Stanford Exploration Project, Report 97, July 8, 1998, pages 353–363

Stanford Exploration Project, Report 97, July 8, 1998, pages 353–363

Helical factorization of the Helmholtz equation

James Rickett and Jon Claerbout[1]

keywords: *helix, wave equation, depth-migration*

ABSTRACT

The accuracy of conventional explicit wavefield extrapolation algorithms at high dips is directly related to the length of the convolution filters: increasing the dip range leads to increased cost. Recursive filters have the advantage over convolutional filters in that short filters can move energy long distances. We discard both Crank-Nicolson and McClellan transforms, and extrapolate waves by factoring the 3-D Helmholtz equation in a helical coordinate system. We show that one of the minimum-phase factors provides a 90° extrapolator, that can be applied recursively in the $(\omega - \mathbf{x})$ domain. By developing a purely recursive wavefield extrapolator, we hope to achieve accuracy at high dips with shorter filters than is possible with explicit methods.

INTRODUCTION

Depth migration algorithms are important for imaging in areas with strong lateral velocity gradients. Wavefield extrapolation algorithms in the $(\omega - \mathbf{x})$ domain have the advantage over Kirchhoff depth migration methods that they are based on finite bandwidth solutions to the wave-equation not asymptotic approximations. Additionally, they have the advantage over $(\omega - \mathbf{k})$ methods that they can easily handle lateral velocity variations in a single migration.

Claerbout (1985) describes implicit 2-D wavefield extrapolation based on the Crank-Nicolson formulation. This rational operator is unitary, and so represents a pure phase-shift. Unfortunately, however, the simple extension to 3-D leads to prohibitive computational complexity.

In practice, wavefield extrapolation in 3-D is usually accomplished with explicit operators and McClellan transforms (Hale, 1990a,b). Unfortunately again, however, accuracy at high dips (large spatial wavenumbers) can only be achieved with long explicit filters.

In this paper, we construct a finite-difference approximation to the Helmholtz operator in the $(\omega - \mathbf{x})$ domain. The helical coordinate system allows us to remap

[1]**email:** james@sep.stanford.edu, jon@sep.stanford.edu

the multi-dimensional operator into one-dimensional space, where we can find two minimum-phase factors using a conventional spectral factorization algorithm.

Each minimum-phase factor provides a recursive filter that we can use to extrapolate the wavefield in depth. Recursive filters move energy longer distances than explicit filters of the same length. By developing a purely recursive wavefield extrapolator, the hope is to achieve accuracy at high dips with shorter filters than is possible with explicit methods.

POISSON'S EQUATION

As a simple illustration of how helical boundary conditions can lead to recursive solutions to partial differential equations, we consider Poisson's equation, which relates potential, u, to source density, f, through the Laplacian operator:

$$\nabla^2 u = f(x, y, z) \tag{1}$$

Poisson's equation crops up in many different branches of physics: for example, in electrostatics, gravity, fluid dynamics (where the fluids are incompressible and irrotational), and steady-state temperature studies. It also serves as a simple analogue to the wave-propagation equations which provide the main interest of this paper.

To solve Poisson's equation on a regular grid (Claerbout, 1997a), we can approximate the Laplacian by a convolution with a small finite-difference filter. Taking the operator, $\mathbf{D}$, to represent convolution with filter, d, Poisson's equation becomes

$$\mathbf{D}\ \mathbf{u} = \mathbf{f}. \tag{2}$$

Although $\mathbf{D}$ itself is a multi-dimensional convolution operator that is not easily invertible, helical boundary conditions (Claerbout, 1997a) allow us to reduce the dimensionality of the convolution to an equivalent one-dimensional filter, which we can factor into the product of a lower-triangular matrix, $\mathbf{L}$, and its transpose, $\mathbf{L}^T$. These triangular matrices represent causal and anti-causal convolution with a minimum-phase filter, in the form

$$\mathbf{D}\ \mathbf{u} = \mathbf{L}^T\ \mathbf{L}\ \mathbf{u} = \mathbf{f}. \tag{3}$$

We can then calculate u directly since $\mathbf{L}$ and its transpose are easily invertible by recursive polynomial division:

$$\mathbf{u} = \mathbf{L}^{-1}\ (\mathbf{L}^T)^{-1}\ \mathbf{f}. \tag{4}$$

THE HELMHOLTZ EQUATION

Starting from the full wave equation in three-dimensions:

$$-\nabla^2 p + \frac{1}{v^2}\frac{\partial^2 p}{\partial t^2} = 0 \tag{5}$$

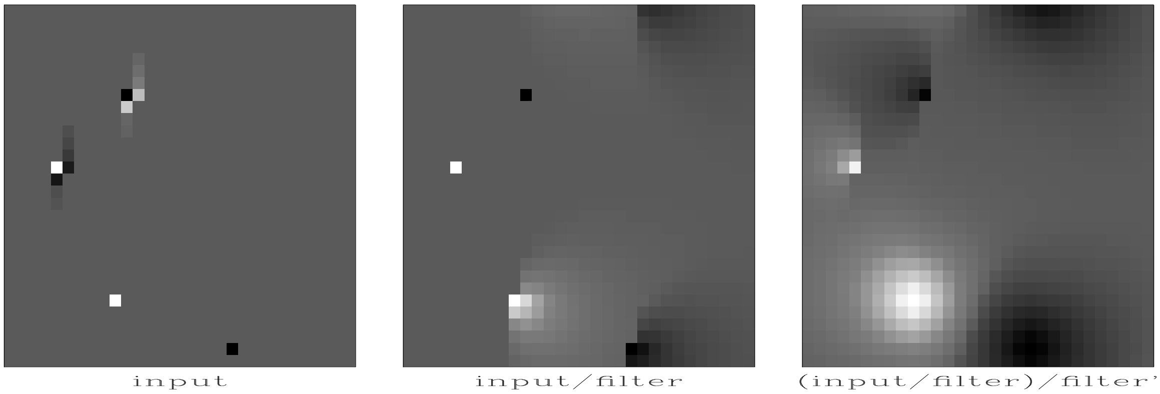

Figure 1: Deconvolution by a filter whose autocorrelation is the two-dimensional Laplacian operator. This amounts to solving the Poisson equation. After Claerbout (1997a). james3-lapfac [CR]

we can Fourier transform the time axis, and look for $(\omega, \mathbf{x})$ solutions of the form:

$$p(\omega, \mathbf{x}) = q(\mathbf{x})\, e^{-i\omega t} \tag{6}$$

For a single frequency, the wave equation therefore reduces to the Helmholtz (time-independent diffusion) equation

$$\left(-\nabla^2 - \alpha^2\right) q(\mathbf{x}) = 0 \tag{7}$$

where $\alpha = \omega/v$.

We aim to factor this equation on a helix, as with the Poisson equation above. However, before we can, we need to ensure that it is a 'level-phase' function (Claerbout, 1998), that is to say the spectrum of the operator does not touch the negative real axis on the complex plane. The spectrum of the Helmholtz operator can be obtained by taking the Fourier transform of equation (7).

$$S(\mathbf{k}) = |\mathbf{k}|^2 - \alpha^2 \tag{8}$$

$S(\mathbf{k})$ clearly becomes negative real for small values of $|\mathbf{k}|$; so as it stands, this equation is not factorable. Fortunately, however, replacing α by $\alpha' = \alpha - \epsilon/i$, where ϵ is a small positive number, successfully stabilizes the spectrum, by pushing the function off the negative real axis. The physical effect of ϵ is to provide damping as the wave propagates, differentiating between the forward and backward extrapolation directions.

Before factorization, equation (7) should therefore be rewritten to include the stabilization term

$$\left[-\nabla^2 + (-i\alpha + \epsilon)^2\right] q(\mathbf{x}) = \left(-\nabla^2 - \alpha'^2\right) q(\mathbf{x}) = 0 \tag{9}$$

Following the helix solution to Poisson's equation above, a simple finite-difference approximation to the Laplacian, $-\nabla^2 \approx \mathbf{D}$, produces the matrix equation:

$$\left(\mathbf{D} - \alpha'^2 \mathbf{I}\right) \mathbf{q} = \mathbf{0} \tag{10}$$

Alternatively, and more accurately, we can form a rational approximation to the Laplacian operator,

$$-\nabla^2 \approx \frac{\mathbf{D}}{\mathbf{I} + \beta \mathbf{D}} \tag{11}$$

where β ($\approx 1/6$) is Claerbout's (1985) adjustable 'one-sixth' parameter, and $\mathbf{D}$ again represents convolution with a simple finite-difference filter, d.

Inserting equation (11) into equation (9) yields a matrix equation of similar form, but with increased accuracy at high spatial wavenumbers:

$$\left(\frac{\mathbf{D}}{\mathbf{I} + \beta \mathbf{D}} - \alpha'^2 \mathbf{I}\right) \mathbf{q} \quad = \quad \mathbf{0} \tag{12}$$

$$\left[(1 - \alpha'^2 \beta)\mathbf{D} - \alpha'^2 \mathbf{I}\right] \mathbf{q} = \mathbf{H}\, \mathbf{q} \quad = \quad \mathbf{0} \tag{13}$$

The operator on the left-hand-side of equation (13) represents a three-dimensional convolution matrix, that can be mapped to an equivalent one-dimensional convolution by applying helical boundary conditions. Although the complex α' coefficients on the main diagonal cause the matrix not to be Hermitian, the spectrum of the matrix is of level-phase. Therefore, for constant α', it can be factored into causal and anti-causal (triangular) components with any spectral factorization algorithm that has been adapted for cross-spectra (Claerbout, 1998).

$$\mathbf{H}\, \mathbf{q} = \mathbf{U}\mathbf{L}\, \mathbf{q} = \mathbf{0} \tag{14}$$

The challenge of extrapolation is to find $\mathbf{q}$ that satisfies both the above equation and our initial conditions, $\mathbf{q}_{z=0}$. Starting from $\mathbf{q}_{z=0}$, we can invert $\mathbf{L}$ recursively to obtain a function that satisfies both the initial conditions, and

$$\mathbf{L}\, \mathbf{q} = \mathbf{0}. \tag{15}$$

Hence $\mathbf{q}$ will also satisfy equation (14).

WAVE EXTRAPOLATION

The basis for wavefield extrapolation is an operator, $W(k)$, that marches the wavefield q, at depth z, down to depth $z + 1$.

$$q_{z+1} = W\, q_z. \tag{16}$$

Gazdag:	$W(\mathbf{k}) = e^{i\sqrt{\alpha^2-\|\mathbf{k}\|^2}}$
Implicit:	$W(\mathbf{k}) = e^{i\alpha}\ \frac{A(\mathbf{k})}{B(\mathbf{k})}$
Implicit with helical factorization:	$W(\mathbf{k}) = e^{i\alpha}\ \frac{U_A(\mathbf{k})L_A(\mathbf{k})}{U_B(\mathbf{k})L_B(\mathbf{k})}$
Explicit:	$W(\mathbf{k}) = e^{i\alpha}\ C(\mathbf{k})$
Helmholtz factorization:	$W(\mathbf{k}) = \frac{1}{L(\mathbf{k})}$

Table 0.1: Comparison of the mathematical form of various wavefield extrapolators

Ideally, $W(\mathbf{k})$, will have the form of the phase-shift operator (Gazdag, 1978),

$$W(\mathbf{k}) = e^{i\sqrt{\alpha^2-|\mathbf{k}|^2}}. \tag{17}$$

Due to lateral velocity variations, and the desire to avoid spatial Fourier transforms, approximations to $W(\mathbf{k})$ are often applied in the $(\omega - \mathbf{x})$ domain. Typically $W(\mathbf{k})$ is split into a 'thin-lens' term that propagates the wave vertically, and a 'diffraction' term that models more complex wave phenomena. In the $(\omega - \mathbf{x})$ domain, the thin-lens term can be applied as a simple phase-shift, while the diffraction term is approximated by a small finite-difference filter. The method of extrapolation determines the nature of the finite-difference filter. The mathematical forms of different extrapolators are summarized in Table 1, and discussed below.

Implicit extrapolation approximates $W(\mathbf{k})$ with a rational form, consisting of a convolutional filter, and an inverse filter. The traditional Crank-Nicolson implicit formulation ensures the pair of convolutional operators, $A(\mathbf{k})$ and $B(\mathbf{k})$, are complex conjugates, and so the resulting extrapolator is unitary. Implicit methods apply an extrapolator of the form

$$W(\mathbf{k}) = e^{i\alpha}\ \frac{A(\mathbf{k})}{B(\mathbf{k})}. \tag{18}$$

Although implicit extrapolation is often the method of choice in 2-D, unfortunately the cost of the matrix inversion means traditional implicit extrapolation is rarely possible in 3-D. Helical boundary conditions facilitate 3-D implicit methods by providing a way to decompose the filters into an upper and lower triangular pair, which can be easily inverted (Rickett et al., 1998). The extrapolator in equation (18), therefore, becomes

$$W(\mathbf{k}) = e^{i\alpha}\ \frac{U_A(\mathbf{k})L_A(\mathbf{k})}{U_B(\mathbf{k})L_B(\mathbf{k})}. \tag{19}$$

Most practical 3-D extrapolation is done with an explicit operator, using McClellan transforms. This approach amounts to approximating $W(\mathbf{k})$ by with a simple convolutional filter, $C(\mathbf{k})$. Explicit extrapolators, therefore, have the form

$$W(\mathbf{k}) = e^{i\alpha}\, C(\mathbf{k}). \tag{20}$$

In contrast to these methods, the minimum-phase factorization of the Helmholtz equation provides appears to provide a recursive depth extrapolator of the different form

$$W(\mathbf{k}) = \frac{1}{L(\mathbf{k})}, \tag{21}$$

where $L(\mathbf{k})$ is a minimum-phase filter.

The inverse of the Helmholtz factorization, therefore, is all poles and no zeros. However, the apparent contradiction that we are approximating the unitary operator in equation (17) with the minimum-phase extrapolator in equation (21) is resolved by examining the impulse response of the operator $\frac{1}{L(\mathbf{k})}$ shown in Figure 2. The inverse of the factor is indeed minimum-phase if you consider the response at depths z and $z+1$. However, we are only interested in the response at depth step $z+1$, i.e. the second bump in the lower panel of Figure 2, which is symmetric and tapers to zero away from the location of the impulse. We are not concerned by the first bump on the lower panel of Figure 2, as this corresponds to the response of the filter at depth step z.

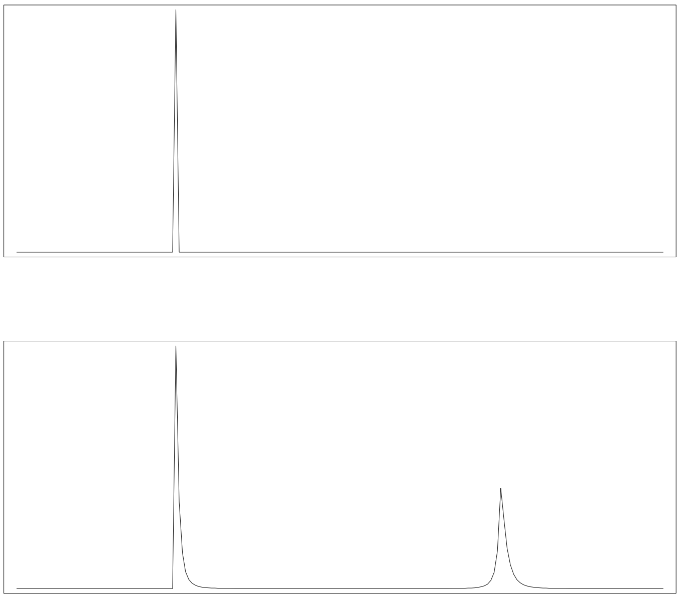

Figure 2: Amplitude of impulse response of polynomial division with minimum-phase factorization of the Helmholtz equation. The top panel shows the location of the impulse. The bottom panel shows the impulse response. Helical boundary conditions mean the second bump in the impulse response corresponds to energy propagating to the next depth step. james3-impresp [CR]

Propagating waves with the `Wavemovie` program

The following pseudo-code provides an algorithm for propagating waves into the Earth with the the new factorization of the wave equation.

```
Fourier Transform input data
```

```
Loop over frequency {
     Initialize wave at z=0
     Factor wave equation for this w/v
     Recursively divide input data by factor
     Fourier Transform back to time-domain
     Sum into output
}
```

Incorporating this code into the `Wavemovie` program (Claerbout, 1985) provides a laboratory for testing the new algorithm.

Figure 3 compares the results of the new extrapolation procedure with the conventional Crank-Nicolson solution to the 45° equation. The new approach has little dispersion since we are using a rational approximation (the 'one-sixth trick') to the Laplacian on the vertical and horizontal axes. In addition, the new factorization retains accuracy up to 90°. The high dip, evanescent energy in the 45° movie, propagates correctly in the new approach.

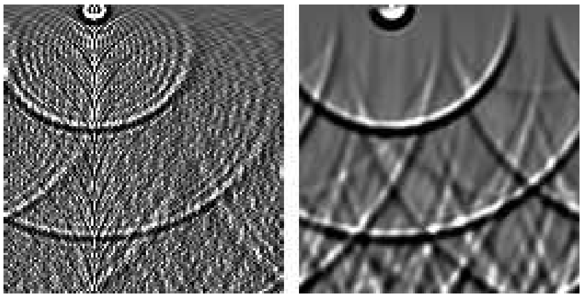

Figure 3: Comparison of the 45° wave equation (left) with the helical factorization of the Helmholtz equation (right). james3-vs45 [ER]

Figure 4 compares different value of the 'one-sixth' parameter, β. For this application, the optimal value seems to be $\beta = 1/12$.

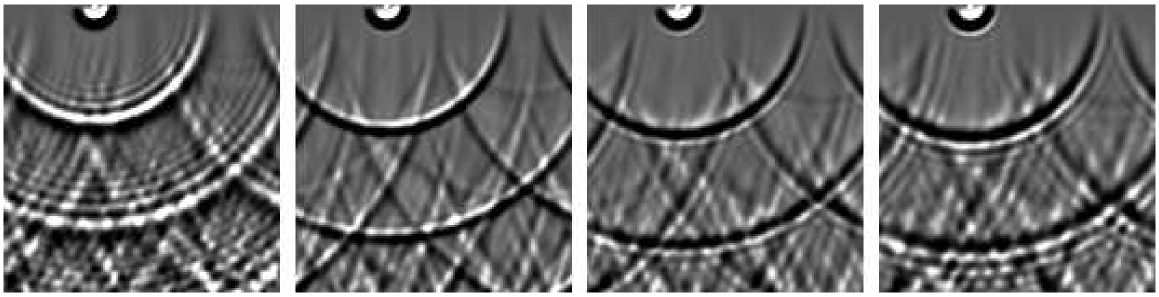

Figure 4: Helmholtz equation factorization with different values for the 'one-sixth' parameter, β. From left, $\beta = 0$, 1/12, 1/8 and 1/6. james3-sixth [CR]

Figure 5 compares different 3×3 finite-difference filters. $\gamma = 0$ corresponds to the conventional 5-point filter, while $\gamma = 0$ corresponds to a rotated 5-point filter. Values in the range $0 < \gamma < 1$ correspond to 9-point filters that are linear combinations of the above. Best results are obtained with $\gamma = 2/3$. The impulse response with $\gamma = 0$ only contains energy on every second grid point, since the rotated filter only propagates energy diagonally: as in the game of a chess, if a bishop starts on a white square, it always stays on white.

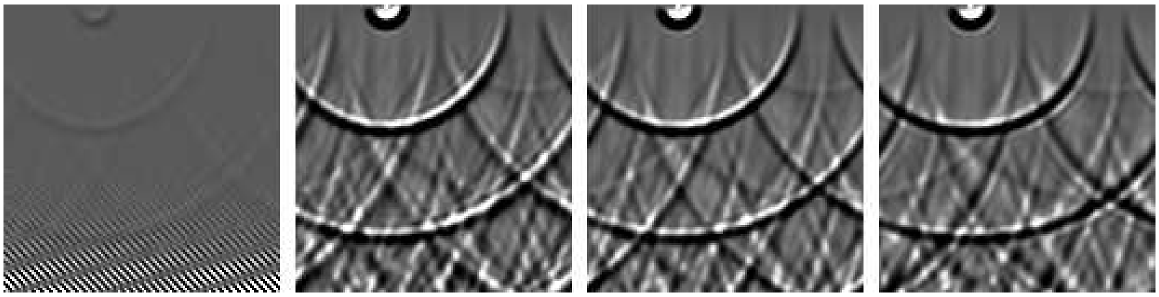

Figure 5: Helmholtz equation factorization with different 3×3 finite-difference representations of the Laplacian. From left, $\gamma = 0$, $1/2$, $2/3$ and 1. james3-laplac [CR]

REDUCING THE FILTER LENGTH

Although mostly zeros, the 9-point filter we factorize in 2-D is actually $2N_x + 3$ points long on a helix, and so its two factors each contain $N_x + 2$ points. For Figures 3, 4 and 5 above, we did not discard any of these filter coefficients: our recursive filters contained the full 128 coefficients. For this extrapolator to compete effectively with other methods (especially in 3-D problems), the number of filter coefficients has to be reduced significantly.

The factor coefficients themselves should be independent of the diameter of the helix. This leads us to expect many zero value coefficients on the helix 'backside'. Fortunately this is observed in practice, and the filter coefficient amplitude drops rapidly from either end.

Unfortunately, however, the operator that we factor has roots very close to the unit circle in the complex plane. The Laplacian already has a pair of roots at $Z = 0$, and the effect of the extra α^2 term on the main diagonal is to destabilize the operator further. The damping rescues us in theory, but in practice we still encounter numerical problems depending on the factorization algorithm, and the two factors are barely minimum-phase.

At the time of this report, we have been using the Fourier-domain Kolmogoroff method (Kolmogoroff, 1939; Claerbout, 1998) to factor the Helmholtz equation. The Kolmogoroff method has two main problems, both due to the proximity of the roots to the unit circle. Firstly, circular boundary conditions require us to pad the cross-correlation function before transforming it to the Fourier domain. With the roots close to the unit circle, extreme amounts of padding are needed: in the 2-D examples above, we need to pad the filters to over 4,000 times their original length. Secondly, the Kolmogoroff method simultaneously computes all the filter coefficients. With roots so close to the unit circle, truncating filter coefficients, even in a reasonable manner, often leads to non-minimum-phase filters and divergent results.

The Wilson-Burg algorithm (Wilson, 1969; Sava et al., 1998) may eventually overcome these problems. By working in the time domain, the algorithm avoids circular boundary conditions, and the number of filter coefficients can be defined at each iteration, providing a best-fit filter with a given number of coefficients. However, the Wilson-Burg algorithm also encounters problems with roots close to the unit circle. Specifically, numerical problems cause filters to lose their minimum-phase nature, causing the algorithm to diverge. With roots very close to the unit circle, this can happen within the first couple of iterations, so even the best result before divergence starts may be unusable.

CONCLUSIONS

We have shown that the minimum-phase helical factorization of the Helmholtz equation can be applied recursively to extrapolate waves up to 90°. The hope is that a recursive extrapolator may image steeper dips than an explicit extrapolator with the same cost. Unfortunately, however, technical problems related to the factoring of functions with roots near the unit circle, are currently preventing this method from competing effectively with conventional extrapolation algorithms.

REFERENCES

Claerbout, J. F., 1985, Imaging the earth's interior: Blackwell Scientific Publications.

Claerbout, J., 1997, Multidimensional recursive filters via a helix: SEP–**95**, 1–13.

Claerbout, J., 1998, Factorization of cross spectra: SEP–**97**, 337–342.

Gazdag, J., 1978, Wave equation migration with the phase-shift method: Geophysics, **43**, no. 7, 1342–1351.

Hale, 1990a, 3-D Depth migration via McClellan transformations: Geophysics, **56**, no. 11, 1778–1785.

Hale, 1990b, Stable explicit depth extrapolation of seismic wavefields: Geophysics, **56**, no. 11, 1770–1777.

Kolmogoroff, A. N., 1939, Sur l'interpolation et l'extrapolation des suites stationnairres: C.R. Acad.Sci., **208**, 2043–2045.

Rickett, J., Claerbout, J., and Fomel, S., 1998, Implicit 3-D depth migration by wavefield extrapolation with helical boundary conditions: SEP–**97**, 1–12.

Sava, P., Rickett, J., Fomel, S., and Claerbout, J., 1998, Wilson-Burg spectral factorization with application to helix filtering: SEP–**97**, 343–352.

Wilson, G., 1969, Factorization of the covariance generating function of a pure moving average process: SIAM J. Numer. Anal., **6**, no. 1, 1–7.