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# Amplitude preserving offset continuation in theory Part 1: The offset continuation equation

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## ABSTRACT

This paper concerns amplitude-preserving kinematically equivalent offset continuation (OC) operators. I introduce a revised partial differential OC equation as a tool to build OC operators that preserve offset-dependent reflectivity in prestack processing. The method of characteristics is applied to reveal the geometric laws of the OC process. With the help of geometric (kinematic) constructions, the equation is proved to be kinematically valid for all offsets and reflector dips in constant velocity media. In the OC process, the angle-dependent reflection coefficient is preserved, and the geometric spreading factor is transformed in accordance with the laws of geometric seismics independently of the reflector curvature.


## INTRODUCTION

Offset continuation (OC) by definition is an operator that transforms common-offset seismic gathers from one constant offset to another (Bolondi et al., 1982). Bagaini et al. (1994) recently identified OC with a whole family of prestack continuation operators, such as shot continuation (Schwab, 1993; Bagaini and Spagnolini, 1993), dip moveout as a particular case of OC to zero offset, and three-dimensional azimuth moveout (Biondi and Chemingui, 1994). Possible practical applications of OC operators include regularizing seismic data by partial stacking prior to prestack migration (Chemingui and Biondi, 1994) and interpolating missing data. Since dip moveout (DMO) represents a particular case of offset continuation to zero offset, the OC concept is also one of the possible approaches to DMO. Another prospective application of prestack continuation operators, pointed out recently by Fabio Rocca (personal communication), is prestack tomography-type velocity analysis.

In the theory of OC operators, two issues need to be addressed. The first is *kinematic equivalence*. We expect seismic sections obtained by OC to contain correctly positioned reflection traveltime curves. The second issue is *amplitude equivalence*. If the traveltimes are positioned correctly, it is wave amplitudes that deserve most of our attention. Since the final outputs of the seismic processing sequence are the migrated sections, the kinematic equivalence of OC concerns preserving the true geometry of seismic images, while the amplitude

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equivalence addresses preserving the desired brightness of the images. Apparently, there can be different definitions of *amplitude-preserving* or *true-amplitude* processing. The most commonly used one (Hubral et al., 1991; Tygel et al., 1992; Black et al., 1993; Goldin, 1992) refers to the reflectivity preservation. According to this definition, amplitude-preserving seismic data processing should make the image amplitudes proportional to the reflection coefficients that correspond to the initial constant-offset gathers. This point of view implies that an amplitude-preserving OC operator tends to transform offset-dependent amplitude factors, except for the reflection coefficient, in accordance with the geometric seismic laws. In this paper I introduce a theoretical approach to constructing different types of OC operators with respect to both kinematic equivalence and amplitude preservation. The first part presents the theory for a revised OC differential equation. As early as in 1982, Bolondi et al. came up with the idea of describing OC as a continuous process by means of a partial differential equation (Bolondi et al., 1982). However, their approximate differential OC operator, built on the results of Deregowski and Rocca's classic paper (1981), turned out to fail in case of steep reflector dips or large offsets. In his famous Ph.D. thesis (1983) Dave Hale wrote:

> The differences between this algorithm [DMO by Fourier transform] and previously published finite-difference DMO algorithms are analogous to the differences between frequency-wavenumber (Stolt, 1978; Gazdag, 1978) and finite-difference (Claerbout, 1976) algorithms for migration. For example, just as finite-difference migration algorithms require approximations that break down at steep dips, finite-difference DMO algorithms are inaccurate for large offsets and steep dips, even for constant velocity.

Continuing this analogy, one can observe that both finite-difference and frequency-domain migration algorithms share a common origin: the wave equation. The new OC equation, presented in this paper and valid for all offsets and dips, can play an analogous role for offset continuation and dip moveout algorithms. The next section begins with a rigorous proof of the revised equation's kinematic validity. Since the OC process belongs to the wave type, it is appropriate to describe it by considering wavefronts (which in this case correspond to the traveltime curves) and ray trajectories (referred to in this paper as *time rays*). The laws of amplitude transport along the time rays illuminate the main dynamic properties of offset continuation and prove the OC equation's amplitude equivalence.

## INTRODUCING THE OFFSET CONTINUATION EQUATION

Most of the contents of this paper refer to the following linear partial differential equation:

$$h \left( \frac{\partial^2 P}{\partial y^2} - \frac{\partial^2 P}{\partial h^2} \right) = t_n \frac{\partial^2 P}{\partial t_n \, \partial h} \; . \tag{1}$$

Equation (1) describes an *imaginary* (nonphysical) process of reflection seismic data transformation in the offset-midpoint-time domain. Here $h$ stands for the half-offset ($h = (r-s)/2$, where $s$ and $r$ are the source and the receiver coordinates), $y$ is the midpoint ($y = (r+s)/2$), and $t_n$ is the time coordinate after normal moveout correction is applied $\left(t_n = \sqrt{t^2 - \frac{4h^2}{v^2}}\right)$. The velocity $v$ is supposed to be constant and known a priori. Equation (1) and the previously published OC equation (Bolondi et al., 1982) differ only with respect to the single term $\frac{\partial^2 P}{\partial h^2}$. However, this difference is substantial. As Appendix A proves, the range of validity for the approximate OC equation

$$h \frac{\partial^2 P}{\partial y^2} = t_n \frac{\partial^2 P}{\partial t_n \, \partial h} \tag{2}$$

can be defined by the inequality

$$h/z \ll \cot\alpha \, , \tag{3}$$

where $z$ is the reflector depth, and $\alpha$ is the dip angle. For example, for a dip of 45 degrees, equation (2) is valid only for offsets that are much smaller than the depth. In order to prove the theoretical validity of equation (1) for all offsets and reflector dips, I apply a simplified version of the ray method technique (Červený et al., 1977; Babich, 1991) and obtain two equations to describe separately wavefront (traveltime) and amplitude transformation in the OC process. According to the formal ray theory, the leading term of the high-frequency asymptotics for a reflected wave, recorded on a seismogram, takes the form

$$P\,(y,h,t_n) \approx A_n(y,h)\, R_n\,(t_n - \tau_n(y,h)) \ , \tag{4}$$

where $A_n$ stands for the amplitude, $R_n$ is the wavelet shape of the leading high-frequency term, and $\tau_n$ is the traveltime curve after normal moveout. Inserting (4) as a trial solution for (1), collecting terms that have the same asymptotic order, and neglecting low-order terms produces a set of two first-order partial differential equations:

$$h \left[ \left( \frac{\partial \tau_n}{\partial y} \right)^2 - \left( \frac{\partial \tau_n}{\partial h} \right)^2 \right] = -\,\tau_n \frac{\partial \tau_n}{\partial h} \ , \tag{5}$$

$$\left( \tau_n - 2h \frac{\partial \tau_n}{\partial h} \right) \frac{\partial A_n}{\partial h} + 2h \frac{\partial \tau_n}{\partial y} \frac{\partial A_n}{\partial y} + h A_n \left( \frac{\partial^2 \tau_n}{\partial y^2} - \frac{\partial^2 \tau_n}{\partial h^2} \right) = 0 \ . \tag{6}$$

Equation (5) describes the transformation of traveltime curve geometry in the OC process analogously to the eikonal equation in the wavefront propagation theory. Thus, what appear to be wavefronts of the wave motion described by (1) are traveltime curves of reflected waves recorded on seismic sections. The law of amplitude transformation for high-frequency wave components, related to those wavefronts, is given by (6). In terms of the theory of partial differential equations, equation (5) is the characteristic equation for (1).

**Proof of kinematic equivalence**

In order to prove the validity of equation (5), it is convenient to transform it to the coordinates of the initial shot gathers: $s = y - h$, $r = y + h$, and $\tau = \sqrt{\tau_n^2 + \frac{4h^2}{v^2}}$. The transformed equation takes the form

$$\left(\tau^2 + \frac{(r-s)^2}{v^2}\right)\left(\frac{\partial \tau}{\partial r} - \frac{\partial \tau}{\partial s}\right) = 2\,(r-s)\,\tau\left(\frac{1}{v^2} - \frac{\partial \tau}{\partial r}\frac{\partial \tau}{\partial s}\right) \; . \tag{7}$$

Now the goal is to prove that any reflection traveltime function $\tau(r,s)$ in a constant velocity medium satisfies equation (7). Let $S$ and $R$ be the source and the reflection locations, and $O$ be a reflection point for that pair. Note that the incident ray $SO$ and the reflected ray $OR$ form a triangle with the basis on the offset $SR$ ($l = |SR| = r - s$). Let $\alpha_1$ be the angle of $SO$ from the vertical axis, and $\alpha_2$ be the analogous angle of $RO$ (Figure 1). Elementary trigonometry (the law of sines) gives us the following explicit relationships between the sides and the angles of the triangle $SOR$:

$$|SO| = |SR|\,\frac{\cos\alpha_1}{\sin(\alpha_2-\alpha_1)} \; , \tag{8}$$

$$|RO| = |SR|\,\frac{\cos\alpha_2}{\sin(\alpha_2-\alpha_1)} \; . \tag{9}$$

Hence, the total length of the reflected ray is

$$v\tau = |SO| + |RO| = |SR|\,\frac{\cos\alpha_1 + \cos\alpha_2}{\sin(\alpha_2-\alpha_1)} = (r-s)\,\frac{\cos\alpha}{\sin\gamma} \; . \tag{10}$$

Here $\gamma$ is the reflection angle ($\gamma = (\alpha_2-\alpha_1)/2$), and $\alpha$ is the central ray angle ($\alpha = (\alpha_2+\alpha_1)/2$) coincident with the local dip angle of the reflector at the reflection point. Recalling the well-known relationships between the ray angles and the first-order traveltime derivatives

$$\frac{\partial \tau}{\partial s} = \frac{\sin\alpha_1}{v} \; , \tag{11}$$

$$\frac{\partial \tau}{\partial r} = \frac{\sin\alpha_2}{v} \; , \tag{12}$$

we can substitute (10), (11), and (12) into (7), which leads to the simple trigonometric equality

$$\cos^2\left(\frac{\alpha_1+\alpha_2}{2}\right) + \sin^2\left(\frac{\alpha_1-\alpha_2}{2}\right) = 1 - \sin\alpha_1\sin\alpha_2 \; . \tag{13}$$

It is now easy to prove that equality (13) is true for any $\alpha_1$ and $\alpha_2$. Thus we have proved that equation (7), equivalent to (5), is valid in constant velocity media independently of the reflector geometry and the offset. This means that high-frequency asymptotic components of the waves, described by the OC equation, are located on the true reflection traveltime curves. The theory of characteristics can provide other ways to prove the kinematic validity of equation (5), as described in (Fomel, 1994; Goldin, 1994).

### Offset continuation geometry: time rays

To study the laws of traveltime curve transformation in the OC process, it is convenient to apply the method of characteristics (Courant, 1962) to the eikonal-type equation (5). The

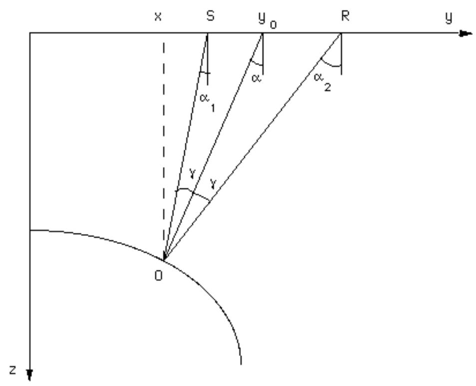


Figure 1: Reflection rays in a constant velocity medium (a scheme). sergey3-ocoray [NR]

characteristics of (5) (*bi*-characteristics with respect to (1)) are the trajectories of the high-frequency energy propagation in the imaginary OC process. Following the formal analogy with seismic rays, let's call those trajectories *time rays*, where the word *time* refers to the fact that time rays describe the traveltime transformation. According to the theory of first-order partial differential equations, time rays are determined by a set of ordinary differential equations (characteristic equations) derived from (5) :

$$\frac{dy}{dt_n} = -\frac{2hY}{t_n H}\,,\ \frac{dY}{dt_n} = \frac{Y}{t_n}\,,$$
$$\frac{dh}{dt_n} = -\frac{1}{H} + \frac{2h}{t_n}\,,\ \frac{dH}{dt_n} = \frac{Y^2}{t_n H}\,, \tag{14}$$

where $Y$ corresponds to $\frac{\partial \tau_n}{\partial y}$ along a ray, and $H$ corresponds to $\frac{\partial \tau_n}{\partial h}$. In this coordinate system, equation (5) takes the form

$$h\,(Y^2 - H^2) = -\,t_n H \tag{15}$$

and serves as an additional constraint for the definition of time rays. System (14) can be solved by standard mathematical methods. Its general solution takes the parametric form, where the time variable $t_n$ is the parameter along a time ray:

$$y(t_n) = C_1 - C_2\,t_n^2 \quad ; \quad h(t_n) = t_n\sqrt{C_2^2 t_n^2 + C_3}\ ; \tag{16}$$
$$Y(t_n) = \frac{C_2\,t_n}{C_3} \quad ; \quad H(t_n) = \frac{h}{C_3\,t_n} \tag{17}$$

and $C_1$, $C_2$, and $C_3$ are independent coefficients, constant along each time ray. To determine the values of these coefficients, we can pose an initial value (Cauchy) problem for the system of differential equations (14). The traveltime curve $\tau_n(y;h)$ for a given common offset $h$ and the first partial derivative $\frac{\partial \tau_n}{\partial h}$ along the same constant offset section provide natural initial conditions for the Cauchy problem. A particular case of those conditions is the zero-offset traveltime curve. If the first partial derivative of traveltime with respect to offset is continuous, it vanishes at zero offset according to the reciprocity principle (traveltime must be an even function of the offset): $t_0(y_0) = \tau_n(y;0)$, $\left.\frac{\partial \tau_n}{\partial h}\right|_{h=0} = 0$. Applying the initial value conditions to the general solution (17) generates the following expressions for the ray invariants:

$$\begin{aligned} C_1 &= y + h\,\frac{Y}{H} = y_0 - \frac{t_0(y_0)}{t_0'(y_0)}\;;\; C_2 = \frac{h\,Y}{\tau_n^2\,H} = -\frac{1}{t_0(y_0)\,t_0'(y_0)}\;; \\ C_3 &= \frac{h}{\tau_n\,H} = -\frac{1}{\left(t_0'(y_0)\right)^2}\,. \end{aligned} \tag{18}$$

Finally, substituting (18) into (17) produces an explicit parametric form of the ray trajectories:

$$\left\{\begin{aligned} y_1(t_1) &= y + \frac{h\,Y}{t_n^2\,H}\left(t_n^2 - t_1^2\right) = y_0 + \frac{t_1^2 - t_0^2(y_0)}{t_0(y_0)\,t_0'(y_0)}\;; \\ h_1^2(t_1) &= \frac{h\,t_1^2}{t_n^3\,H}\left(t_n^2 + t_1^2\,\frac{h\,Y^2}{t_n\,H}\right) = t_1^2\,\frac{t_1^2 - t_0^2(y_0)}{\left(t_0(y_0)\,t_0'(y_0)\right)^2}\,. \end{aligned}\right. \tag{19}$$

Here $y_1$, $h_1$, and $t_1$ are the coordinates of the continued seismic section. The first of equations (19) indicates that the time ray projections to a common-offset section have a parabolic form. Time rays don't exist for $t_0'(y_0) = 0$ (a locally horizontal reflector), because in this case post-NMO offset continuation transform is not required. The actual parameter that determines a particular time ray is the reflection point location. This important conclusion follows from the known parametric equations

$$\left\{\begin{aligned} t_0(x) &= t_v \sec\alpha = t_v(x)\sqrt{1 + u^2\left(t_v'(x)\right)^2}\,, \\ y_0(x) &= x + u t_v \tan\alpha = x + u^2\,t_v(x) t_v'(x)\,, \end{aligned}\right. \tag{20}$$

where $x$ is the reflection point, $u$ is half of the wave velocity ($u = v/2$), $t_v$ is the vertical time (reflector depth divided by $u$), and $\alpha$ is the local reflector dip. Taking into account that the derivative of the zero-offset traveltime curve is

$$\frac{dt_0}{dy_0} = \frac{t_0'(x)}{y_0'(x)} = \frac{\sin\alpha}{u} = \frac{t_v'(x)}{\sqrt{1 + u^2\left(t_v'(x)\right)^2}} \tag{21}$$

and substituting (20) into (19), we get

$$\left\{\begin{aligned} y_1(t_1) &= x + \frac{t_1^2 - t_v^2(x)}{t_v(x)\,t_v'(x)}\;; \\ u^2 t^2(t_1) &= t_1^2\,\frac{t_1^2 - t_v^2(x)}{\left(t_v(x)\,t_v'(x)\right)^2}\,, \end{aligned}\right. \tag{22}$$

where $t^2(t_1) = t_1^2 + h_1^2(t_1)/u^2$. To visualize the concept of time rays, let's consider some simple analytic examples of its application to geometric analysis of the offset continuation process. The simplest and most important example is the case of a plane dipping reflector. Putting the origin of the $y$ axis at the reflector plane intersection with the surface, we can express the reflection traveltime after NMO in the form

$$\tau_n(y,h) = p\sqrt{y^2 - h^2}\,, \tag{23}$$

where $p = 2\,\frac{\sin\alpha}{v}$, and $\alpha$ is the dip angle. The zero-offset traveltime in this case is a straight line:

$$t_0\,(y_0) = p\,y_0\,. \tag{24}$$

According to (19), time rays are defined by

$$y_1\,(t_1) = \frac{t_1^2}{p^2\,y_0}\;;\; h_1^2\,(t_1) = t_1^2\,\frac{t_1^2 - p^2\,y_0^2}{p^4\,y_0^2}\;;\; y_0 = \frac{y^2 - h^2}{y}\,. \tag{25}$$

The geometry of the OC transformation is shown in Figure 2. The second example is the case

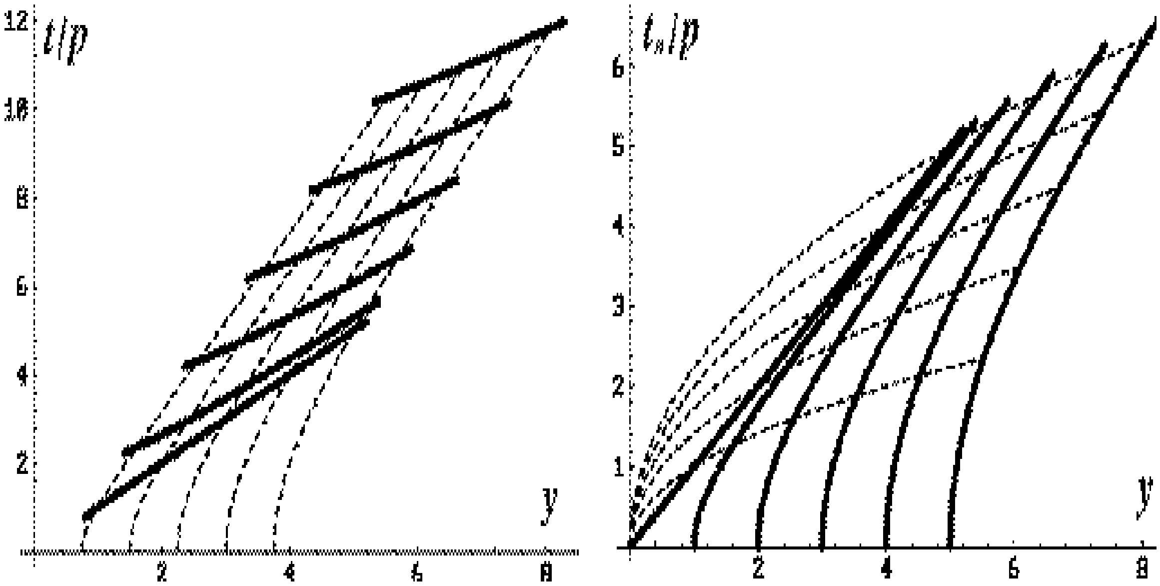


Figure 2: Transformation of the reflection traveltime curves in the OC process: the case of a plane dipping reflector. Left: Time coordinate before the NMO correction. Right: Time coordinate after NMO. Solid lines indicate traveltime curves; dashed lines, time rays. sergey3-ocopln [NR]

of a point diffractor (the left side of Figure 3). Without loss of generality, the origin of the midpoint axis can be put above the diffraction point. In this case the zero-offset reflection traveltime curve has the well-known hyperbolic form

$$t_0\,(y_0) = \frac{\sqrt{z^2 + y_0^2}}{u}\,, \tag{26}$$

where $z$ is the depth of the diffractor, and $u = v/2$ is half of the wave velocity. Time rays are defined according to (19), as follows:

$$y_1\,(t_1) = \frac{u^2\,t_1^2 - z^2}{y_0}\ ;\ u^2\,t^2\,(t_1) = u^2\,t_1^2 + h_1^2\,(t_1) = u^2\,t_1^2\,\frac{u^2\,t_1^2 - z^2}{y_0^2}\ . \tag{27}$$

The third curious example (the right side of Figure 3) is the case of a focusing elliptic reflector.

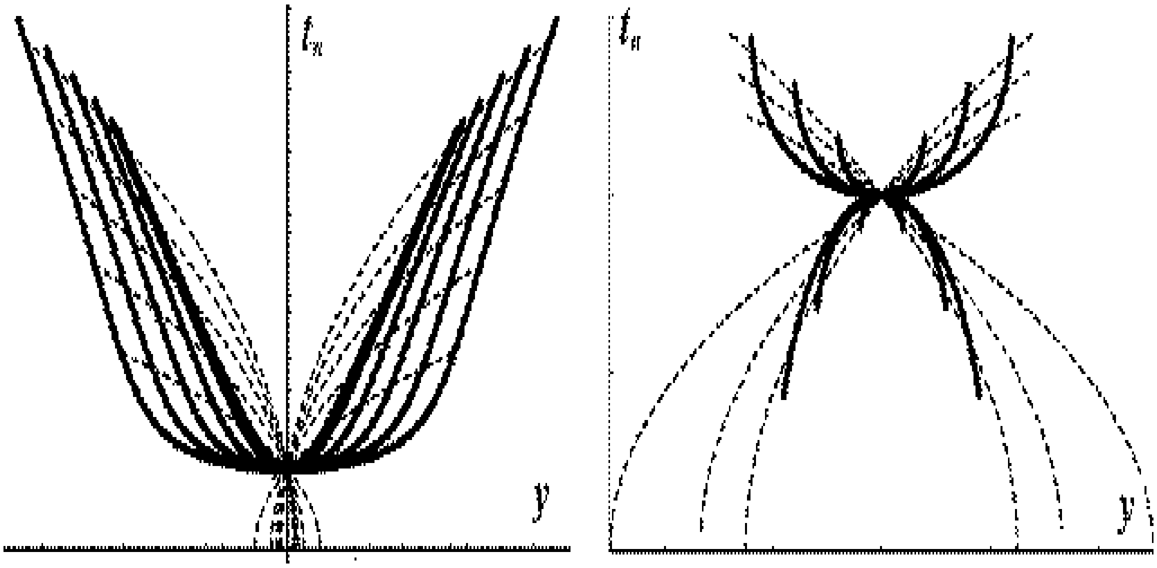


Figure 3: Transformation of the reflection traveltime curves in the OC process. Left: the case of a diffraction point. Right: the case of an elliptic reflector. Solid lines indicate traveltime curves; dashed lines, time rays. sergey3-ococrv [NR]

Let $y$ be the center of the ellipse and $h$ be half the distance between the focuses of the ellipse. If both focuses are on the surface, the zero-offset traveltime curve is defined by the so-called "DMO smile" (Deregowski and Rocca, 1981):

$$t_0\,(y_0) = \frac{t_n}{h}\,\sqrt{h^2 - (y - y_0)^2}\ , \tag{28}$$

where $t_n = 2\,z/v$, and $z$ is the small semi-axis of the ellipse. The time ray equations are

$$y_1\,(t_1) = y + \frac{h^2}{y - y_0}\,\frac{t_1^2 - t_n^2}{t_n^2}\ ;\ h_1^2\,(t_1) = h^2\,\frac{t_1^2}{t_n^2}\left(1 + \frac{h^2}{(y - y_0)^2}\,\frac{t_1^2 - t_n^2}{t_n^2}\right)\ . \tag{29}$$

When $y_1$ coincides with $y$, and $h_1$ coincides with $h$, the source and the receiver are in the focuses of the elliptic reflector, and the traveltime curve degenerates to a point $t_1 = t_n$. This remarkable fact is the actual basis of the geometric theory of dip moveout (Deregowski and Rocca, 1981).

### Proof of amplitude equivalence

This section discusses the connection between the laws of traveltime transformation and the laws of the corresponding amplitude transformation. The change of the wave amplitudes in the OC process is described by the first-order partial differential transport equation (6). The general solution of this equation can be found by applying the method of characteristics. It takes the explicit integral form

$$A_n\left(t_n\right) = A_0\left(t_0\right) \exp\left(\int_{t_o}^{t_n} h \left(\frac{\partial^2 \tau_n}{\partial y^2} - \frac{\partial^2 \tau_n}{\partial h^2}\right) \left(\tau_n \frac{\partial \tau_n}{\partial h}\right)^{-1} d\tau_n\right) . \tag{30}$$

The integral in (30) is defined on a curved time ray, and $A_n(t_n)$ stands for the amplitude transported along this ray. In the case of a plane dipping reflector, the ray amplitude can be immediately evaluated by substituting the explicit traveltime and time ray formulas from the preceding section into (30). The amplitude expression in this case takes the simple form

$$A_n\left(t_n\right) = A_0\left(t_0\right) \frac{t_0}{t_n} . \tag{31}$$

In order to consider the more general case of a curvilinear reflector, we need to take into account a connection between the traveltime derivatives in (30) and the geometric quantities of the reflector. As follows directly from the trigonometry of the incident and reflected rays triangle (Figure 1),

$$h = \frac{r-s}{2} = D \frac{\cos\alpha \, \sin\gamma \, \cos\gamma}{\cos^2\alpha - \sin^2\gamma} , \tag{32}$$

$$y = \frac{r+s}{2} = x + D \frac{\cos^2\alpha \, \sin\alpha}{\cos^2\alpha - \sin^2\gamma} , \tag{33}$$

$$y_0 = x + D \sin\alpha , \tag{34}$$

where $D$ is the length of the normal ray. Let $\tau_0 = 2\,D/v$ be the zero-offset reflection traveltime. Combining (32) and (34) with (10) we can get the following relationship:

$$a = \frac{\tau_n}{\tau_0} = \frac{\cos\alpha \, \cos\gamma}{\left(\cos^2\alpha - \sin^2\gamma\right)^{1/2}} = \left(1 + \frac{\sin^2\alpha \, \sin^2\gamma}{\cos^2\alpha - \sin^2\gamma}\right)^{1/2} = \frac{h}{\sqrt{h^2 - (y-y_0)^2}} , \tag{35}$$

which interprets the "DMO smile" (28) found by Deregowski and Rocca (1981) in geometric terms. Equation (35) allows a convenient change of variables in (30). Let the reflection angle $\gamma$ be a parameter monotonously increasing along a time ray. In this case, each time ray is uniquely determined by the position of the reflection point, which in turn is defined by the values of $D$ and $\alpha$. According to this change of variables we can differentiate (35) along a time ray to get

$$\frac{d\tau_n}{\tau_n} = -\frac{\sin^2\alpha}{2\cos^2\gamma \left(\cos^2\gamma - \sin^2\alpha\right)} d\left(\cos^2\gamma\right) . \tag{36}$$

Note also that the quantity $h \left(\tau_n \frac{\partial \tau_n}{\partial h}\right)^{-1}$ in (30) coincides exactly with the time ray invariant $C_3$ found in (18). Therefore its value is constant along each time ray and equals

$$h \left(\tau_n \frac{\partial \tau_n}{\partial h}\right)^{-1} = -\frac{v^2}{4 \sin^2 \alpha} \; . \tag{37}$$

Finally, as shown in Appendix B,

$$\tau_n \left(\frac{\partial^2 \tau_n}{\partial y^2} - \frac{\partial^2 \tau_n}{\partial h^2}\right) = 4 \frac{\cos^2 \gamma}{v^2} \left(\frac{\sin^2 \alpha + DK}{\cos^2 \gamma + DK}\right) \; , \tag{38}$$

where $K$ is the reflector curvature at the reflection point. Substituting (36), (37), and (38) into (30) transforms the integral to the form

$$\int_{t_o}^{t_n} h \left(\frac{\partial^2 \tau_n}{\partial y^2} - \frac{\partial^2 \tau_n}{\partial h^2}\right) \left(\tau_n \frac{\partial \tau_n}{\partial h}\right)^{-1} d\tau_n =$$

$$= -\frac{1}{2} \int_{\cos^2 \gamma_0}^{\cos^2 \gamma} \left(\frac{1}{\cos^2 \gamma' - \sin^2 \alpha} - \frac{1}{\cos^2 \gamma' + DK}\right) d\left(\cos^2 \gamma'\right) \tag{39}$$

which we can evaluate analytically. The final formula for the amplitude transformation takes the form

$$\begin{aligned} A_n &= A_0 \frac{\sqrt{\cos^2 \gamma - \sin^2 \alpha}}{\sqrt{\cos^2 \gamma_0 - \sin^2 \alpha}} \left(\frac{\cos^2 \gamma_0 + DK}{\cos^2 \gamma + DK}\right)^{1/2} = \\ &= A_0 \frac{\tau_0 \cos \gamma}{\tau_n \cos \gamma_0} \left(\frac{\cos^2 \gamma_0 + DK}{\cos^2 \gamma + DK}\right)^{1/2} \; . \end{aligned} \tag{40}$$

In case of a plane reflector, the curvature $K$ is zero, and (40) coincides with (31). Equation (40) can be rewritten as

$$A_n = \frac{c \cos \gamma}{\tau_n \sqrt{\cos^2 \gamma + DK}} \; , \tag{41}$$

where $c$ is constant along each time ray (it may vary with the reflection point location on the reflector but not with the offset). We should compare equation (41) with the known expression for the reflection wave amplitude of the leading ray series term in 2.5-D media:

$$A = \frac{C_R(\gamma) \Psi}{G} \; , \tag{42}$$

where $C_R$ stands for the angle-dependent reflection coefficient, $G$ is the geometric spreading

$$G = v \tau \frac{\sqrt{\cos^2 \gamma + DK}}{\cos \gamma} \; , \tag{43}$$

and $\Psi$ includes other possible factors (such as the source directivity) that we can either correct or neglect in the preliminary processing. It is evident that the curvature dependence of the amplitude transformation (41) coincides completely with the true geometric spreading factor (43), and that the angle dependence of the reflection coefficient is not provided by the offset continuation process. If the wavelet shape of the reflected wave on seismic sections ($R_n$ in (4)) is described by the delta function, then, as follows from the known properties of this function,

$$A\,\delta\left(t-\tau(y,h)\right)=\left|\frac{dt_n}{dt}\right|\,A\,\delta\left(t_n-\tau_n(y,h)\right)=\frac{t}{t_n}\,A\,\delta\left(t_n-\tau_n(y,h)\right)\;, \tag{44}$$

which leads to the equality

$$A_n = A\,\frac{t}{t_n}\;. \tag{45}$$

Combining (45) with (42) and (41) allows us to evaluate the amplitude after continuation from some initial offset $h_0$ to another offset $h_1$, as follows:

$$A_1=\frac{C_R(\gamma_0)\Psi_0}{G_1}\;. \tag{46}$$

Equation (46) indicates that the OC process described by equation (1) is amplitude-preserving in the sense that corresponds to the so-called Born DMO (Liner, 1991; Bleistein, 1990). This means that the geometric spreading factor from the initial amplitudes is transformed to the true geometric spreading on the continued section, while the reflection coefficient stays the same. This remarkable dynamic property allows AVO (amplitude versus offset) analysis to be performed by a dynamic comparison between true constant-offset sections and the sections transformed by OC from different offsets. With a simple trick, the offset coordinate is transferred to the reflection angles for the AVO analysis. As follows from (35) and (10),

$$\frac{\tau_n^2}{\tau\,\tau_0}=\cos\gamma\;. \tag{47}$$

If we include the $\frac{t_n^2}{t\,t_0}$ factor in the DMO operator (continuation to zero offset) and divide the result by the DMO section obtained without this factor, the resultant amplitude of the reflected events will be directly proportional to $\cos\gamma$, where the reflection angle $\gamma$ corresponds to the initial offset. Of course, this conclusion is rigorously valid for constant-velocity 2.5-D media only. Black et al. (1993) recently suggested a definition of true-amplitude DMO different from that of Born DMO. The difference consists of two important components:

1. *True-amplitude DMO addresses preserving the peak amplitude of the image wavelet instead of preserving its spectral density.* In the terms of this paper, the peak amplitude corresponds to the initial amplitude $A$ instead of the spectral density amplitude $A_n$. A simple correction factor $\frac{t}{t_n}$ would help us take the difference between the two amplitudes into account. Multiplication by $\frac{t}{t_n}$ can be easily done at the NMO stage.

2. *Seismic sections are multiplied by time to correct for the geometric spreading factor prior to DMO (or in our case, offset continuation) processing.*

As follows from (43), multiplication by $t$ is a valid geometric spreading correction for plane reflectors only. It is amplitude-preserving offset continuation based on the OC equation (1) that is able to correct for the curvature-dependent factor in the amplitude. To take into account the second aspect of Black's definition, we can consider the wave field $\hat{P}$ such that

$$\hat{P}\,(y,h,t_n) = t\,P\,(y,h,t_n)\;. \tag{48}$$

Substituting (48) into the OC equation (1) transforms the latter to the form

$$h\left(\frac{\partial^2 \hat{P}}{\partial y^2} - \frac{\partial^2 \hat{P}}{\partial h^2}\right) = t_n\,\frac{\partial^2 \hat{P}}{\partial t_n\,\partial h} - \frac{\partial \hat{P}}{\partial h}\,. \tag{49}$$

Equations (49) and (1) differ only with respect to the first-order term $\frac{\partial \hat{P}}{\partial h}$. This term affects the amplitude behavior but not the traveltimes, since the eikonal-type equation (5) depends on the second-order terms only. Offset continuation operators based on (49) conform to Black's definition of true-amplitude processing.

## CONCLUSIONS

I have introduced a partial differential equation (1) and proved that the process described by it provides for a kinematically and dynamically equivalent offset continuation transform. Kinematic equivalence means that in constant velocity media the reflection traveltimes are transformed to their true locations on different offsets. Dynamic equivalence means that the geometric spreading term in the amplitudes of reflected waves transforms in accordance with the geometric seismics laws, while the angle-dependent reflection coefficient stays the same in the OC process. The amplitude properties of amplitude-preserving OC may find an important application in the seismic data processing connected with AVO interpretation .

The offset continuation equation can be applied directly to design OC operators of the finite-difference type. Other types of operators are related to different forms of the solutions of the OC equation. Part 2 of this paper will describe integral-type offset continuation operators based on the initial value problem associated with equation (1). Other important topics in the theory of offset continuation include

- Connection between OC and amplitude-preserving frequency-domain DMO
- Connection between OC and true-amplitude prestack migration
- Generalizing the OC concept to 3-D azimuth moveout (AMO)

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## APPENDIX A

## RANGE OF VALIDITY FOR BOLONDI'S OC EQUATION

From the OC characteristic equation (5) we can conclude that the first-order traveltime derivative with respect to offset decreases with a decrease of the offset. At zero offset the derivative equals zero, as predicted by the principle of reciprocity (reflection traveltime has to be an *even* function of offset). Neglecting $\frac{\partial \tau_n}{\partial h}$ in (5) leads to the characteristic equation

$$h \left(\frac{\partial \tau_n}{\partial y}\right)^2 = -\,\tau_n\,\frac{\partial \tau_n}{\partial h}\;, \tag{A-1}$$

which corresponds to the approximate OC equation (2) of Bolondi et al. (1982). Comparing (A-1) and (5), note that approximation (A-1) is valid only if

$$\left(\frac{\partial \tau_n}{\partial h}\right)^2 \ll \left(\frac{\partial \tau_n}{\partial y}\right)^2\;. \tag{A-2}$$

To find the geometric constraints implied by inequality (A-2), we can express the traveltime derivatives in geometric terms. As follows from expressions (11) and (12),

$$\frac{\partial \tau}{\partial x} = \frac{\partial \tau}{\partial s} + \frac{\partial \tau}{\partial r} = \frac{2\sin\alpha\cos\gamma}{v}\;; \tag{A-3}$$

$$\frac{\partial \tau}{\partial h} = \frac{\partial \tau}{\partial r} - \frac{\partial \tau}{\partial s} = \frac{2\cos\alpha\sin\gamma}{v}\;. \tag{A-4}$$

Expression (10) allows transforming (A-3) and (A-4) to the form

$$\tau_n\,\frac{\partial \tau_n}{\partial y} = \tau\,\frac{\partial \tau}{\partial y} = 4h\,\frac{\sin\alpha\cos\alpha\cot\gamma}{v^2}\;; \tag{A-5}$$

$$\tau_n\,\frac{\partial \tau_n}{\partial h} = \tau\,\frac{\partial \tau}{\partial h} - \frac{4h}{v^2} = -4h\,\frac{\sin^2\alpha}{v^2}\;. \tag{A-6}$$

Without loss of generality, we can assume $\alpha$ to be positive. Consider a plane tangent to the true reflector at the reflection point (Figure A-1). The traveltime of the wave, reflected from the plane, has the well-known explicit expression

$$\tau = \frac{2}{v}\sqrt{L^2 + h^2\cos^2\alpha}\;, \tag{A-7}$$

where $L$ is the length of the normal ray from the midpoint. As follows from combining (A-7) and (10),

$$\cos\alpha\cot\gamma = \frac{L}{h}\;. \tag{A-8}$$

We can then combine equalities (A-8), (A-5), (A-6) (A-3), and (A-4) to transform inequality (A-2) to the form

$$h \ll \frac{L}{\sin\alpha} = z\cot\alpha\;, \tag{A-9}$$

where $z$ is the depth of the plane reflector under the midpoint. The proven inequality (A-9) coincides with (3) in the main text.

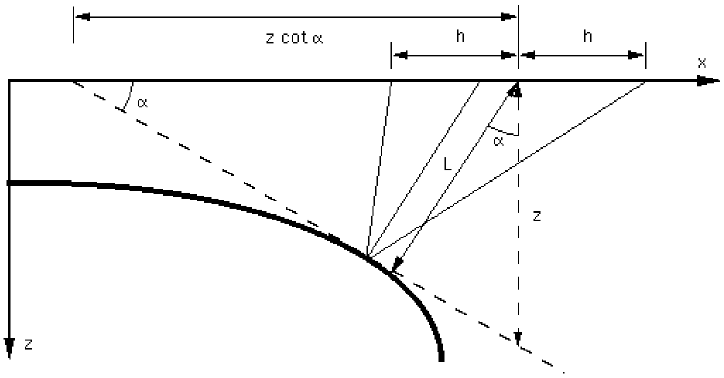


Figure A-1: Reflection rays and tangent to the reflector in a constant velocity medium (a scheme). sergey3-ocobol [NR]

## APPENDIX B

## SECOND-ORDER REFLECTION TRAVELTIME DERIVATIVES

In this appendix I derive formulas connecting second-order partial derivatives of the reflection traveltime with the geometric properties of the reflector in a constant velocity medium. These formulas are used in the main text of the paper for the amplitude behavior description. Let $\tau(s,r)$ be the reflection traveltime from the source $s$ to the receiver $r$. Consider a formal equality

$$\tau(s,r) = \tau_1(s, x(s,r)) + \tau_2(x(s,r), r) \ , \tag{B-1}$$

where $x$ is the reflection point parameter, $\tau_1$ corresponds to the incident ray, and $\tau_2$ corresponds to the reflected ray. Differentiating (B-1) with respect to $s$ and $r$ yields

$$\frac{\partial \tau}{\partial s} = \frac{\partial \tau_1}{\partial s} + \frac{\partial \tau}{\partial x}\frac{\partial x}{\partial s} \ , \tag{B-2}$$

$$\frac{\partial \tau}{\partial r} = \frac{\partial \tau_2}{\partial r} + \frac{\partial \tau}{\partial x}\frac{\partial x}{\partial r} \ . \tag{B-3}$$

According to Fermat's principle, the two-point reflection ray path must correspond to the traveltime extremum. Therefore

$$\frac{\partial \tau}{\partial x} \equiv 0 \tag{B-4}$$

for any $s$ and $r$. Taking into account (B-4) while differentiating (B-2) and (B-3), we get

$$\frac{\partial^2 \tau}{\partial s^2} = \frac{\partial^2 \tau_1}{\partial s^2} + B_1 \frac{\partial x}{\partial s} \ , \tag{B-5}$$

$$\frac{\partial^2 \tau}{\partial r^2} = \frac{\partial^2 \tau_2}{\partial r^2} + B_2 \frac{\partial x}{\partial r} \;, \tag{B-6}$$

$$\frac{\partial^2 \tau}{\partial s \partial r} = B_1 \frac{\partial x}{\partial r} \; = B_2 \frac{\partial x}{\partial s} \;, \tag{B-7}$$

where

$$B_1 = \frac{\partial^2 \tau_1}{\partial s \partial x} \; ; \; B_2 = \frac{\partial^2 \tau_2}{\partial r \partial x} \; .$$

Differentiating (B-4) gives us the additional pair of equations

$$C \frac{\partial x}{\partial s} + B_1 = 0 \;, \tag{B-8}$$

$$C \frac{\partial x}{\partial r} + B_2 = 0 \;, \tag{B-9}$$

where

$$C = \frac{\partial^2 \tau}{\partial x^2} = \frac{\partial^2 \tau_1}{\partial x^2} + \frac{\partial^2 \tau_2}{\partial x^2} \; .$$

Solving the system (B-8) - (B-9) for $\frac{\partial x}{\partial s}$ and $\frac{\partial x}{\partial r}$ and substituting the result into (B-5) - (B-7) produces the following set of expressions:

$$\frac{\partial^2 \tau}{\partial s^2} = \frac{\partial^2 \tau_1}{\partial s^2} - C^{-1} \, B_1^2 \; ; \tag{B-10}$$

$$\frac{\partial^2 \tau}{\partial r^2} = \frac{\partial^2 \tau_2}{\partial r^2} - C^{-1} \, B_2 \; ; \tag{B-11}$$

$$\frac{\partial^2 \tau}{\partial s \partial r} = -C^{-1} \, B_1 \, B_2 \; . \tag{B-12}$$

In the case of a constant velocity medium, expressions (B-10) to (B-12) can be applied directly to the explicit formula for the two-point eikonal

$$\tau_1(y,x) = \tau_2(x,y) = \frac{\sqrt{(x-y)^2 + z^2(x)}}{v} \; . \tag{B-13}$$

Differentiating (B-13) and taking into account the trigonometric relationships for the incident and reflected rays (Figure 1), one can evaluate all the quantities in (B-10) to (B-12) explicitly. After some heavy algebra, the resultant expressions for the traveltime derivatives take the form

$$\frac{\partial \tau}{\partial s} = \frac{\partial \tau_1}{\partial s} = \frac{\sin\alpha_1}{v} \; ; \; \frac{\partial \tau}{\partial r} = \frac{\partial \tau_2}{\partial r} = \frac{\sin\alpha_2}{v} \; ; \tag{B-14}$$

$$\frac{\partial \tau_1}{\partial x} = \frac{\sin\gamma}{v\cos\alpha}\;;\;\frac{\partial \tau_2}{\partial x} = -\frac{\sin\gamma}{v\cos\alpha}\;; \tag{B-15}$$

$$B_1 = \frac{\partial^2 \tau_1}{\partial s\,\partial x} = \frac{\cos\alpha_1}{v\,D\,\cos\alpha}\left(-1 - \frac{\sin\gamma}{\cos\alpha}\,\sin\alpha_1\right)\;; \tag{B-16}$$

$$B_2 = \frac{\partial^2 \tau_2}{\partial r\,\partial x} = \frac{\cos\alpha_2}{v\,D\,\cos\alpha}\left(-1 + \frac{\sin\gamma}{\cos\alpha}\,\sin\alpha_2\right)\;; \tag{B-17}$$

$$B_1\,B_2 = \frac{\cos^6\gamma}{v^2\,D^2\,a^4}\;;\;B_1 + B_2 = -2\,\frac{\cos^3\gamma}{v\,D\,a^2}\,\left(2\,a^2 - 1\right)\;; \tag{B-18}$$

$$\frac{\partial^2 \tau_1}{\partial x^2} = \frac{\cos^2\gamma + D\,K}{v\,D\,\cos^3\alpha}\,\cos\alpha_1\;;\;\frac{\partial^2 \tau_2}{\partial x^2} = \frac{\cos^2\gamma + D\,K}{v\,D\,\cos^3\alpha}\,\cos\alpha_2\;; \tag{B-19}$$

$$C = \frac{\partial^2 \tau_1}{\partial x^2} + \frac{\partial^2 \tau_2}{\partial x^2} = 2\,\cos\gamma\,\frac{\cos^2\gamma + D\,K}{v\,D\,\cos^3\alpha}\;. \tag{B-20}$$

Here $D$ is the length of the normal (central) ray, $\alpha$ is its dip angle ($\alpha = \frac{\alpha_1+\alpha_2}{2}$, $\tan\alpha = z'(x)$), $\gamma$ is the reflection angle $\left(\gamma = \frac{\alpha_2-\alpha_1}{2}\right)$, $K$ is the reflector curvature at the reflection point $\left(K = z''(x)\cos^3\alpha\right)$, and $a$ is the nondimensional function of $\alpha$ and $\gamma$ defined in (35). The formulas derived in this appendix were used to get the formula

$$\tau_n\left(\frac{\partial^2 \tau_n}{\partial y^2} - \frac{\partial^2 \tau_n}{\partial h^2}\right) = 4\left(\tau\,\frac{\partial^2 \tau}{\partial s\,\partial r} + \frac{\cos^2\gamma}{v^2}\right) = 4\,\frac{\cos^2\gamma}{v^2}\left(\frac{\sin^2\alpha + DK}{\cos^2\gamma + DK}\right)\,, \tag{B-21}$$

which coincides with (38) in the main text.