Stanford Exploration Project, Report 82, May 11, 2001, pages 1–144

Shortest-path calculation of first arrival traveltimes by expanding wavefronts

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ABSTRACT

A new approach to computing traveltimes and ray paths by solving the shortest path problem is presented. The technique is based on a partitioning of the shortest path optimization problem into smaller problems. We recursively evaluate the solution on expanding wavefronts instead of finding the global shortest paths from the source. To solve the local minimization, we apply a modified version of the Bellman-Ford optimization algorithm because of its suitability for a parallel implementation in three dimensions.

INTRODUCTION

Traveltime calculations and ray path information play an important role in many methods of seismic data processing such as migration, tomography and modeling. For any three-dimensional application of these processing tools, an efficient use of computer resources becomes paramount. A process such as prestack migration, for example, requires calculating Green functions, which depend on the traveltimes between survey points on the surface and depth points in the velocity model for every surface (source or receiver) location. Moreover, in nonlinear seismic tomography and velocity inversion, ray tracing is the most time-consuming step. Traveltimes and ray paths are often computed by ray tracing (Červený et al., 1977), ray bending (Julian and Gubbins, 1977), or shooting rays (Dines and Lytle, 1979). These methods can produce the correct answer but are computationally intensive, frequently encounter shadow zones, and sometimes pick the wrong ray path as the first arrival. Several new methods for calculating first-arrival traveltimes and ray paths have recently been published. Vidale (1990), Podvin and Lecomte (1991), and van Trier and Symes (1991) used different versions of finite-difference approximations to the eikonal equation to estimate first arrival traveltimes on a regular grid. Moser (1991) and Fischer and Lees (1993) calculate seismic ray paths by using graph theory to find the shortest path traveltimes on a network that represents the model of the earth. The advantages of this approach is that it always finds the global minima. Problems associated with convergence to local minima, as in ray bending, are thus avoided. Inspired by this new method of shortest path ray tracing, we present an efficient, shortest path traveltime computation method that will determine accurate first arrival travel-

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times through arbitrary, discrete, and discontinuous velocity models. The method is a modified version of the Bellman-Ford algorithm (see Bertsekas and Tsitsiklis, 1989) used for solving shortest path problems. We chose to use this algorithm because of its suitability for a parallel implementation in three dimensions. Our traveltime computation procedure advances radially across a polar grid, computing first arrival traveltimes and ray paths, shell by shell, as it goes. At any stage during the mapping, only the outer traveltimes of a shell are used to calculate new traveltimes of the next shell. The traveltime computation scheme and our modification to the Bellman-Ford algorithm, which decrease the computational requirements, are presented in detail in this paper from a theoretical standpoint. Implementations of this method will be discussed in a future SEP report.

THE SHORTEST PATH PROBLEM AND SEISMIC RAY PATHS

The shortest path problem is a classic and important combinatorial problem that arises in many contexts. There are many applications of the shortest path problem. As Moser (1991) points out, "They are usually of a discrete nature: there are a finite number of objects, and the exact solution of the problem can be found in a finite number of steps." The *shortest path problem* is defined in terms of a directed graph consisting of n nodes, which are numbered $0,\ldots,n-1$. Node 0 is called the "source". We are given a scalar a_{ij} for each arc (i,j), which we call the weight or "length" of (i,j). The *length* of a path $(i,i_1,i_2,\ldots,i_k,j)$ from node i to node j with arcs $(i,i_1),(i_1,i_2),\ldots,(i_k,j)$ is defined to be the sum of the arc lengths $(a_{ii_1}+a_{i_1i_2}+\cdots+a_{i_kj})$. The problem is to find a path of minimum length (or shortest path) from the source to every node i. Seismic ray paths are approximated, based on Fermat's principle, by shortest paths for the problem of computing first arrival traveltimes. The model of the earth is represented by a *network*, consisting of nodes connected by arcs. Each arc is assigned a length equal to the ray's traveltime along the arc. The networks used to represent velocity models in this paper are *sparse*; i.e., each node is connected with a restricted number of nodes in its neighborhood but not with points that lie farther away. This neighborhood is called the node's *adjacency-list* and contains all the nodes such that there exists an arc that connects them. The shortest path problem can be solved for any parameterization of the velocity model and any arrangement of the nodes as long as there exists a path from every node $i=1,\ldots,n-1$ to the source node. In our implementation of the shortest path problem we have chosen to use a polar parameterization, assigning a constant velocity inside each polar cell. Just as Fischer and Lees (1993), we distributed the nodes regularly at the boundaries of the cells (Figure 1). Two nodes are connected only when there is no cell boundary between them. The traveltime between two connected nodes is defined as their Euclidean distance multiplied with the slowness of the cell in between. As suggested by Fischer and Lees, the choice of models with constant velocity cells has the welcome advantage of particularly simple and fast calculation of traveltimes between connected nodes; a lookup table can be used.

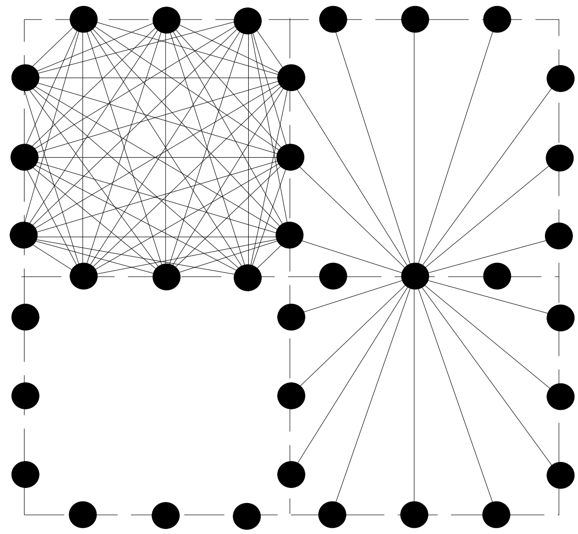

Figure 1: Network of nodes and arcs. Solid dots represent nodes in the graph, which are placed on cell boundaries (dotted lines). Arcs are represented by solid lines. The upper-left cell shows all the arcs that lie within that cell. On the right side we show the adjacency-list of the node in the middle. Adapted from Fischer et. al. (1993). hector-network [NR]

THE BELLMAN-FORD ALGORITHM

Moser, and Fischer and Lees solved the single source shortest path problem using the *modified Dijkstra algorithm* (see Cormen et. al., 1990). Dijkstra's algorithm maintains a priority queue that contains nodes for which the shortest path from the source has not yet been computed. In the modified Dijkstra algorithm this priority queue is implemented as a binary heap. In both methods, the algorithm inflicts a heavy toll on computer resources, both in memory and time, which may prevent it from being used seriously in 3-D applications. The method we have chosen to solve the single source shortest path is the *Bellman-Ford algorithm*, with which we address these difficulties.

Bertsekas and Tsitsiklis (1989) show that the shortest path lengths x_i^*, $i = 0, \ldots, n-1$ are the unique solutions of the system

$$x_i^* = \min_{j \in A(i)} (a_{ij} + x_j^*), \qquad i = 1, \ldots, n-1 \tag{1}$$

$$x_0^* = 0 \tag{2}$$

(known as *Bellman's equations*). $A(i)$ is the adjacency-list of node i. Furthermore, they also prove that the iteration

$$x_i^k = \min_{j \in A(i)} (a_{ij} + x_j^{k-1}), \qquad i = 1, \ldots, n-1 \tag{3}$$

$$x_0^k = 0 \tag{4}$$

(also known as the *Bellman-Ford algorithm*) converges to this solution for an arbitrary initial vector x with $x_0 = 0$. The algorithm is said to have *converged after k iterations* if $x_i^k = x_i^{k-1}$

for all i. Note that the Bellman-Ford algorithm is particularly well suited for parallel and distributed implementations since the iteration for each node i can be carried out simultaneously with the iteration for every other node. Regarding initial conditions, one possible set of initial conditions in the Bellman-Ford algorithm is $x_i^0 = \infty$ for $i \neq 0$ and $x_0^0 = 0$. Another possible set of initial conditions can be obtained for certain applications where the shortest path problem must be solved repeatedly, since the arc lengths change by small increments. A small change in the arc lengths implies a small change in the shortest path lengths, so it may be advantageous in terms of convergence speed to restart the Bellman-Ford algorithm, using as initial conditions the previous shortest path lengths. This would be the case in which we have a certain number of sources spaced closely enough so that the paths the seismic rays follow change by small increments for two contiguous source locations.

Modified Bellman-Ford Algorithm

We derive a more efficient Bellman-Ford algorithm by iterating equation (3) only on the nodes that belong to a certain shell. Once the algorithm converges on this shell it goes on to the next outer shell. Figure 2 depicts the steps followed by the modified Bellman-Ford algorithm. Equations (5) and (6) define the new algorithm

$$x_i^k = \min_{j \in A(i)} (a_{ij} + x_j^{k-1}), \qquad i \in \mathrm{S}(l) \tag{5}$$

$$x_0^k = 0 \tag{6}$$

where $S(l)$ contains the set of nodes that belong to shell l. The traveltimes to the first ring of nodes (Figure 2a) are computed by tracing straight rays from the source to these nodes. At each step, the algorithm computes the shortest path traveltimes to the nodes of one particular shell. At the next step, the algorithm finds the traveltimes to the nodes belonging to the next contiguous outer shell. In this implementation, nodes are only connected if they obey the following two rules:

- there are no cell boundaries between them
- and, either both belong to the same shell or one of them sits in the outer boundary of the previous shell.

Thus, at each iteration, a node sitting on the outer boundary of a shell cannot connect with nodes that lie radially further away. This condition reduces the size of the adjacency-list of these nodes, which translates into fewer iterations for finding the minimum function in equation (5). A drawback is that we are eliminating overturned rays. However, we reduce this problem by using a polar coordinate frame. The use of polar coordinates has the disadvantage of the extra cost of mapping velocity and traveltime fields to and from polar coordinates. Nevertheless, polar coordinates have been successfully used in finite difference traveltime calculation (van Trier and Symes, 1991). These coordinates have the advantage of providing a frame that matches the constant velocity wavefronts. Even in variable velocity models, the computational grid for polar coordinates better matches wavefronts than a rectangular

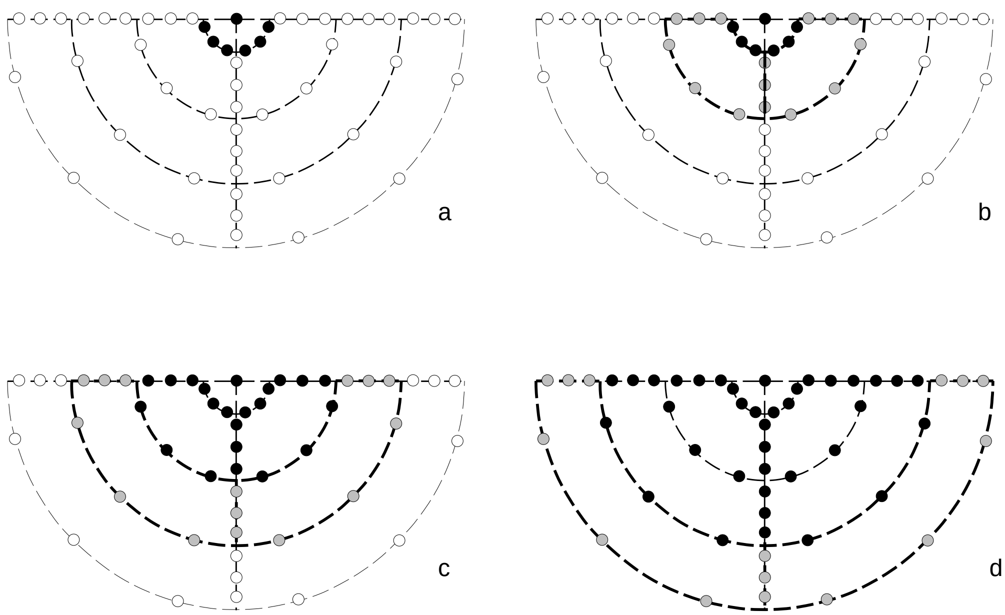

Figure 2: Four steps (a,b,c,d) with the modified Bellman-Ford implementation for a very simple model. The model is represented by two polar cells per shell. Step (a) shows the initial conditions. Solid dots represent nodes for which we have calculated shortest path traveltimes. White dots represent nodes for which we have not computed its shortest path traveltimes. In steps (b,c, & d) the algorithm iterates for each shell to find the shortest path traveltimes of the nodes that belong to that shell (polka-dotted dots). hector-iter [NR]

grid does. In implementing the Bellman-Ford algorithm in a "shell by shell" way, we reduce the amount of memory resources that the algorithm needs, since the modified version needs to keep in memory at one time only the information concerning the nodes of a shell, while in the original version or in Dijkstra's algorithm, we need to allocate memory for all the nodes of the model. For a parallel implementation of the modified Bellman-Ford algorithm, only the iterations of nodes that belong to a particular shell are computed simultaneously. A comparison of the Bellman-Ford algorithm with the modified version shows that the latter has the advantage of needing only to broadcast a smaller copy of the vector x^{k-1} to each running processor. While in the original algorithm this vector has the size of the total number of nodes, in the modified algorithm it only requires the nodes per shell.

Traveltimes at receivers

In order to generate the traveltime tables, the algorithm takes only the values of the shortest path traveltime computed for those nodes that lie on curved boundaries (see Figure 3). Since these nodes are regularly distributed on a polar grid, as shown in the Figure, the algorithm outputs these values directly. Finally, the output of the algorithm is piped into a separate program that does a bilinear interpolation of the traveltime tables in order to map them from polar coordinates to rectangular coordinates. Note that radially aligned nodes are only used in the algorithm to allow rays to bend, while they are not used in the generation of the traveltime tables.

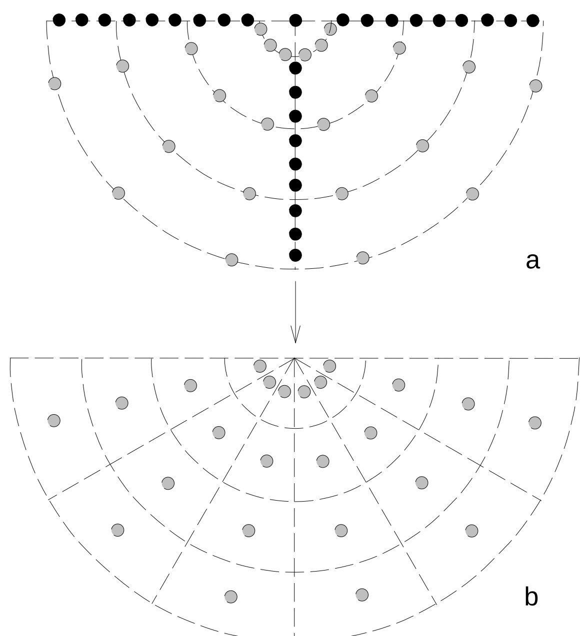

Figure 3: Final traveltime table (b) on a polar frame. (a) shows a set of nodes for which we have already computed the shortest path traveltimes. Out of these nodes we pick those that lie on the curved boundaries (polka-dotted dots) to build the final traveltime table. Radially aligned nodes (solid dots) are necessary for rays to bend. hector-inter [NR]

CONCLUSIONS

The modified Bellman-Ford algorithm allows an efficient computation of ray paths and traveltimes by computing them over expanding wavefronts. By partitioning the global problem of finding the shortest paths from a source, into smaller problems, we obtained an algorithm that is more computationaly efficient. The method constructs a global traveltime and ray field to all points in space, so there are no problems with shadow zones. A parallel implementation of the algorithm should prove very useful for three dimensional applications.

REFERENCES

Bertsekas, D. P., and Tsitsiklis, J. N., 1989, Parallel and distributed computation: numerical methods: Prentice-Hall, Inc.

Červený, V., Molotkov, I. A., and Pšenčik, I., 1977, Ray methods in seismology: Univ. of Karlova Press.

Dines, K. A., and Lytle, R. J., 1979, Computerized geophysical tomography: Proc. IEEE, **67**, no. 7, 1065–1073.

Fischer, R., and Lees, J. M., 1993, Shortest path ray tracing with sparse graphs: Geophysics, **58**, no. 7, 987–996.

Julian, B. R., and Gubbins, D., 1977, Three-dimensional seismic ray tracing: J. Geophys., **43**, 95–114.

Moser, T. J., 1991, Shortest path calculation of seismic rays: Geophysics, **56**, no. 1, 59–67.

Podvin, P., and Lecomte, I., 1991, Finite-difference computation of traveltimes in very contrasted velocity model: A massively parallel approach and its associated tools: Geophys. J. Int., **105**, 271–284.

van Trier, J., and Symes, W. W., 1991, Upwind finite-difference calculation of traveltimes: Geophysics, **56**, no. 6, 812–821.

Vidale, J. E., 1990, Finite-difference calculation of traveltimes in three dimensions: Geophysics, **55**, no. 5, 521–526.