did not repeat it here. The final expression for the variable ω is found to be

$$\omega \equiv \omega_0\left[1 + \frac{v^2 k_h^2}{4\omega_0^2 - v^2 k_y^2}\right]^{\frac{1}{2}}, \tag{19}$$

where ω_0 is considered variable and k_y, k_h are constant. Substituting ω in equation (16), the downward continuation phase $k_z(\omega, k_y, k_h)$ is transformed into

$$k_z \equiv -2 \, \text{sign}(\omega_0)\sqrt{\frac{\omega_0^2}{v^2} - \frac{k_y^2}{4}} \tag{20}$$

which now has the same form as the phase in equation (18). The somewhat lengthy but straightforward algebraic proof is shown in Appendix A.

In order to isolate the zero-offset migration operator, after substituting the expression of ω given by equation (19) in the prestack migration equation (15, the order of integration is changed between ω_0 and k_h. The integration boundaries have to be observed carefully as they are modified after each change of variables and integration order. However, for the sake of simplicity, I will ignore in the following demonstration the integration limits, which are discussed in Appendix D. By substituting the variable ω in equation (15) with its new expression (19) as a function of ω_0, and changing the integration order between ω_0 and k_h the prestack migration equation becomes

$$\begin{aligned} p(t=0,k_y,h=0,z) &= \int dk_h \int d\omega\, e^{ik_z(\omega,k_y,k_h)z} p(\omega,k_y,k_h,z=0) \\ &= \int d\omega_0\, e^{ik_z(\omega_0,k_y)z} \int dk_h \left[\frac{d\omega}{d\omega_0}\right] p^*(\omega_0,k_y,k_h) \qquad (21) \\ &= \int d\omega_0\, e^{ik_z(\omega_0,k_y)z} p_0(\omega_0,k_y). \end{aligned}$$

The new field $p^*(\omega_0,k_y,k_h)$ represents a remapping (interpolation) from ω to ω_0 of the field $p(\omega,k_y,k_h,z=0)$. Each value in the new field $p^*(\omega_0,k_y,k_h)$ with coordinates (ω_0,k_y,k_h) corresponds to the value in the field $p(\omega,k_y,k_h,z=0)$ with coordinates $(\omega=\omega_0\sqrt{1+\frac{v_h^2}{\omega_0^2-v_y^2}},k_y,k_h)$, where for simplicity I define the variables:

$$v_h = \frac{vk_h}{2}; v_y = \frac{vk_y}{2}.$$

The field $p_0(\omega_0,k_y)$ defined as

$$p_0(\omega_0,k_y) = \int dk_h \left[\frac{d\omega}{d\omega_0}\right] p^*(\omega_0,k_y,k_h) \qquad (22)$$

represents the zero-offset field. The Jacobian in equation (22) obtained from the change of coordinates from ω to ω_0 is shown in Appendix B to be:

$$J = \left[\frac{d\omega}{d\omega_0}\right] = (1+\frac{v_h^2}{\omega_0^2-v_y^2})^{-\frac{1}{2}} \left[1-\frac{v_h^2 v_y^2}{(\omega_0^2-v_y^2)^2}\right]. \qquad (23)$$

The last equation in (21) is of course the zero-offset migration equation (17), the classic zero-offset downward continuation and imaging described by Gazdag (1978) or Stolt (1978). Equation (22) represents a way of obtaining the zero-offset section from prestack data in midpoint-offset coordinates. So far the operations needed to obtain the zero-offset stacked section from the prestack field are:

1. Fourier transform the prestack field $p(t,y,h) \rightarrow p(\omega,k_y,k_h)$.

2. Remap (interpolate) the data field from ω into ω_0.
3. Multiply by the Jacobian.
4. Integrate over k_h.
5. Inverse Fourier transform $p_0(\omega_0, k_y) \rightarrow p_0(t_0, y)$.

However I want to go further and replace the remapping step with an operation that does not require the interpolation of the initial data. The problem to be solved here is very similar to the one confronted in Stolt migration. After our data is evenly sampled by an FFT, we need to interpolate it for a different variable.

MZO as phase shift

The interpolated field $p^*(\omega_0, k_y, k_h)$ in equation (22) represents the values of the field $p(\omega, k_y, k_h)$ after remapping from ω to ω_0. It is obtained by first Fourier transforming the initial prestack field along all three (time, midpoint and offset) axes: $p(t, y, h) \rightarrow p(\omega, k_y, k_h)$, and second interpolating from ω to ω_0. As in Stolt migration (Popovici *et al.* 1993), we can replace the two steps of

1. Fourier transform with even sampling in ω,
2. interpolation from ω to ω_0,

by a single step of slow Fourier transform with uneven sampling in ω. We assume that the initial field is already Fourier transformed in the offset and midpoint coordinates: $p(t, y, h) \rightarrow p(t, k_y, k_h)$. Formally we inverse Fourier transform in time equation (22) to have

$$\begin{aligned} p_0(t_0, k_y) &= \int d\omega_0\, e^{-i\omega_0 t_0} \int dk_h\; J\; p^*(\omega_0, k_y, k_h) \\ &= \int dk_h \int d\omega_0\, e^{-i\omega_0 t_0} J\; p^*(\omega_0, k_y, k_h). \end{aligned} \tag{24}$$

In this formulation we can reinterpolate back from ω_0 to ω and drop the original remapping step. For this, we change the integration variable from ω_0 back to ω. The field $p^*(\omega_0, k_y, k_h)$ is reverted to the original field $p(\omega, k_y, k_h)$. In Appendix C the expression of ω_0 function of ω is found to be

$$\omega_0 = \frac{1}{2}\mathrm{sign}(\omega)\left[\sqrt{(\omega - v_y)^2 - v_h^2} + \sqrt{(\omega + v_y)^2 - v_h^2}\right]. \tag{25}$$

Substituting the variable ω_0 with the new expression in ω, and simplifying the Jacobian in equation (24) we have

$$p_0(t_0, k_y) = \int dk_h \int d\omega\, e^{-\frac{i}{2}\,\mathrm{sign}(\omega)\left[\sqrt{(\omega - v_y)^2 - v_h^2} + \sqrt{(\omega + v_y)^2 - v_h^2}\right] t_0}\, p(\omega, k_y, k_h). \tag{26}$$

Equation (26) represents a new form for migration to zero offset. It is analytically derived from the wave equation and therefore it handles correctly not only the kinematics of the DMO+NMO operator, but also the amplitudes. It is very similar in form to the DSR equation, as the complex exponential operator has the sum of two square roots in its phase. However downward continuation is performed in time in the case of the MZO operator and not in depth as is the case for DSR migration. This in turn suggests the use of a V_{RMS} velocity in the case of variable velocity, instead of the interval velocity, which could be a more convenient process as the V_{RMS} velocity is information obtained from surface data and makes less assumptions about structure.

The only drawback so far to equation (26) is that it performs a Fourier transform and later a summation over the offset variable. I will show in the next chapter how the offset variable can be separated and as a result MZO can be applied to distinct common-offset sections. Once MZO is applied to separate common-offset sections I isolate the conventional NMO and DMO processes. I will further show how equation (26) can be applied to variable velocity media, via a phase shift algorithm similar to Gazdag migration, and PSPI or split-step.

REFERENCES

Claerbout, J. C., 1985, Imaging the Earth's Interior: Blackwell Scientific Publications.

Gazdag, J., 1978, Wave equation migration with the phase shift method: Geophysics, **43**, 1342–1351.

Gazdag, J., and Sguazzero, P., 1984, Migration of seismic data by phase shift plus interpolation: Geophysics, **49**, 124–131.

Hale, I. D., 1983, Dip-moveout by Fourier transform: Ph.D. Thesis, Stanford University.

Popovici, A. M., Blondel, P., & Muir, F., 1993, Interpolation in Stolt migration: SEP–**79**, 261–264.

Stoffa, P. L., Fokkema, J. T., de Luna Freire, R. M., and Kessinger, W. P., 1990, Split-step Fourier Migration: Geophysics, **55**, 410–421.

Stolt, R.H., 1978, Migration by Fourier transform: Geophysics, **43**, 23–48.

Yilmaz, O.,1979, Prestack partial migration: Ph.D. thesis, Stanford University.

Zauderer, E., 1989, Partial Differential Equations of Applied Mathematics: Wiley-Interscience.

APPENDIX A

In this appendix I show that by writing the variable ω function of ω_0, the double square-root (DSR) phase used in prestack migration is transformed to a new form corresponding to the phase used for zero-offset migration. The transformation from ω to ω_0 as defined in equation (19) is:

$$\omega \equiv \omega_0 \left[1 + \frac{v^2 k_h^2}{4\omega_0^2 - v^2 k_y^2} \right]^{\frac{1}{2}} \equiv \omega_0 \left[1 + \frac{v_h^2}{\omega_0^2 - v_y^2} \right]^{\frac{1}{2}},$$

where v_h and v_y are defined as

$$\begin{aligned} v_h &= \frac{v k_h}{2} \\ v_y &= \frac{v k_y}{2}. \end{aligned}$$

The DSR phase is transformed from

$$k_z(\omega, k_y, k_h) \equiv -\text{sign}(\omega) \left[\sqrt{\frac{\omega^2}{v^2} - \frac{1}{4}(k_y + k_h)^2} + \sqrt{\frac{\omega^2}{v^2} - \frac{1}{4}(k_y - k_h)^2} \right],$$

to

$$k_z(\omega_0, k_y) \equiv -2\ \text{sign}(\omega_0) \sqrt{\frac{\omega_0^2}{v^2} - \frac{k_y^2}{4}}.$$

Hale (1983) in the Appendix A of his thesis proves an equivalent assertion, with a different logic. Comparing the DSR phase with the phase of the zero-offset migration (defined as a single square root), Hale finds the expression of ω_0 which transforms the former into the latter. I was tempted to refer the reader to his appendix as an indirect proof, but decided to include a thorough derivation, for completeness. Using the identity

$$\sqrt{a} + \sqrt{b} \equiv \sqrt{a + b + 2\sqrt{ab}}; \ \text{ for } a \geq 0, b \geq 0,$$

I rewrite the DSR phase as

$$\begin{aligned} k_z &= -\frac{1}{v}\text{sign}(\omega) \left\{ \left[\omega^2 - (v_y + v_h)^2\right]^{\frac{1}{2}} + \left[\omega^2 - (v_y - v_h)^2\right]^{\frac{1}{2}} \right\} \\ &= -\frac{\sqrt{2}}{v}\ \text{sign}(\omega) \left[\omega^2 - v_y^2 - v_h^2 + \sqrt{(\omega^2 - v_y^2 - v_h^2)^2 - 4 v_y^2 v_h^2} \right]^{\frac{1}{2}}. \end{aligned} \qquad \text{(A-1)}$$

Examine the expression under the second square root (SSR) in equation (A-1)

$$SSR = (\omega^2 - v_y^2 - v_h^2)^2 - 4 v_y^2 v_h^2$$

and substitute for ω the expression in ω_0. The expression under the second square root becomes:

$$
\begin{aligned}
SSR &= (\omega_0^2 + \frac{\omega_0^2 v_h^2}{\omega_0^2 - v_y^2} - v_y^2 - v_h^2)^2 - 4 v_y^2 v_h^2 \\
&= \left[\omega_0^2 - v_y^2 + \frac{\omega_0^2 v_h^2 - \omega_0^2 v_h^2 + v_y^2 v_h^2}{\omega_0^2 - v_y^2} \right]^2 - 4 v_y^2 v_h^2 \\
&= \left[\frac{(\omega_0^2 - v_y^2)^2 + v_y^2 v_h^2}{\omega_0^2 - v_y^2} \right]^2 - 4 v_y^2 v_h^2 \\
&= \frac{1}{(\omega_0^2 - v_y^2)^2} \left[(\omega_0^2 - v_y^2)^4 - 2 v_y^2 v_h^2 (\omega_0^2 - v_y^2)^2 + v_y^4 v_h^4 \right] \\
&= \left[\frac{(\omega_0^2 - v_y^2)^2 - v_y^2 v_h^2}{\omega_0^2 - v_y^2} \right]^2 .
\end{aligned}
\tag{A-2}
$$

The DSR becomes:

$$
\begin{aligned}
k_z &= -\frac{\sqrt{2}}{v} \operatorname{sign}(\omega_0) \left[\omega_0^2 + \frac{\omega_0^2 v_h^2}{\omega_0^2 - v_y^2} - v_y^2 - v_h^2 + \sqrt{SSR} \right]^{\frac{1}{2}} \\
&= -\frac{\sqrt{2}}{v} \operatorname{sign}(\omega_0) \left[\omega_0^2 + \frac{\omega_0^2 v_h^2}{\omega_0^2 - v_y^2} - v_y^2 - v_h^2 + \omega_0^2 - v_y^2 - \frac{v_y^2 v_h^2}{\omega_0^2 - v_y^2} \right]^{\frac{1}{2}} \\
&= -\frac{\sqrt{2}}{v} \operatorname{sign}(\omega_0) \left[2\omega_0^2 - 2 v_y^2 + \frac{\omega_0^2 v_h^2 - \omega_0^2 v_h^2 + v_y^2 v_h^2 - v_y^2 v_h^2}{\omega_0^2 - v_y^2} \right]^{\frac{1}{2}} \\
&= -\frac{2}{v} \operatorname{sign}(\omega_0) \left[\omega_0^2 - v_y^2 \right]^{\frac{1}{2}} \\
&= -2 \operatorname{sign}(\omega_0) \left[\frac{\omega_0^2}{v^2} - \frac{k_y^2}{4} \right],
\end{aligned}
\tag{A-3}
$$

which is the same equation as (18).

APPENDIX B

The purpose of this appendix is to evaluate the Jacobian of the transformation from ω to ω_0:

$$\begin{aligned} J &= \left[\frac{d\omega}{d\omega_0}\right] \\ &= (1+\frac{v_h^2}{\omega_0^2-v_y^2})^{-\frac{1}{2}}\left[1-\frac{v_h^2 v_y^2}{(\omega_0^2-v_y^2)^2}\right]. \end{aligned}$$

Starting with the transformation of variable

$$\omega \equiv \omega_0\left[1+\frac{v^2k_h^2}{4\omega_0^2-v^2k_y^2}\right]^{\frac{1}{2}} \equiv \omega_0\left[1+\frac{v_h^2}{\omega_0^2-v_y^2}\right]^{\frac{1}{2}},$$

and differentiating we have

$$\begin{aligned} d\omega &= \left\{\left[1+\frac{v_h^2}{\omega_0^2-v_y^2}\right]^{\frac{1}{2}} - \frac{\omega_0^2 v_h^2}{(\omega_0^2-v_y^2)^2\left[1+\frac{v_h^2}{\omega_0^2-v_y^2}\right]^{\frac{1}{2}}}\right\} d\omega_0 \\ &= (1+\frac{v_h^2}{\omega_0^2-v_y^2})^{-\frac{1}{2}}\left[1+\frac{v_h^2}{\omega_0^2-v_y^2}-\frac{v_h^2\omega_0^2}{(\omega_0^2-v_y^2)^2}\right] d\omega_0 \\ &= (1+\frac{v_h^2}{\omega_0^2-v_y^2})^{-\frac{1}{2}}\left[1+\frac{v_h^2\omega_0^2-v_h^2v_y^2-v_h^2\omega_0^2}{(\omega_0^2-v_y^2)^2}\right] d\omega_0 \\ &= (1+\frac{v_h^2}{\omega_0^2-v_y^2})^{-\frac{1}{2}}\left[1-\frac{v_h^2v_y^2}{(\omega_0^2-v_y^2)^2}\right] d\omega_0, \end{aligned}$$

and therefore the Jacobian is:

$$J=\left[\frac{d\omega}{d\omega_0}\right]=(1+\frac{v_h^2}{\omega_0^2-v_y^2})^{-\frac{1}{2}}\left[1-\frac{v_h^2v_y^2}{(\omega_0^2-v_y^2)^2}\right].$$

APPENDIX C

The purpose of this appendix is to find the inverse of the transformation $\omega \to \omega_0$, or to express ω_0 function of ω. Starting with the original transformation of variable we have

$$\omega \equiv \omega_0\left[1+\frac{v^2k_h^2}{4\omega_0^2-v^2k_y^2}\right]^{\frac{1}{2}} \equiv \omega_0\left[1+\frac{v_h^2}{\omega_0^2-v_y^2}\right]^{\frac{1}{2}},$$

and note that ω has always the same sign as ω_0. Square the equation to obtain

$$\omega^2 = \omega_0^2 + \frac{\omega_0^2 v_h^2}{\omega_0^2 - v_y^2}$$

and after isolating the terms in ω_0 we have the equation

$$\omega_0^4 - \omega_0^2(\omega^2 + v_y^2 - v_h^2) + \omega^2 v_y^2 = 0 \tag{C-1}$$

which can be solved in ω_0^2. The solutions are:

$$\omega_{0_{1,2}}^2 = \frac{1}{2}(\omega^2 + v_y^2 - v_h^2 \pm \sqrt{(\omega^2 + v_y^2 - v_h^2)^2 - 4\omega^2 v_y^2}). \tag{C-2}$$

The discriminant Δ is

$$\begin{aligned} \Delta &= (\omega^2 + v_y^2 - v_h^2 - 2\omega v_y)(\omega^2 + v_y^2 - v_h^2 + 2\omega v_y) \\ &= (\omega - v_y - v_h)(\omega - v_y + v_h)(\omega + v_y - v_h)(\omega + v_y + v_h). \end{aligned}$$

The existence conditions for k_z

$$|\,\omega\,| \quad \geq \quad |\,v_y\,| + |\,v_h\,|$$

ensure that Δ is always positive and therefore ω_0^2 is always real within the ω existence limits. The choice of a positive sign for the discriminant in equation (C-2) is assisted by the observation that for $v_h = 0$, the case of a zero-offset data field, the equation becomes an identity as it is expected. Chosing the positive sign for the discriminant, equation (C-2) becomes

$$\omega_0^2 = \frac{1}{2}(\omega^2 + v_y^2 - v_h^2 + \sqrt{(\omega^2 + v_y^2 - v_h^2)^2 - 4\omega^2 v_y^2}),$$

and using the observation that ω has the same sign as ω_0 we have:

$$\omega_0 = \text{sign}(\omega)\left[\frac{1}{2}(\omega^2 + v_y^2 - v_h^2 + \sqrt{(\omega^2 + v_y^2 - v_h^2)^2 - 4\omega^2 v_y^2})\right]^{\frac{1}{2}}, \tag{C-3}$$

which can be written in a simpler form using the identity

$$\sqrt{a} + \sqrt{b} \equiv \sqrt{a + b + 2\sqrt{ab}}; \ \text{ for } a \geq 0, b \geq 0.$$

We have

$$\begin{aligned} \omega_0 &= \text{sign}(\omega)\frac{1}{2}\left[2\omega^2 + 2v_y^2 - 2v_h^2 + 2\sqrt{(\omega^2 - 2\omega v_y + v_y^2 - v_h^2)(\omega^2 + 2\omega v_y + v_y^2 - v_h^2)}\right]^{\frac{1}{2}} \\ &= \text{sign}(\omega)\frac{1}{2}\left[\sqrt{(\omega - v_y)^2 - v_h^2} + \sqrt{(\omega + v_y)^2 - v_h^2}\right] \\ &= \text{sign}(\omega)\frac{v}{4}\left[\sqrt{(\frac{2\omega}{v} - k_y)^2 - k_h^2} + \sqrt{(\frac{2\omega}{v} + k_y)^2 - k_h^2}\right]. \end{aligned} \tag{C-4}$$

The second part of equation (C-4), in a double square-root form, is of particular importance in the phase of the MZO operator.

APPENDIX D

In this appendix I follow the integration boundaries for all the integral transformations from equation (15) to equation (26). In equation (15) the values of the constant k_z, given by equation (16), have to be real. This requires the conditions

$$|\,\omega\,| \quad \geq \quad |\,v_y + v_h\,|$$

$$|\,\omega\,| \quad \geq \quad |\,v_y - v_h\,|$$

to be satisfied simultaneously. Considering all four possible sign cases for v_y and v_h represented in Figure **??**, and the interval of existence for ω displayed in the shaded area, the two requirements can be reduced to the condition

$$|\,\omega\,| \quad \geq \quad |\,v_y\,| + |\,v_h\,| \ . \tag{D-1}$$

In Figure **??** the shaded area represents the region of integration established by equation ((D-1)

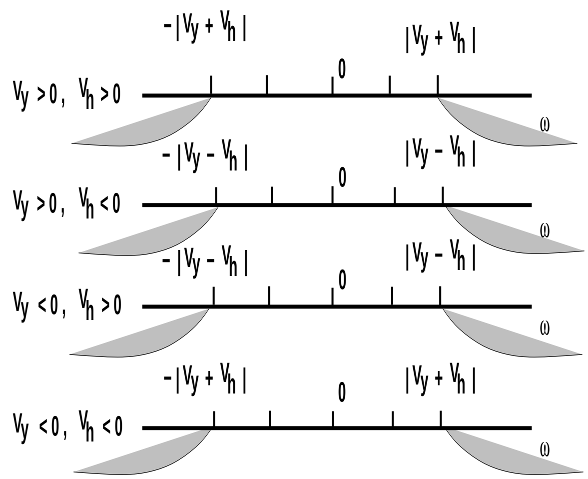

Figure D-1: Four possible cases for the values of v_y and v_h and the interval of existence of ω. mihai1-DSRbound [NR]

for a constant k_y. The existence condition for v_h in equation (D-1) requires the integration boundaries in equation (15) to be as follows:

$$p(t=0,k_y,h=0,z) = \int_{-\infty}^{\infty} d\omega \int_{-|\frac{2\omega}{v}|+|k_y|}^{|\frac{2\omega}{v}|-|k_y|} dk_h[...].$$

After the change of variable from ω to ω_0 in equation (19)

$$\omega \equiv \omega_0 \left[1 + \frac{v_h^2}{\omega_0^2 - v_y^2} \right]^{\frac{1}{2}},$$

we need to determine the new integration boundaries. In equation (21) the new variable ω_0 takes values from $-\infty$ to ∞, but the boundary values for k_h have to be expressed now function

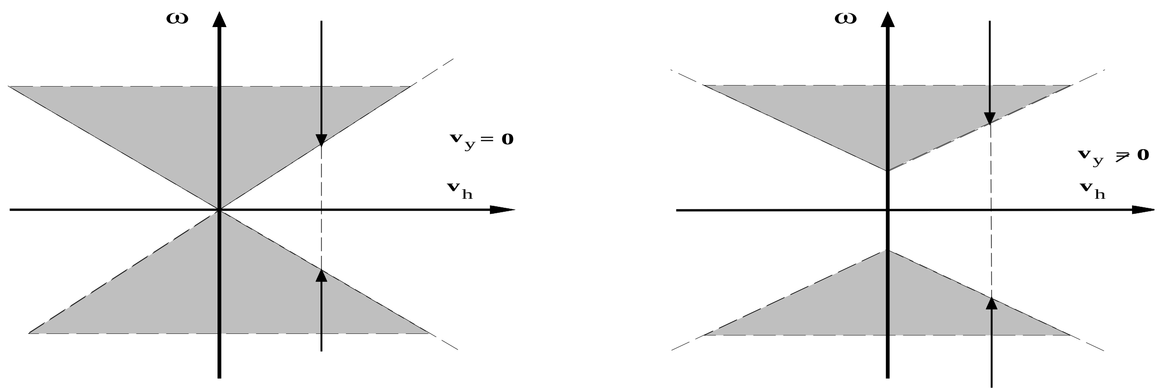

Figure D-2: Regions of integration. mihai1-khkyomega [NR]

of the new variable ω_0. Starting with the initial boundary equation (D-1) and squaring it we have

$$\omega^2 = v_h^2 + v_y^2 + 2 \mid v_y v_h \mid$$

and replacing ω with its expression in ω_0 we obtain

$$\omega_0^2 + \frac{v_h^2}{\omega_0^2 - v_y^2} = v_h^2 + v_y^2 + 2 \mid v_y v_h \mid .$$

After multiplying by $\omega_0^2 - v_y^2$ and grouping the terms we have

$$(\omega_0^2 - v_y^2)^2 - 2(\omega_0^2 - v_y^2) \mid v_y v_h \mid + v_h^2 v_y^2 = 0$$

which is transformed in the condition for k_h:

$$\mid k_h \mid \leq \frac{2}{v} \frac{\omega_0^2 - v_y^2}{\mid v_y \mid}.$$

Therefore the second line in equation (21) should have the integration boundaries:

$$p(t=0, k_y, h=0, z) = \int_{-\infty}^{\infty} d\omega_0 \int_{-\frac{2}{v}\frac{\omega_0^2 - v_y^2}{|v_y|}}^{\frac{2}{v}\frac{\omega_0^2 - v_y^2}{|v_y|}} dk_h [...],$$

and subsequently equation (22) has the same integration boundaries in k_h

$$p_0(\omega_0, k_y) = \int_{-\frac{2}{v}\frac{\omega_0^2 - v_y^2}{|v_y|}}^{\frac{2}{v}\frac{\omega_0^2 - v_y^2}{|v_y|}} dk_h [...].$$

Finally, the change of integration variable from ω_0 back to ω from equation (24) to equation (26) will restore the initial condition for k_h:

$$k_h \in (- \mid \frac{2\omega}{v} \mid + \mid k_y \mid, \mid \frac{2\omega}{v} \mid - \mid k_y \mid).$$

However, since in equation (26) there is an interchange in the order of integration variables, the integration boundaries become

$$p(t_0, k_y) = \int_{-\infty}^{\infty} dk_h \int_{-\frac{v}{2}(|k_y|+|k_h|)}^{\frac{v}{2}(|k_y|+|k_h|)} d\omega[...].$$