WAVE EQUATION MULTIPLE SUPPRESSION: CORRESPONDENCE WITH TIME SERIES

Jon F. Claerbout

Wave equation methods should enable a correct prediction of the topographically caused features of multiple reflections. This theoretical potential has not yet been routinely realized in production processing. The reasons are a subject of current research. Thus it is a valuable exercise to specialize the wave equation methods to the case of flat horizontal layers. Then the theory reduces to conventional time series analysis so that noises and errors can be analyzed by conventional methods.

The Source and the Free Surface

Techniques for processing multiple reflections are an extension of those for processing primaries. However, in addition to a computer mesh on which to hold the upcoming wave U(x,t) we need a second computer mesh on which to hold a downgoing wave D(x,t). Also we need a free surface condition that upcoming waves change polarity and reflect back down into the earth.

$$D(x,t,z=0) \quad = \quad \text{Source} - U(x,t,z=0) \tag{1}$$

The source function may be fairly well known, or maybe it isn't. Anyway, in the final analysis, the sensitivity of the predicted multiples to error in knowledge of the source waveform will be important. In modeling problems, the source may be taken to be a delta function on a plane wave (or a spherical wave) and any waveform can be convolved on later.

Interior Equations

Next we need to choose some interior equations. Since our final objective is data processing, not forward modeling, we naturally choose extrapolation equations instead of the full scalar wave equation. A good place to start is with equation (10-5-1) on page 216 of ***Fundamentals of Geophysical Data Processing*** (my book, 1976), which is

$$\frac{d}{dz}\begin{bmatrix} U \\ D \end{bmatrix} = \begin{bmatrix} -iab & 0 \\ 0 & iab \end{bmatrix}\begin{bmatrix} U \\ D \end{bmatrix} - \frac{1}{2}\frac{Y_z}{Y}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} U \\ D \end{bmatrix} \tag{2}$$

along with the definitions on pages 171-172 of FGDP, namely

$$k_z = ab = \omega\left(\frac{\rho}{k} - \frac{k_x^2}{\omega^2}\right)^{\frac{1}{2}} = \frac{\omega}{v}\cos\theta \tag{3}$$

$$Y = \frac{b}{a} = \frac{1}{\rho v}\left(1 - \frac{v^2 k_x^2}{\omega^2}\right)^{\frac{1}{2}} = \frac{\cos\theta}{\rho v} \tag{4}$$

From equation (2) we deduce that if the impedance Y is independent of depth z, then the upcoming wave U and the downgoing wave D are uncoupled. In the analysis of primary reflections, the cross term multiplying Y_z in (2) is usually neglected. Now we are trying to include some of the effects of this term. A way to introduce free surface multiples, and they are the multiples of most practical importance, is to introduce the assumption that the downgoing wave D is much larger than the upcoming wave U. This happens when the reflection coefficient $c \approx Y_z/Y$ is small compared to unity. We are assuming

$$U \ll D \tag{5a}$$

$$c = -\frac{1}{2}\frac{Y_z}{Y} \ll 1 \tag{5b}$$

Introducing (5) into (2) and neglecting the assumed small quantities we have

$$\frac{d}{dz} D = +iab\ D \tag{6a}$$

$$\frac{d}{dz} U = -iab\ U + cD \tag{6b}$$

We can bring (6) from the two-dimensional (ω, k_x) frequency domain to the time and space domain if the square root is approximated by a ratio of polynomials, and then substitute $-i\omega = \partial_t$ and $ik_x = \partial_x$. Experience with migration of primary reflections indicates that we may be able to satisfactorily describe lateral variations in material properties, say $v = v(x,z)$ and $c = c(x,z)$ with (6), even though it was actually derived with lateral Fourier transforms. I believe that the most significant resulting error in (6) will turn out to be in the angular dependence of the reflection coefficient from a dipping bed, but that this error will not have substantial practical consequences.

Diffraction and Slanting Effects

Continuing to abandon those complexities which appear not to be at the heart of the problem, we next expand the square roots in (6b), keeping only the 15-degree term

$$U_z = \frac{1}{v} U_t - \frac{v}{2} U_{xx}^{t} + c(x,z)\ D(x,z,t) \tag{7}$$

As a practical matter equation (7) no longer has the ability to deal with large offset. We might like to handle large offsets but small dips. A good way to deal with larger offsets is to do the 15-degree expansion about some Snell p parameter other than $p = 0$. This leads to some lateral shifts in the cD term, an interesting subject developed by Morley and Claerbout in SEP-15 (p. 191). But this, along with the U_{xx}^{t} diffraction term, bring us outside the realm of time series analysis, so we abandon all these terms, knowing that we can come back to get them when we have the courage.

Retardation

Our downgoing wave equation is now

$$D_z = -\frac{1}{v} D_t \tag{8}$$

Define retarded time t':

$$t' = t - \int_0^z \frac{dz}{v}$$

$$z' = z$$

The chain rule for partial differentiation states

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'} + \frac{\partial t'}{\partial z}\frac{\partial}{\partial t'} = \partial_{z'} - \frac{1}{v}\partial_{t'} \tag{9a}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \frac{\partial z'}{\partial z}\frac{\partial}{\partial z'} = \partial_{t'} \tag{9b}$$

The resulting equation for the downgoing wave is

$$\frac{\partial}{\partial z'} D' = 0 \tag{10}$$

For the upcoming wave U we need the retardation transformation

$$t'' = t + \int_0^z \frac{dz}{v} \tag{11a}$$

$$z'' = z \tag{11b}$$

which by the chain rule implies a wave equation for upcoming waves

$$\frac{\partial}{\partial z''} U'' = c(z'') D''(z'',t'')$$

$$= c''(z'') D'\left(z',t' = t'' - 2 \int_0^{z''} \frac{dz}{v}\right)$$

Travel Time-Depth Leads to Convolution

Things will be clarified and simplified if we now introduce a travel time-depth τ:

$$\tau = 2 \int \frac{dz}{v} \quad ; \qquad \frac{dz}{d\tau} = \frac{v}{2} \tag{12}$$

With this definition and dropping all the primes, our basic equations become

$$D_\tau = 0 \tag{13a}$$

$$U_\tau = \frac{v(\tau)}{2} c(\tau) D(\tau,t-\tau)$$

$$= c'(\tau) D(\tau,t-\tau) \tag{13b}$$

Integrating (13b) with respect to τ and noting that (13a) says we may ignore the first argument of D, we get a convolutional form

$$U = \int c(\tau) D(t - \tau) d\tau \tag{14}$$

Forward and Inverse Problems

Let us now re-express the concept embodied by (14) in terms of Z-transforms:

$$U(Z) = C(Z)\,D(Z) = C(Z)[S(Z) - U(Z)] \qquad (15)$$

Supposing that the source function S may be regarded as an impulse S(Z) = 1, we are left with

$$C(Z) = \frac{U(Z)}{1 - U(Z)} = \text{data processing} \qquad (16a)$$

$$U(Z) = \frac{C(Z)}{1 + C(Z)} = \text{modeling} \qquad (16b)$$

An interesting interpretation of (16a) is that reflectivity is defined as the ratio of the up- to the downgoing wave. In this way it is equivalent to Don C. Riley's "Noah" method.

Recapitulation

The theory developed here suppresses free surface multiples. We started with a very general theory and made one approximation after another until it all reduced to time series analysis. In time series analysis there are well-developed procedures for analysis of noise and stability. A way to apply this theory rather directly would be to make vertical incidence wave stacks (called SIMPLANS by Seiscom Delta, Inc.) of your field data. There is little hope that any of this will work on CDP stacks. If the source waveform can be satisfactorily handled as Riley did and if cable truncation phenomena are not too severe, it seems reasonable to hope that Morley's extension of this theory might work on slanted wave stacks, i.e. on Snell waves. No doubt there are also many other ingenious ways of applying this theory to field problems.