INTRODUCTION TO WAVEFIELD EXTRAPOLATION

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The single square root equation, whose function is to extrapolate wavefields down into the earth, is the principal basis for the migration process of reflection seismology. The purpose of this article is to develop the single square root wave extrapolation equation from a minimal background of mathematical physics.

Plane Waves

Figure 1 is a plot of the equation of a straight line in the x-z plane. The line depicts a seismic wavefront.
Multiplying through the equation by a cosine and rearranging, we have

$$z \cos \theta + x \sin \theta = \text{const} \tag{1}$$

Assuming that the right hand constant increases with time t at speed v, we have a moving line which in three-dimensional Cartesian space is the equation of a moving plane:

$$z \cos \theta + x \sin \theta = vt \tag{2}$$

An important case occurs when $z = 0$, because our geophysical measurements are ordinarily constrained to be on the earth's surface. Equation (2) with $z = 0$ now gives the horizontal speed x/t of the intersection

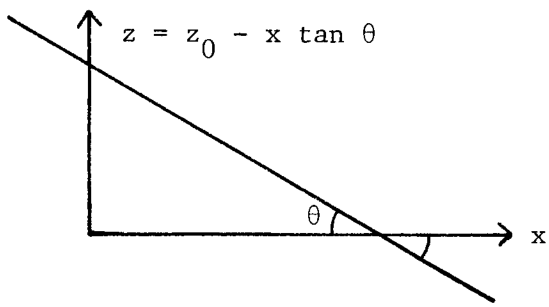

FIG. 1. The straight line wavefront.

between the planewave and the earth's surface, namely

$$\frac{x}{t} = \frac{v}{\sin \theta} \geq v \qquad (3a)$$

This horizontal speed is called the ***apparent velocity*** or the ***horizontal phase velocity*** of the wave. Notice that this speed generally ***exceeds*** the speed v of the wave itself. We occasionally have borehole measurements in which the apparent velocity is the ***vertical*** phase velocity. In this case we substitute $x = 0$ into (2), getting

$$\frac{z}{t} = \frac{v}{\cos \theta} \geq v \qquad (3b)$$

Let $f(t)$ denote some arbitrary waveform function of time. Then $f(t - t_0)$ represents the same waveform shifted by t_0 in time. Now if we let t_0 be the value of t defined by (2), we have an expression for an arbitrary waveform f moving on a plane wave

$$f\left[t - \frac{1}{v}(z \cos \theta + x \sin \theta)\right] = \text{waveform on planewave} \qquad (4)$$

Snell Waves

A major property of the earth's crust is that the velocity of seismic waves increases with depth. The effect of depth-dependent velocity $v(z)$ is to bend rays by Snell's law, so that along any ray $\{\sin[\theta(z)]\}/v(z)$ is a constant. This constant is called Snell's parameter $p = (\sin\theta)/v$. We have defined θ as the angle from a wavefront to the ***horizontal***. Rays are defined as being perpendicular to wavefronts, so in terms of a ray, θ is the angle from the ray to the ***vertical***. The inverse of (3) is

$$\frac{t}{x} = \frac{\sin[\theta(z)]}{v(z)} = p \neq p(z) \tag{5a}$$

$$\frac{t}{z} = \frac{\cos\theta}{v} \tag{5b}$$

Equation (5b) is not valid when v depends on z. The validity of the concept of a *local* plane wave is expressed by the equations

$$\frac{\Delta t}{\Delta x} \approx \frac{dt}{dx} = \frac{\sin\theta}{v} = p \tag{6a}$$

$$\frac{\Delta t}{\Delta z} \approx \frac{dt}{dz} = \frac{\cos\theta}{v} = \frac{[1 - p^2 v(z)^2]^{\frac{1}{2}}}{v(z)} = \left[\frac{1}{v(z)^2} - \left(\frac{dt}{dx}\right)^2\right]^{\frac{1}{2}} \tag{6b}$$

To obtain the right members of (6b) we used (5a), (6a), and the fact that $\cos\theta$ is equal to $(1 - \sin^2\theta)^{\frac{1}{2}}$. Equations (6a) and (6b) depict the local situation; the global situation is given by

$$t(x,z) = px + \int_0^z \frac{\left[1 - p^2 v(z)^2\right]^{\frac{1}{2}}}{v(z)} dz \tag{7}$$

as may be verified by the substitution of (7) into (6). Equation (7) tells you at what time t a wavefront will pass you if you are located at position (x,z). We can use this as a reference time t_0 on an arbitrary waveform f.

$$f(t - t_0) = f\left[t - px - \int_0^z \frac{\left[1 + p^2 v(z)^2\right]^{\frac{1}{2}}}{v(z)} dz\right] \qquad (8)$$

$$= \text{arbitrary signal } f \text{ following ray path } t_0(x,z)$$

An expression like (8) will be called a ***Snell wave***.

Shifting Equations

An important task is to predict the wavefield inside the earth given the waveform at the surface. For a downgoing plane wave this can be done by the time-shifting partial differential equation

$$\frac{\partial P}{\partial z} = -\frac{1}{v}\frac{\partial P}{\partial t} \qquad (9)$$

as may be readily verified by substituting the trial solutions

$$P = f\left(t - \frac{z}{v}\right) \quad \text{for constant } v \qquad (10a)$$

or

$$P = f\left(t - \int_0^z \frac{dz}{v(z)}\right) \quad \text{for } v(z) \qquad (10b)$$

Heeding some important restrictions, this also works for non-vertically incident waves with the partial differential equation

$$\frac{\partial P}{\partial z} = -\frac{dt}{dz}\frac{\partial P}{\partial t} \qquad (11)$$

which has the solution

$$P = f(t - px - \int_0^z \frac{dt}{dz} dz) \qquad (12)$$

In interpreting (11) and (12) we recall that dz/dt is the apparent velocity in a borehole. The partial derivative of wavefield P with respect to depth z is taken at constant x, i.e. the wave is extrapolated down the borehole. The idea that downward extrapolation can be achieved by merely time-shifting clearly holds only for situations in which a single Snell wave is present, that is, the *same* arbitrary time function must be seen at all locations. Substitution from (6) also enables us to rewrite (11) in the form

$$\frac{\partial P}{\partial z} = -\left[\frac{1}{v(z)^2} - \left(\frac{dt}{dx}\right)^2\right]^{\frac{1}{2}} \frac{\partial P}{\partial t} \tag{13}$$

Since $dt/dx = p$ can be measured along the surface of the earth, it seems that equation (13), along with an assumed velocity $v(z)$ and some observed data $P(x,t)$, would enable us to determine $\partial P/\partial z$, which is the necessary first step of downward continuation. But we must not forget that we are dealing by assumption with a single Snell wave and not a superposition of several Snell waves. Superposition of different waveforms on different Snell paths will cause different time functions to be seen at different places. Then a mere time-shift will not achieve downward continuation. Luckily, a complicated wavefield that is variable from place to place may be decomposed, by mathematical techniques not yet discussed, into many Snell waves, each of which can be downward extrapolated with the differential equation (13) or its solution (12). The most well-known decomposition technique is Fourier analysis.

Multidimensional Complex Exponential

A sinusoidal disturbance in space and time is conveniently described by the complex exponential function

$$f = e^{-i\omega t + ik_x x + ik_z z} \tag{14}$$

This function plays an important role in Fourier transformation, where the angular frequency ω is called the Fourier dual to time t and vice versa. Likewise, the space coordinates (x,y,z) have Fourier duals

(k_x, k_y, k_z). In equation (14) we have already set k_y equal to zero. If we specialize the arbitrary waveform of equation (8) to a sinusoid, it will begin to resemble equation (14):

$$f = e^{-i\omega\left(t - px - \int_0^z \frac{\left[1-p^2 v(z)^2\right]^{\frac{1}{2}}}{v(z)} dz\right)} \tag{15}$$

At the earth's surface $z = 0$ we can reconcile the moving wavefront (15) with the Fourier function (14) by the identification of

$$\frac{k_x}{\omega} = p \tag{16a}$$

which, incorporating (5a), is

$$\sin\theta = \frac{vk_x}{\omega} \tag{16b}$$

Replacing the time derivative $\partial/\partial t$ by $-i\omega$ and utilizing the substitutions (6) and (16), the extrapolation equation (11) or (13) can be put in the following various forms:

$$\frac{dP}{dz} = \frac{i\omega}{v(z)} \left[1 - p^2 v(z)^2\right]^{\frac{1}{2}} P \tag{17a}$$

$$\frac{dP}{dz} = \frac{i\omega}{v(z)} \left[1 - \left(\frac{v(z)k_x}{\omega}\right)^2\right]^{\frac{1}{2}} P \tag{17b}$$

$$\frac{dP}{dz} = \frac{i\omega}{v} \cos(\theta)\ P \tag{17c}$$

Equation (17c) is the most intuitive form because it says that upon downward extrapolation at a fixed x the phase changes at a rate of the frequency ω divided by vertical phase velocity $v/\cos\theta$. Equation (17b) is most important in practice: An observed surface waveform is a function of t and x. Two-dimensional Fourier transformation converts it to a function of ω and k_x. Then equation (17b) forms the basis of various algorithms to downward extrapolate each point in (ω, k_x)-space.

An analytical solution may be found as follows: Evaluate (15) at $z = z_1$; evaluate it again at $z = z_2$; divide and rearrange it, obtaining

$$f(z_2) = f(z_1)\, e^{i\omega \int_{z_1}^{z_2} \frac{\left[1 - p^2 v(z)^2\right]^{\frac{1}{2}}}{v(z)} dz} \tag{18a}$$

$$f(z_2) = f(z_1)\, e^{i\omega \int_{z_1}^{z_2} \frac{1}{v(z)} \left[1 - \left(\frac{v(z)k_x}{\omega}\right)^2\right]^{\frac{1}{2}} dz} \tag{18b}$$

$$f(z_2) = f(z_1)\, e^{i\omega \int_{z_1}^{z_2} \frac{\cos[\theta(z)]}{v(z)} dz} \tag{18c}$$

Again (18c) is the most intuitive, while (18b) is most commonly used. Equation (18a) forms the basis of various migration techniques known as ***dip domain***, ***slant stack***, ***wave stack***, ***Snell stack***, or (in the Russian literature) ***Controlled Directional Recording*** (CDR) methods.

The Double Square Root (DSR) Equation

The DSR equation forms the basis for imaging and velocity analysis in reflection seismology. In fact, if reflection seismology were to be summarized in a single equation, the DSR equation would be a good choice. It can best be understood after studying a variety of techniques to implement the single square root equation [equation (13)], but it is always nice to get a bird's eye view of a territory before a detailed one, even if the first, big picture is not fully understood.

The basic seismic shooting and recording geometry is shown in figure 2.

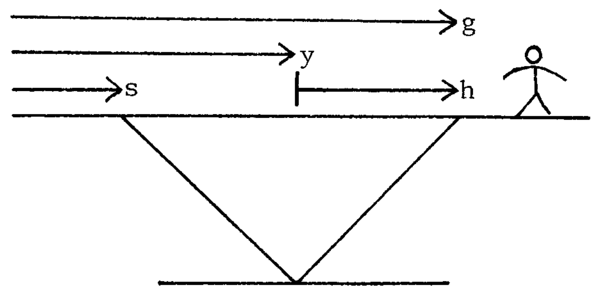

FIG. 2. Conventional reflection seismology geometry.

Data is taken along the x-axis where shot points are located at all possible positions $x = s$; likewise, recording geophones are at all $x = g$. The midpoint $y = (g+s)/2$ is halfway between the geophone and the shot and we use $h = (g-s)/2$ to denote ***half*** the offset between the shot and geophone. The transformation between (g,s) recording parameters and (y,h) interpretation parameters is

$$y = \frac{g+s}{2} \tag{19a}$$

$$h = \frac{g-s}{2} \tag{19b}$$

Traveltime t may be parameterized in (g,s)-space or (y,h)-space. Differential relations for converting are given by the chain rule for derivatives, namely

$$\frac{dt}{dg} = \frac{dt}{dy}\frac{dy}{dg} + \frac{dt}{dh}\frac{dh}{dg} = \frac{1}{2}\left(\frac{dt}{dy} + \frac{dt}{dh}\right) \tag{20a}$$

$$\frac{dt}{ds} = \frac{dt}{dy}\frac{dy}{ds} + \frac{dt}{dh}\frac{dh}{ds} = \frac{1}{2}\left(\frac{dt}{dy} - \frac{dt}{dh}\right) \tag{20b}$$

Let the geophones descend a distance dz_g into the earth. How will their traveltime change? Equation (6b) gives

$$\frac{dt}{dz_g} = -\left[\frac{1}{v^2} - \left(\frac{dt}{dg}\right)^2\right]^{\frac{1}{2}} \tag{21a}$$

The minus sign has been introduced because equation (6b) described a downgoing wave, but we now wish to see an upcoming wave. Suppose the shots had been let off at depth dz_s instead of at $z = 0$. By similar reasoning equation (6b) gives

$$\frac{dt}{dz_s} = -\left[\frac{1}{v^2} - \left(\frac{dt}{ds}\right)^2\right]^{\frac{1}{2}} \tag{21b}$$

We also need a minus sign here because the traveltime in the experiment must decrease as the shots are pushed downward.

Now suppose we simultaneously downward project both the shots and geophones by an identical amount $dz = dz_g = dz_s$. The traveltime change is the sum of (21a) and (21b), namely

$$dt = \frac{dt}{dz_g}\, dz_g + \frac{dt}{dz_s}\, dz_s = \left(\frac{dt}{dz_g} + \frac{dt}{dz_s}\right) dz \tag{22a}$$

or

$$\frac{dt}{dz} = -\left\{\left[\frac{1}{v^2} - \left(\frac{dt}{dg}\right)^2\right]^{\frac{1}{2}} + \left[\frac{1}{v^2} - \left(\frac{dt}{ds}\right)\right]^{\frac{1}{2}}\right\} \tag{22b}$$

This expression for dt/dz may be substituted into the time shifting partial differential equation which operates on the wavefield $P(x,z,t)$, namely equation (11):

$$\frac{\partial P}{\partial z} = -\frac{dt}{dz}\frac{\partial P}{\partial z} \tag{11}$$

$$\frac{\partial P}{\partial z} = -\left\{\left[\frac{1}{v^2} - \left(\frac{dt}{dg}\right)^2\right]^{\frac{1}{2}} + \left[\frac{1}{v^2} - \left(\frac{dt}{ds}\right)^2\right]^{\frac{1}{2}}\right\} \frac{\partial P}{\partial t} \qquad (23)$$

Finally we can express everything in midpoint-offset coordinates by substituting from (20):

$$\frac{\partial P}{\partial z} = -\left\{\left[\frac{1}{v^2} - \frac{1}{4}\left(\frac{dt}{dy} + \frac{dt}{dh}\right)^2\right]^{\frac{1}{2}} + \left[\frac{1}{v^2} - \frac{1}{4}\left(\frac{dt}{dy} - \frac{dt}{dh}\right)^2\right]^{\frac{1}{2}}\right\} \frac{\partial P}{\partial t} \qquad (24)$$

Equation (24) is the heralded DSR equation. We will interpret it in two simplified cases. First take the case of a flat, layered earth. The traveltime is then independent of the midpoint, so we may set dt/dy equal to 0, obtaining

$$\frac{\partial P}{\partial z} = -2\left[\frac{1}{v^2} - \frac{1}{4}\left(\frac{dt}{dh}\right)^2\right]^{\frac{1}{2}} \frac{\partial P}{\partial t} \qquad (25)$$

We see that except for some scaling factors of 2, we are back to the single square root equation (6b).

What do single square root equations do? Basically, they describe waves. Throwing a pebble into a pond, we see an expanding spherical wave. In (x,z,t)-space, it is a cone with circles in (x,z) and hyperbolae in (x,t). Ordinary wave equations take an (x,z)-space picture and let it evolve in time, showing, for example, an expanding spherical wave. Single square root equations take an (x,t)-space picture and let it evolve in depth, showing, for example, expanding and contracting hyperboloids.

A second interpretation of equation (24), the DSR equation, is for zero-offset sections. A common midpoint gather is a symmetrical function of offset so that at zero-offset we may take dt/dh to be zero. Then (24) also becomes a single square root equation.

$$\frac{\partial P}{\partial z} \;=\; -2\left[\frac{1}{v^2} - \frac{1}{4}\left(\frac{dt}{dy}\right)^2\right]^{\frac{1}{2}} \frac{\partial P}{dt} \tag{26}$$

Eventually we will see that properly formulated, equation (26) can convert hyperbolae to smaller hyperbolae and finally focus them to a point. Such a process is called the migration of a seismic section.

Having seen the big picture with the double square root equation and having shown that important applications lead to the single square root equation, we will now proceed to a variety of techniques for implementing single square root equations. These techniques are the tools of reflection seismic data analysis.