COMPUTING LOVE WAVES IN LATERALLY VARYING MEDIA BY WAVE EXTRAPOLATION

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Introduction

One-way wave extrapolation equations provide an economical and accurate method for certain types of wave modeling. In this paper we consider the extrapolation with the scalar wave equations, and illustrate the method with the extrapolation of Love wave normal modes in laterally varying media.

The basic restriction on extrapolation modeling is that the problem contains no backscattered waves. This condition arises because waves moving backward to the direction of extrapolation are not included in the solution unless they are explicitly coupled with reflection coefficients. However, the method can be used to model refracted body waves, head waves, and surface waves.

The economy of extrapolation methods is essentially one of storage. Finite difference solutions of the full wave equation require storage proportional to the product of the x and z grid dimensions. The one-way equations, in their monochromatic form, only require storage proportional to one of the dimensions.

Modeling with one-way equations has a further advantage in that boundary conditions at layer interfaces need not be matched explicitly. If the correct form of the variable velocity one-way equation is used,

$$[Q] = 0 \qquad \text{and} \qquad [b\hat{n}\cdot\nabla Q] = 0 \tag{2}$$

where $\hat{n}$ is the unit vector normal to the interface and the square brackets denote differences across the interface. For a horizontally layered medium the boundary conditions reduce to

$$[Q] = 0 \qquad \text{and} \qquad [b\partial_z Q] = 0 \ . \tag{3}$$

The one-way extrapolation equations are defined by the differential equation

$$(\partial_x - \frac{1}{v} S_{n+1})\, Q = 0 \tag{4}$$

where S_{n+1} is a square root approximation generated recursively by Muir's relation

$$S_{n+1} = \frac{i\omega S_n + (a\partial_z b\partial_z - \omega^2)}{i\omega + S_n}, \qquad S_0 = i\omega \tag{5}$$

The square root approximations given by equation (5) are well known to model accurately waves traveling within some cone of the extrapolation direction. The dispersion curves for the first and second approximations (15- and 45-degree) equations are shown in figure 2. The approximations have no evanescent zone in the k_z direction ($|k_z| > \omega/v$), but they do model behavior in the k_x evanescent zone ($|k_x| > \omega/v$). This is shown in the right panel of figure 2, which is simply a replotting of the dispersion relation to show its behavior for both real and imaginary values of k_x.

The square root operator in equation (5) will match the boundary conditions in (3) exactly for flat layers. This is because the z-differentials have the same form as the z-differentials in the full scalar wave equation. However, since the true boundary conditions are actually given by (2), the boundary conditions implied by the one-way

which is an operator with strictly positive real eigenvalues. Following Muir's rules for causal positive real operators, the approximation S_{n+1} can be shown to be unconditionally stable. Thus, to guarantee the stability for displacement solutions, the density has to be independent of z; while for pressure solutions, the bulk modulus has to be independent of z. It is conjectured that the form of equation (5) will be stable for at least mildly varying values of a.

Love Wave Modes

As an example of extrapolation modeling we have computed Love wave mode shapes in simple laterally varying media. The example is useful for testing the modeling method because the modes contain both a propagating and an evanescent component. Also, because the analytic solution for horizontal layers is simple (the mode shape does not change), it is easy to check whether the program is producing the correct answer for the layered case.

As mentioned earlier, the SH-type displacements are governed by equation (1) with a as the inverse of density and b the shear modulus. Love waves (SH surface waves) have a solution to this equation of the form (see Achenbach, p. 218-220)

$$u(x,z,t) = U(z) \exp i\omega(t - \frac{x}{c}) \qquad (6)$$

where u(x,z,t) is the displacement in the y direction, U(z) is the mode shape, and c is the phase velocity of the mode. For a layered earth structure the boundary conditions to be satisfied are:

1) $\partial_z U = 0$ at z=0 (zero free surface stress),
2) $[U] = 0$ at layer boundaries (continuity of displacement),
3) $[\mu \partial_z U] = 0$ at layer boundaries (continuity of stress).

For the simple case of a layer of thickness h with properties ρ_1 and μ_1 over a half-space with properties ρ_2 and μ_2, the mode shape is given by

The boundary conditions also impose the further restriction that

$$\cot(\nu_1 h) = \frac{\mu_1}{\mu_2} \frac{\nu_1}{i\nu_2} . \tag{9}$$

This equation is the period equation for the simplest type of Love waves, and it has solutions in the range

$$\frac{\omega}{\beta_1} > |k_x| > \frac{\omega}{\beta_2} .$$

With k_x in this range, the mode is propagating in the layer and evanescent in the half space. The period equation determines the relationship between ω and k_x, and since it is nonlinear, the modes are dispersive (velocity depends on frequency). The propagating speed of the mode is given by

$$c(\omega) = \frac{\omega}{k_x(\omega)} . \tag{10}$$

To test that the extrapolation works for a layered case, an initial mode shape was specified according to equation (7). The solution was then extrapolated in the x direction with a monochromatic 45 degree equation. The density was assumed constant. This allowed the guaranteed stable $\partial_z v^2 \partial_z$ form of the z differential to be used. At each step in the x direction the solution was multiplied before plotting by the demodulation factor

$$\exp i\frac{\omega}{c(\omega)} x$$

with $c(\omega)$ determined analytically from equation (10). This tests two aspects of the solution. First, multiplication by the demodulation factor effectively cancels the x-dependence of equation (6), leaving only the mode shape. Consequently, the solution if done correctly should be the same at all x-steps. Second, if the solution after multiplication by the demodulation factor remains invariant with respect to x, then it