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Analysis of Thomsen parameters for finely-layered VTI media

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ABSTRACT

Since the work of Postma (1955) and Backus (1962), much has been learned about elastic constants in vertical transversely isotropic (VTI) media when the anisotropy is due to fine layering of isotropic elastic materials. Nevertheless, there has continued to be a degree of uncertainty about the possible range of Thomsen's anisotropy parameters ϵ and δ for such media. We show that ϵ lies in the range $-3/8 \leq \epsilon \leq \frac{1}{2}\left[\left\langle v_p^2\right\rangle\left\langle v_p^{-2}\right\rangle - 1\right]$, for finely layered media having constant density; smaller positive and all negative values of ϵ occur for media with large fluctuations in the Lamé parameter λ. We show that δ can also be either positive or negative, and that for constant density media $\text{sign}(\delta) = \text{sign}\left(\left\langle v_p^{-2}\right\rangle - \left\langle v_s^{-2}\right\rangle\left\langle v_s^2/v_p^2\right\rangle\right)$. Among all theoretically possible random media, positive and negative δ are equally likely in finely layered media limited to two types of constituent layers. Layered media having large fluctuations in Lamé λ are the ones most likely to have positive δ. Since Gassmann's results for fluid-saturated porous media show that the effects of fluids influence only the λ Lamé constant, not the shear modulus μ, these results suggest that positive δ occurring together with positive but small ϵ may be indicative of changing fluid content in layered earth.

INTRODUCTION

Two primary goals of seismic reflection processing are: (1) to image geologic structure and (2) to provide information about lithology for interpretation. The process used to achieve the second goal is made complex by the fact that the same seismic velocity may result from several different combinations/mixtures of materials in the earth, *i.e.*, the possible causes of the observed behavior are often nonunique. It is therefore necessary to explore the possible range of seismic velocities that can occur within the set of circumstances deemed most likely to occur in the earth at the site of interest.

Fine horizontal layering (*i.e.*, layers with thickness small compared to the wave-

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length of the seismic wave) is known to result in vertical transverse isotropy (VTI) – wherein wave speeds vary with propagation angle in such media, but are uniquely determined by the angle from the vertical. There has continued to be some doubt about the range of anisotropy parameters possible in such media. Here we will correct some common errors found in the literature. We show that Thomsen's parameter ϵ can be negative and actually has a greater negative range than indicated in previous published work. We also show that Thomsen's parameter δ can be positive in finely layered media (contrary to some erroneous claims that have appeared in the literature), and furthermore that regions having both positive δ and negative (or small) ϵ are likely to be regions of rapid spatial variation in fluid content. We use Monte Carlo simulations to establish the existence of both positive δ and negative ϵ, and analysis of Backus averaging formulas to clarify when such behavior occurs.

NOTATION FOR ANISOTROPIC ELASTIC MEDIA

In tensor notation, the relationship between components of stress σ_{ij} and strain $u_{k,l}$ is given by

$$\sigma_{ij} = c_{ijkl}u_{k,l}, \tag{1}$$

where c_{ijkl} is the adiabatic stiffness tensor. Repeated indices on the right hand side of (1) are summed. In (1), u_k is the kth Cartesian component of the displacement vector $\mathbf{u}$, and $u_{k,l} = \partial u_k/\partial x_l$. Whereas for an isotropic elastic medium the stiffness tensor has the form

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{2}$$

depending on only two parameters (the Lamé constants, λ and μ). This tensor can have up to 21 independent constants for general anisotropic elastic media. The stiffness tensor has pairwise symmetry in its indices such that $c_{ijkl} = c_{jikl}$ and $c_{ijkl} = c_{ijlk}$, which will be used later to simplify the resulting equations. The requirement that straining the medium must produce a positive change in its internal energy provides the additional constraint that $c_{ijkl} = c_{klij}$.

The general equation of motion for wave propagation through an anisotropic elastic medium is given by

$$\rho\ddot{u}_i = \sigma_{ij,j} = c_{ijkl}u_{k,lj}, \tag{3}$$

where $\ddot{u}_i$ is the second time derivative of the ith Cartesian component of the displacement vector $\mathbf{u}$ and ρ is the density (assumed constant). Equation (3) is a statement that the product of mass times acceleration of a particle is determined by the internal stress force $\sigma_{ij,j}$.

A commonly used simplification of the notation for elastic analysis is given by introducing the strain tensor, where

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{4}$$

Then, using one version of the Voigt convention, in which the pairwise symmetries of the stiffness tensor indices are used to reduce the number of indices from 4 to 2 using the rules $11 \to 1$, $22 \to 2$, $33 \to 3$, 23 or $32 \to 4$, 13 or $31 \to 5$, and 12 or $21 \to 6$, we have

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & & & \\ c_{12} & c_{22} & c_{23} & & & \\ c_{13} & c_{23} & c_{33} & & & \\ & & & 2c_{44} & & \\ & & & & 2c_{55} & \\ & & & & & 2c_{66} \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} . \tag{5}$$

Although the Voigt convention introduces no restrictions on the stiffness tensor, we have chosen to limit discussion to the form in (5), which is not completely general. Of the 36 coefficients (of which 21 are generally independent), we choose to treat only those cases for which the 12 coefficients shown (of which nine are generally independent) are nonzero. This form includes all orthorhombic, cubic, hexagonal, and isotropic systems, while excluding triclinic, monoclinic, trigonal, and some tetragonal systems, since each of the latter contains additional off-diagonal constants that may be nonzero. Nevertheless, we will restrict the discussion to (5) or to the still simpler case of transversely isotropic (TI) materials.

For TI materials, $c_{11} = c_{22} \equiv a$, $c_{12} \equiv b$, $c_{13} = c_{23} \equiv f$, $c_{33} \equiv c$, $c_{44} = c_{55} \equiv l$, and $c_{66} \equiv m$. There is also one further constraint on the constants that $a = b + 2m$, following from rotational symmetry in the x_1x_2-plane. In such materials, (5) may be replaced by

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} a & b & f & & & \\ b & a & f & & & \\ f & f & c & & & \\ & & & 2l & & \\ & & & & 2l & \\ & & & & & 2m \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix} , \tag{6}$$

in which the matrix has the same symmetry as hexagonal systems and of which isotropic symmetry is a special case (having $a = c = \lambda + 2\mu$, $b = f = \lambda$, and $l = m = \mu$).

BACKUS AVERAGING

Backus (1962) presents an elegant method of producing the effective constants for a thinly layered medium composed of either isotropic or anisotropic elastic layers. This method applies either to spatially periodic layering or to random layering, by which we mean either that the material constants change in a nonperiodic (unpredictable) manner from layer to layer or that the layer thicknesses may also be random. For simplicity, we will assume that the layers are isotropic, in which case equation (5) becomes

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & 2\mu & & \\ & & & & 2\mu & \\ & & & & & 2\mu \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{pmatrix}. \tag{7}$$

Although the materials forming the thin layers that make up the VTI medium do not need to be isotropic materials, any VTI medium that is anisotropic due to thin layering can be represented using a model with layers composed of a minimum of three isotropic materials (Backus, 1962). The key idea presented by Backus is that these equations can be rearranged into a form where rapidly varying coefficients multiply slowly varying stresses or strains. This analysis is well-known so we will not repeat it here.

The results are

$$a = c_{11} = \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} + 4\left\langle \frac{\mu(\lambda+\mu)}{\lambda+2\mu} \right\rangle, \tag{8}$$

$$b = c_{12} = \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} + 2\left\langle \frac{\lambda\mu}{\lambda+2\mu} \right\rangle, \tag{9}$$

$$c = c_{33} = \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1} \tag{10}$$

$$f = c_{13} = \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1}, \tag{11}$$

$$l = c_{44} = \left\langle \frac{1}{\mu} \right\rangle^{-1}, \tag{12}$$

and

$$m = c_{66} = \langle \mu \rangle . \tag{13}$$

One very important fact that is known about these equations is that they reduce to isotropic results with $a = c$, $b = f$, and $l = m$, if the shear modulus is a constant, regardless of the behavior of λ.

KNOWN INEQUALITIES AND THOMSEN PARAMETER γ

Since the stress-strain relation (6) is derivable from an energy functional, it is not hard to show that the matrix must be nonnegative or the material will be mechanically unstable. Nonnegativity of the matrix implies that all its principal minors must be nonnegative, which in turn implies the following inequalities:

$$a = b + 2m \geq 0, \qquad c \geq 0, \tag{14}$$

$$l \geq 0, \qquad m \geq 0, \tag{15}$$

and

$$(a^2 - b^2)/4m = b + m \geq 0, \qquad ac - f^2 \geq 0, \tag{16}$$

and

$$\left[a(ac - 2f^2) - b(bc - 2f^2)\right]/4m = (b + m)c - f^2 \geq 0. \tag{17}$$

The second inequality in (16) follows from (17), (15), and the second inequality in (14) and is therefore often omitted from such listings. Similarly, the inequality for a follows from those for m and $b + m$. All of these inequalities must be satisfied regardless of the source of the anisotropy.

The formulas (8)-(13) can be used to derive some very simple relations among the constants. For example,

$$c \geq f \tag{18}$$

follows directly from (10) and (11), simply noting that $\lambda/(\lambda + 2\mu) \leq 1$ in every layer. The inequality

$$c \geq \frac{4}{3}l \tag{19}$$

is derived directly from the fact that

$$\left\langle \frac{1}{\lambda + 2\mu} \right\rangle \leq \left\langle \frac{1}{4\mu/3} \right\rangle = \frac{3}{4}\left\langle \frac{1}{\mu} \right\rangle, \tag{20}$$

which follows from the fact that the bulk modulus must be nonnegative in each layer so that $\lambda + 2\mu/3 \geq 0$ everywhere. The two shear moduli must satisfy

$$1 \leq \langle \mu \rangle \left\langle \frac{1}{\mu} \right\rangle = \frac{m}{l}, \tag{21}$$

because the well-known Cauchy-Schwartz inequality $\langle \alpha\beta \rangle^2 \leq \langle \alpha^2 \rangle \langle \beta^2 \rangle$ gives this result when $\alpha = \mu^{1/2}$ and $\beta = 1/\mu^{1/2}$. Equality applies in the Cauchy-Schwartz inequalities only when $\alpha = const \times \beta$, which implies in the present circumstances that μ must be constant for $l = m$. But this is precisely the condition mentioned earlier for the layer equations to be isotropic, so we generally exclude this case from consideration. The implication of (21) for Thomsen's γ is that

$$\gamma = \frac{m - l}{2l} \geq 0. \tag{22}$$

THOMSEN PARAMETER ϵ

An important anisotropy parameter for qP-waves is Thomsen's parameter ϵ, defined by

$$\epsilon = \frac{a-c}{2c}. \tag{23}$$

Formula (8) for a may be rewritten as

$$a = \left\langle \frac{(\lambda+2\mu)^2 - \lambda^2}{\lambda+2\mu} \right\rangle + \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1}, \tag{24}$$

which can be rearranged into the convenient and illuminating form

$$a = \langle \lambda+2\mu \rangle - \left[\left\langle \frac{\lambda^2}{\lambda+2\mu} \right\rangle \left\langle \frac{1}{\lambda+2\mu} \right\rangle - \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \right] \left\langle \frac{1}{\lambda+2\mu} \right\rangle^{-1}. \tag{25}$$

This formula is very instructive because the term in square brackets is again in Cauchy-Schwartz form, so this factor is nonnegative. Furthermore, the magnitude of this term depends principally on the fluctuations in the λ Lamé constant, largely independent of μ. Clearly, if $\lambda = constant$, then this factor vanishes identically, regardless of the behavior of μ. Large fluctuations in λ will tend to make this term large. If in addition we now consider the combination

$$2\epsilon = \left[\langle \lambda+2\mu \rangle \left\langle \frac{1}{\lambda+2\mu} \right\rangle - 1 \right] - \left[\left\langle \frac{\lambda^2}{\lambda+2\mu} \right\rangle \left\langle \frac{1}{\lambda+2\mu} \right\rangle - \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle^2 \right], \tag{26}$$

we find that the first bracket on the right hand side is again in Cauchy-Schwartz form, which makes it always positive unless $\lambda + 2\mu = constant$, in which case it vanishes. Similarly, the second term always makes a negative contribution unless $\lambda = constant$, in which case it vanishes.

The properties of the isotropic layers making up the anisotropic rocks depend on mineralogy, porosity, and fluid content. We can gain some insight into how the relative sizes of λ and μ are related to mineralogy by considering the measured values of these stiffnesses for quartz and calcite. Simmons and Brace (1965) report measured values of about 28 to 31 GPa for the shear modulus of fused quartz, and measured values of about .027 to .032 GPa^{-1} for the compressibility, corresponding to values of about 13 to 16 GPa for λ. (These measurements were made for confining pressures of 0.1 MPa to 10 GPa.) Wilkens *et al.* (1984) gave ultrasonic velocity values measured in single crystals of quartz and calcite that yield values of about 44 GPa for the shear modulus of quartz, 8.4 GPa for λ for quartz, 31 GPa for the shear modulus of calcite, and 55 GPa for λ for calcite. (These measurements were made without accounting for the anisotropy in quartz and calcite crystals. See Aleksandrov and Ryzhova (1961) for measured values of all the elastic stiffnesses, for single calcite crystals and quartz crystals.) Both the static and the dynamic measurements show that μ values in quartz and calcite are similar, whereas the λ values differ considerably for the two minerals, with λ being larger than μ for calcite and smaller than μ for quartz. We also know that v_p/v_s ratios in sandstones are close to 1.5, and v_p/v_s ratios are about 1.8 to 2.0 for sedimentary rocks containing calcite, dolomite, or feldspar (*e.g.*, Wilkens *et al.*, 1984; Castagna *et al.*, 1985). If λ is approximately equal to μ, the v_p/v_s ratio is 1.7. Rocks having λ larger than μ have v_p/v_s ratios larger than 1.7, and rocks having λ smaller than μ have v_p/v_s ratios smaller than 1.7.

Although the stiffnesses of clay minerals can be found in the literature (*e.g.*, Katahara, 1996), it is more useful to consider instead the velocities of rocks containing clays (*e.g.*, Tosaya and Nur, 1982; Han *et al.*, 1986) because the clays themselves typically contain micropores and the location of the clays in the microstructure (e.g., lining pores, or between grains) is more important than mineralogy, for elastic properties (Caruso *et al.*, 1985; Wilkens *et al.*, 1986). Porosity does not directly affect λ and μ values of the grains, but since it affects the velocities, it affects the v_p/v_s ratio. Clay content and the presence of pores both tend to increase v_p/v_s ratios (Castagna *et al.*, 1985). If randomly-oriented flat cracks are present, they will also tend to increase the v_p/v_s ratio, since v_s is more sensitive than v_p to flat cracks (Wilkens *et al.*, 1984).

Fluctuations in λ in the earth have important implications for oil and gas exploration. Gassmann's well-known results (Gassmann, 1951) show that, when isotropic porous elastic media are saturated with any fluid, the fluid has no mechanical effect on the shear modulus μ, but can have a significant effect on the bulk modulus $K = \lambda + \frac{2}{3}\mu$, and therefore on λ. Thus, observed variations in μ have no direct information about fluid content, while observed variations in λ, especially if they are large variations, may contain imporant clues about variations in fluid content. So the observed structure of ϵ in (26) strongly suggests that small positive and all negative values of ϵ may be important indicators of significant fluctuations in fluid content.

If the finely layered medium is composed of only two distinct types of isotropic elastic materials and they appear in the layering sequence with equal spatial frequency, then we find that

$$2\epsilon = (\mu_2 - \mu_1)\frac{(\lambda_2 - \lambda_1) + (\mu_2 - \mu_1)}{(\lambda_1 + 2\mu_1)(\lambda_2 + 2\mu_2)}. \tag{27}$$

This result agrees with Postma (1955) except for an obvious typographical error in the denominator of his published formula. This formula shows clearly that if $\mu_1 = \mu_2$ then Thomsen parameter ϵ is identically equal to zero as expected. Also, if $\mu_1 \neq \mu_2$ but $\lambda_1 = \lambda_2$, then (27) implies $\epsilon \geq 0$, as we inferred from (26).

Now, we can use this formula to deduce the smallest possible value of the right hand side of (27). The shear moduli must not be equal, so without loss of generality we suppose that $\mu_2 > \mu_1$. Then, the numerator is seen to become negative by taking λ_2 towards negative values and $\lambda_1 \to +\infty$. The smallest value λ_2 can take is determined by the bulk modulus bound $\lambda_2 + \frac{2}{3}\mu_2 \geq 0$. So we may set $\lambda_2 = -\frac{2}{3}\mu_2$ in both the numerator and denominator. This choice also makes the factor $\lambda_2 + 2\mu_2 = \frac{4}{3}\mu_2$ as small as possible in the denominator, thus helping to magnify the effect of the negative numerator as much as possible. The result so far is that

$$2\epsilon = \frac{3}{4}\left(\frac{\mu_2 - \mu_1}{\mu_2}\right)\left(\frac{-\lambda_1 + \mu_2/3 - \mu_1}{\lambda_1 + 2\mu_1}\right). \tag{28}$$

The parameter λ_1 may vary from $-\frac{2}{3}\mu_1$ to plus infinity. At $\lambda_1 = -\frac{2}{3}\mu_1$, the second expression in parentheses is positive; but, this expression is also a monotonically decreasing function of λ_1 and approaches -1 as $\lambda_1 \to +\infty$. Thus, the smallest value of Thomsen's parameter ϵ is given by

$$\epsilon = -\frac{3}{8}\frac{\mu_2 - \mu_1}{\mu_2} \geq -\frac{3}{8}. \tag{29}$$

This result differs by a factor of 2 from the corresponding result of Postma (1955), which was obtained improperly by allowing three of the four elastic constants to vanish and also using a physically motivated but unnecessary restriction that both λ_1 and λ_2 must be nonnegative. If we had used the nonnegativity constraint on the λ's, the present result would have changed to

$$\epsilon = 1 - \frac{\mu_2 - \mu_1}{2\mu_2} \geq -\frac{1}{4}, \tag{30}$$

which is the same *inequality* as that found by Postma, but his *equality* differed from that in (30) and was in fact improperly obtained.

As a final point about the formula (25), we note that it implies in general that

$$a \leq \langle \lambda + 2\mu \rangle , \tag{31}$$

so we have an upper bound on ϵ in finely layered media stating that

$$2\epsilon \leq \left(\langle \lambda + 2\mu \rangle \left\langle \frac{1}{\lambda + 2\mu} \right\rangle - 1\right) = \left(\left\langle v_p^2 \right\rangle \left\langle v_p^{-2} \right\rangle - 1\right), \tag{32}$$

where the final form is true for layered media having constant density.

Before concluding this section, we want to note one further identity for ϵ. The general formula can be rearranged to give

$$\epsilon = 2\left[\left\langle \frac{\mu(\lambda + \mu)}{\lambda + 2\mu} \right\rangle \left\langle \frac{1}{\lambda + 2\mu} \right\rangle - \left\langle \frac{\lambda + \mu}{\lambda + 2\mu} \right\rangle \left\langle \frac{\mu}{\lambda + 2\mu} \right\rangle\right]. \tag{33}$$

This formula is not in Cauchy-Schwartz form, but is nevertheless probably the simplest form of the result for this anisotropy parameter. In particular, it is easy to see from this form that if either $\mu = constant$ or $\lambda + \mu = constant$, then the right hand side vanishes identically. The first result is well-known and the second has been known since Postma's (1955) work to be true for two-constituent periodic layered media [also see (27)]. The present result generalizes Postma's observation in this case.

THOMSEN PARAMETER δ

Thomsen's parameter δ is defined by

$$\delta = \frac{(f+l)^2 - (c-l)^2}{2c(c-l)}. \tag{34}$$

This parameter is considerably more difficult to analyze than either γ or ϵ for various reasons, some of which we will enumerate shortly.

Because of the controversy surrounding the sign of δ for finely layered media, we have performed a series of Monte Carlo simulations with the purpose of establishing the existence or nonexistence of layered models having positive δ. The simulations to be presented here were performed on models limited to two types of layers. Another set of simulations was performed having three types of layers, since it is known (Backus, 1962) that this is the most general case that needs to be considered, but we will not present that work here. The set of constant density models was chosen randomly by specifying the possible values of v_p to lie in some range $v_p^{min} \leq v_p \leq v_p^{max}$. The values chosen for the limiting wave speeds in the example shown in Figures 1 and 2 were $v_p^{min} = 2.5$ km/sec and $v_p^{max} = 5.5$ km/sec. The range of the shear wave velocity was similarly specified by constraining the ratio v_s/v_p to lie within a range $r_{min} \leq v_s/v_p \leq r_{max}$, where for the case being presented $r_{min} = 0.35$ and $r_{max} = 0.65$. The density was chosen to be $\rho = 2670$ kg/m^3, but any choice of constant density would have resulted in the same dimensionless Thomsen parameters. For comparison, note that the density of quartz is about 2650 kg/m^3 and the density of calcite is about 2720 kg/m^3 (*e.g.*, Wilkens *et al.*, 1984).

Figure 1 shows the Monte Carlo results for δ plotted versus η from 500 random models, where

$$\eta = \frac{\epsilon - \delta}{1 + 2\delta}. \tag{35}$$

The parameter η has been shown to be useful in seismic processing by Alkhalifah and Tsvankin (1995). It is particularly useful in the present context because it is known that for layered media this parameter will always be positive because $\epsilon - \delta$ will always be positive (Postma, 1955; Backus, 1962; Berryman, 1979). We see that most models are clustered around the origin $(\eta, \delta) = (0, 0)$, but there is no question that a large fraction of the models have positive δ. For this particular method of generating the models, we find that about 25% of the two-layer models have positive δ.

Encouraged by the Monte Carlo simulation results, we have analyzed δ in light of the Backus formulas in order to determine whether it was possible to understand those circumstances in which $\delta > 0$ are most likely to occur. Using the formulas (8)-(13) derived by Backus (1962), it is not hard to show that one natural form of the expression for δ is

$$\delta = 2\left\langle \frac{\lambda+\mu}{\lambda+2\mu} \right\rangle \left\langle \frac{\lambda+\mu}{\mu(\lambda+2\mu)} \right\rangle^{-1} \left[\left\langle \frac{1}{\mu} \right\rangle \left\langle \frac{\lambda+\mu}{\lambda+2\mu} \right\rangle - \left\langle \frac{\lambda+\mu}{\mu(\lambda+2\mu)} \right\rangle \right]. \tag{36}$$

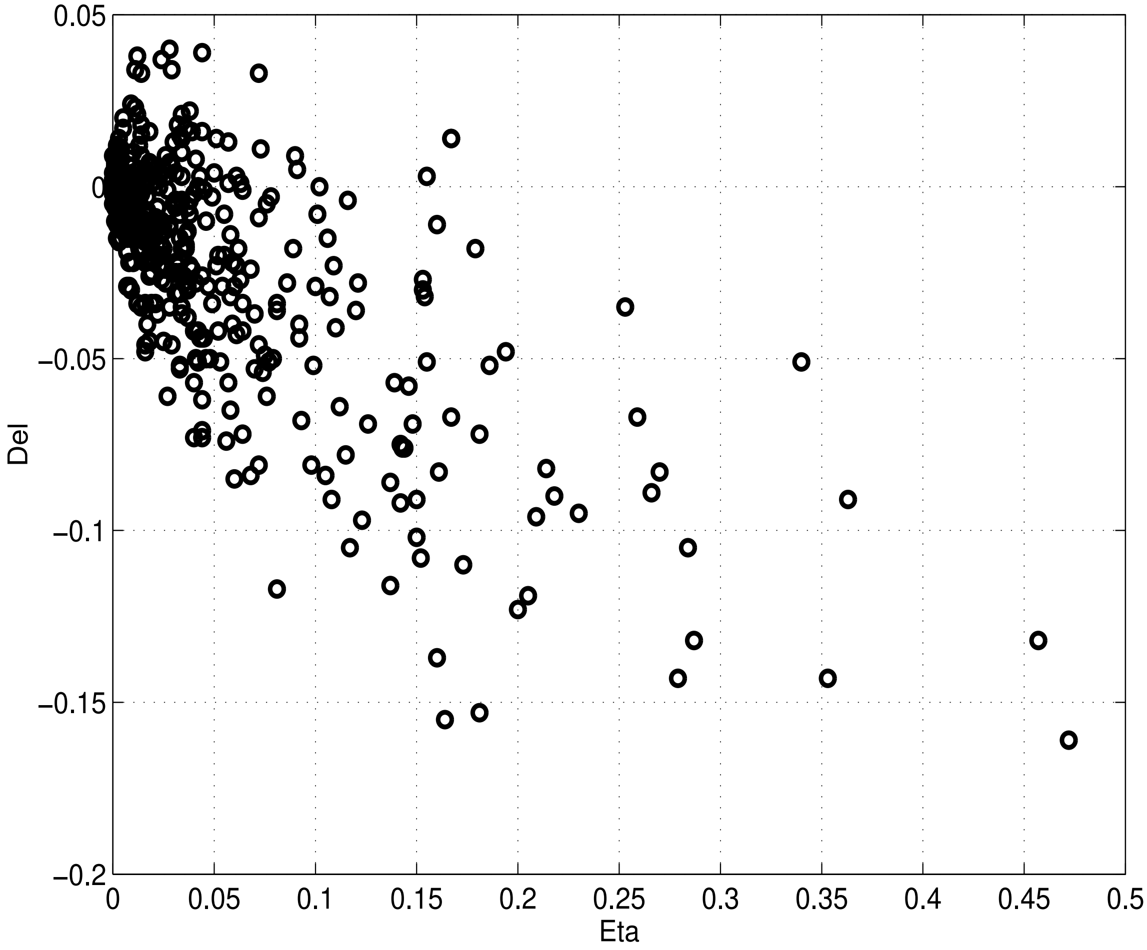

Figure 1: Crossplot of δ versus η for 500 random examples of layered media according to the algorithm in the text. jim1-eta.del.500 [NR]

It is easy to show that the prefactors are all always positive. So the sign of δ is determined by the expression in brackets. Using simple algebraic manipulations, this expression can be rewritten in a number of different but useful forms, including

$$
\begin{aligned}
&\left\langle \frac{1}{\mu} \right\rangle \left\langle \frac{\lambda+\mu}{\lambda+2\mu} \right\rangle - \left\langle \frac{\lambda+\mu}{\mu(\lambda+2\mu)} \right\rangle = \\
&\qquad \left\langle \frac{1}{\lambda+2\mu} \right\rangle \left\langle \frac{\lambda}{\lambda+2\mu} \right\rangle - \left\langle \frac{\lambda}{\mu(\lambda+2\mu)} \right\rangle \left\langle \frac{\mu}{\lambda+2\mu} \right\rangle = \\
&\qquad \left\langle \frac{1}{\lambda+2\mu} \right\rangle \left\langle \frac{\lambda+\mu}{\lambda+2\mu} \right\rangle - \left\langle \frac{\lambda+\mu}{\mu(\lambda+2\mu)} \right\rangle \left\langle \frac{\mu}{\lambda+2\mu} \right\rangle = \\
&\qquad\qquad \left\langle \frac{1}{\lambda+2\mu} \right\rangle - \left\langle \frac{1}{\mu} \right\rangle \left\langle \frac{\mu}{\lambda+2\mu} \right\rangle .
\end{aligned} \tag{37}
$$

The manipulations required to arrive at these results will be described in more detail elsewhere. The important point about this sequence of equalities is that each successive one can be used to prove something general about the sign of δ for layered media. In each case, we note that if some multiplicative factor is constant in all layers of the finely layered medium, then that factor may be removed from the averages. Using this technique, the first transformation shows that if $\lambda = \mathit{constant}$, then the expression and therefore δ is nonpositive. Similarly, the second equality in the sequence shows that if $\lambda + \mu = \mathit{constant}$, then δ is also nonpositive. The third equality shows that if $\lambda + 2\mu = \mathit{constant}$, then again δ is nonpositive. The very first expression may also be used in a slightly different way to arrive at a general result, $\delta = 0$ if Poisson's ratio is constant (Thomsen, 1986), for in that case the ratio λ/μ is also constant and so $(\lambda+\mu)/(\lambda+2\mu)$ is constant.

It is straightforward to show for layered media having only two constituents that the last expression in (37) implies that δ will be positive if

$$\frac{\lambda_1}{\lambda_2} > \frac{\mu_1}{\mu_2} > 1, \tag{38}$$

or if

$$\frac{\lambda_1}{\lambda_2} < \frac{\mu_1}{\mu_2} < 1, \tag{39}$$

showing that the fluctuations in the λ Lamé constant must be greater than those in the shear modulus μ to observe positive δ.

As discussed earlier, v_s/v_p ratios are decreased by the presence of pores, flat cracks, or the addition of clay, dolomite, feldspar, or calcite to silicic rocks. Any of these factors can cause fluctuations in λ without greatly changing μ.

For layered media with constant density, the final expression in (37) shows that

$$\mathrm{sign}(\delta) = \mathrm{sign}\left(\left\langle v_p^{-2} \right\rangle - \left\langle v_s^{-2} \right\rangle \left\langle v_s^2/v_p^2 \right\rangle \right). \tag{40}$$

Because our analysis establishes a similarity in the circumstances between the occurrence of positive δ and the occurrence of small positive or negative ϵ (*i.e.*, both occur when Lamé λ is fluctuating greatly from layer to layer), we have plotted δ versus ϵ in Figure 2. The Monte Carlo results used to produce Figure 2 are exactly the same as those used to produce Figure 1. We see that, as expected from the analysis, the positive values of δ are in fact most highly correlated with the small but positive values of ϵ. We should also keep in mind the fact that $\epsilon - \delta \geq 0$ is always true for layered models and this fact also plays a role in Figure 2, determining the unoccupied upper left hand corner.

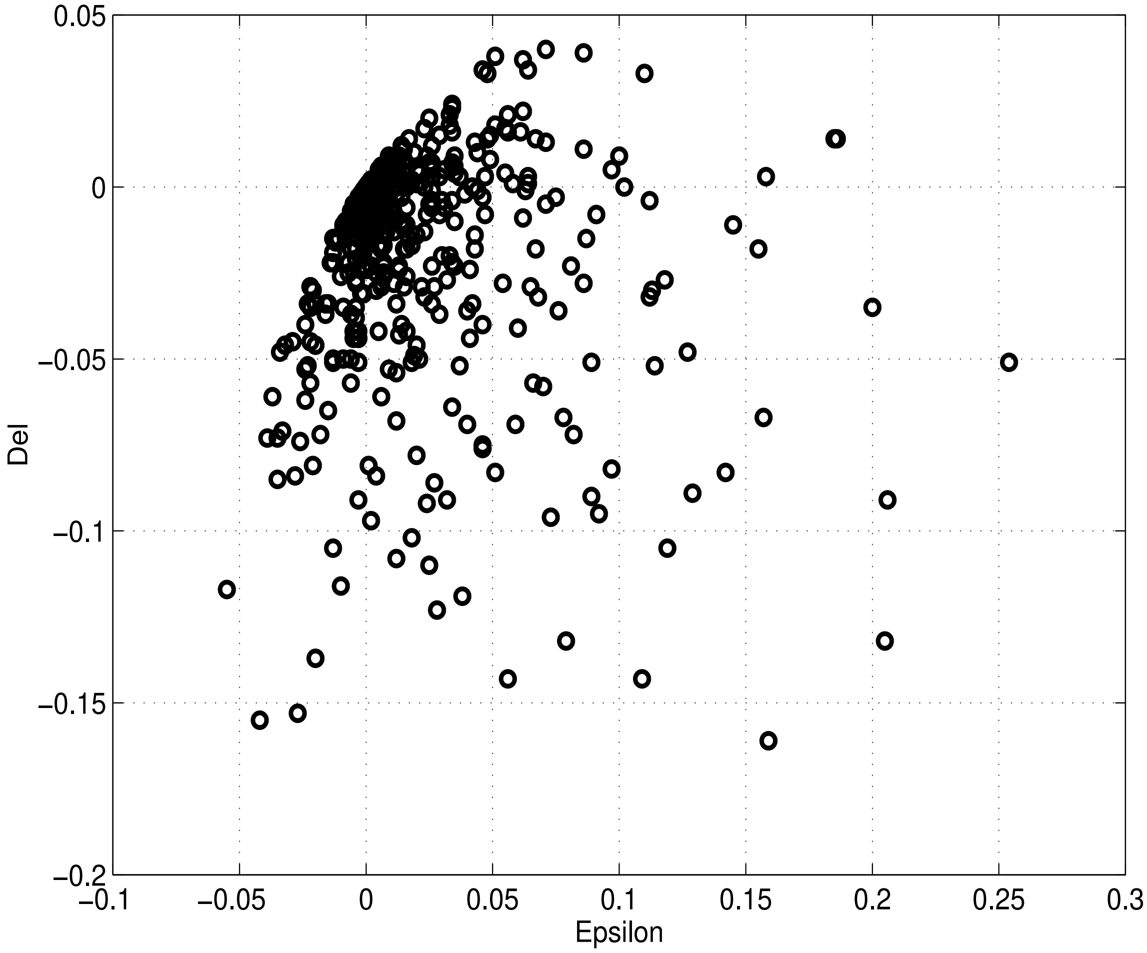

Figure 2: Crossplot of δ versus ϵ for 500 random examples of layered media according to the algorithm in the text. jim1-eps.del.500 [NR]

SUMMARY OF RESULTS ON THOMSEN PARAMETER BEHAVIOR

The results concerning the signs of the Thomsen parameters obtained in the preceding sections are summarized in Table 1.

TABLE 1. Behavior of anisotropy parameters as the layer material elastic parameters vary. All parameters vanish when $\mu = const$.

Anisotropy Parameters	$\lambda = const$ $\mu \neq const$	$\lambda + \mu = const$ $\mu \neq const$	$\lambda + 2\mu = const$ $\mu \neq const$	$\nu = const$ $\lambda, \mu \neq const$
ε	≥ 0	0	≤ 0	≥ 0
δ	≤ 0	≤ 0	≤ 0	0
γ	≥ 0	≥ 0	≥ 0	≥ 0
$\varepsilon - \delta$	≥ 0	≥ 0	≥ 0	≥ 0

The nonnegativity of γ and $\epsilon - \delta$ for layered models is well-known. The fact that ϵ can be either positive or negative, and the circumstances leading to negative values have been little appreciated before. A quick glance at the Table seems to indicate that δ is either zero or negative and this is perhaps why there has been so much confusion about the possibility of $\delta > 0$ for layered models. We have shown that the correct inference about positive δ follows rather simply from the result expressed in the last column of the Table. Since $\delta = 0$ when Poisson's ratio is constant or when $\mu = constant$, it is clear that we must have (for example) $\mu_1/\mu_2 > 1$ and $\lambda_1/\mu_1 \neq \lambda_2/\mu_2$ if δ is to be positive. The second expression can be rearranged to $\lambda_1/\lambda_2 \neq \mu_1/\mu_2 > 1$. Now we see that there are two possible cases, either $\lambda_1/\lambda_2 > \mu_1/\mu_2 > 1$ or $\lambda_1/\lambda_2 < \mu_1/\mu_2$. The first case results in positive δ, while the second case does not when $\mu_1/\mu_2 > 1$. This also shows that in perfectly random layered media one expects $\delta > 0$ to account for 50% of the models. In our simulation we found $\delta > 0$ accounted for only about 25% of the models produced, but this apparent discrepancy has been traced to a bias in the particular algorithm we used to generate the models. In any case, the earth does not have to obey perfectly random statistics and there is no reason to suppose that real layered earth will conform to these statistical considerations. Our main point is merely that $\delta > 0$ is entirely possible and quite understandable for layered earth models, and so it is not at all surprising that $\delta > 0$ is often observed in real data examples.

DISCUSSION

The results obtained here and in particular (25) and (26) show in general terms that Thomsen's parameter ϵ is smallest when the variation in the layer λ Lamé parameter is large, independent of the variation in the shear modulus μ. This result is therefore important for applications to porous layers containing pore fluids for which Gassmann's (1951) equation shows that the effects of fluids influence only the λ Lamé constants, not μ. Similarly, we find that δ is positive in finely layered media having large variations in λ. Thus, the regions of positive but small ϵ when occurring together with positive δ may be useful indicators of rapid spatial changes in fluid content in the layers. This result cannot, however, be considered in isolation, as there may be other effects in nonlayered media that produce similar values of the Thomsen parameters, and these should be explored and enumerated before any definitive statement can be made.

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