ROOTS OF SEISMIC Z-TRANSFORMS

Bert Jacobs

Abstract

The z-transform of a seismic trace can be factored, each factor yielding a single root. These roots can, in theory, be used in the deconvolution of seismic data.

Introduction

Predictive deconvolution constructs a minimum phase inverse from the power spectrum of a seismic trace. This may not be a good way in which to construct an estimate of the inverse wavelet. One reason for this is that the physical waveform may not have a minimum phase inverse. Another is that predictive deconvolution is applicable only when dealing with datasets which are wide-sense stationary.

Various nonlinear, iterative schemes have been proposed as a way of overcoming the limitations of predictive deconvolution. In particular, the wavelet estimate is not constrained to be minimum delay. The problem with these methods is that they are unstable when applied to some datasets. They also tend to pay too much attention to big spikes in the input time series.

If the locations of the roots of a seismic traces could be found accurately and quickly then an inverse wavelet could be constructed from the roots shared by all the traces. Alternatively, and less ambitiously, if one had a minimum phase inverse from a predictive deconvolution program and could factor it, then it would be possible to vary the phase spectrum while keeping the power spectrum constant by appropriately adjusting root positions.

Traces, Wavelets, and Their Roots

If noise is not too serious a problem, then the Z-transform convolution thereom guarantees that the roots corresponding to the shot waveform will appear in the factorization of a seismic reflection trace.

Seismic traces are time series. A typical series can be represented by the sequence of real numbers:

$$y_0 , y_1 , y_2 , y_2 , \cdots , y_{Ny}$$

This sequence has a Z-transform

$$\sum_{k=0}^{Ny} y_k z^{-k}$$

We choose to factor the roots of the product of this transform with z^{Ny}

$$Y(z) = \sum_{k=0}^{Ny} y_k z^{Ny-k}$$

Each trace y_i of a common-shot gather can be considered to be the convolution of a waveform b with a reflectivity series x_i where i is the trace index. When we take Z-transforms, these convolutions change to products. This yields N equations - one for each seismic trace.

$$Y_i(z) = X_i(z)\ B(z)$$

$$= (z-x_{i1})\ (z-x_{i2})\ldots(z-x_{iNx})\ (z-b_i)\ldots(z-b_{Nb})$$

where the x_{ij} are the Nx roots of the ith reflectivity series and the b_j are the Nb roots of the shot waveform. Both sets of roots come in complex conjugated pairs since the trace, reflectivity series, and shot waveform are all real time series. When written in this way it is easy to see why the waveform factors can be found from the set of trace factorizations. If the roots from all N traces are listed then the wavelet can be identified with those roots which reoccur.

This simple model is complicated by the presence of additive noise. The effect of noise on roots is considered next.

The Condition and Distribution of Seismic Trace Roots

Noise enters this scheme in two forms. The first is the familiar additive noise. The other source of noise is the truncation at the far end of the trace.

The effect of noise is not as serious as one might think. This is because the zeros of a random polynomial are well conditioned. To help substantiate this claim it is useful to consider three thereoms from the literature on polynomial factorization.

> Thereom 1: If we consider a zero x of a polynomial f(z) and perturb this polynomial by the addition of a polynomial $\epsilon \cdot g(z)$, then x will be perturbed by a number of size $(\epsilon\{[m!g(x)]/[f^m(x)]\})^{1/m}$ where $f^m(x)$ is the mth derivative of f(z) evaluated at z = x and m is the multiplicity of the zero at x.
>
> Thereom 2: Consider a polynomial $f(z) = a_0 + \ldots + a_n z^n$ Let Nab denote the number of roots within the angular sector between $\theta = a$ and $\theta = b$, $0 \le a < b \le 2\pi$. Then

$$\left| n \frac{b-a}{2\pi} - Nab \right| < 16 \sqrt{n \log \frac{|a_0| + \ldots + |a_n|}{|a_0 a_n|^{\frac{1}{2}}}}$$

Thereom 3: If $g(z) = 1+b_1z+\ldots+b_nz^n+\ldots$ has the unit circle as a circle of convergence, then every point of this circle is a cluster point of zeros of the partial sums $s(z) = 1+b_1z+\ldots+b_nz^n$. For an arbitrary ϵ, we have an infinite sequence of indices n_ν

$$a_{n_\nu} > (1 - \epsilon^2)^{n_\nu}$$

$$n_1 < n_2 < \cdots$$

We also have $|a_n| < (1 + \epsilon^2)^n$ for sufficiently large n. The number of roots of $s_{n_\nu}(z)$ outside the annulus $1-\epsilon < |z| < 1+\epsilon$ is less than $7\,\epsilon n_\nu$.

The first thereom is due to Wilkinson. It says that isolated roots are well conditioned. Multiple roots or near-multiple roots can be expected to be ill-conditioned.

The second thereom is the work of Erdos and Turan. It says that the roots of polynomials are more or less evenly distributed around the origin. This means that a random polynomial with a suitably large first and last coefficient will not have many multiple roots. We will soon consider a real example which suggests that most random polynomials have this even distribution of roots.

The third thereom was proved by R. Jentzsch and G. Szego. It says that the roots of a random polynomial tend to be found within an annular region about the unit circle. This will be true when our seismic traces are properly gained. This fact is important because the zeros which lie far away from the unit circle are the ones most affected by aliasing. Among the roots of unit multiplicity, these are also the roots which one expects to be most affected by the presence of noise.

From these three thereoms one can draw the conclusion that random polynomials will have roots which are, in general, insensitive to noise, evenly distributed angularly, and close to the unit circle.

Examples of Factorizations

With this theoretical background, a plot of the roots of a seismic trace will make more sense. In the following few diagrams, the root locations in the complex Z-plane will be represented by crosses. The Fourier transform is the value of the Z-transform on the unit circle, with the frequency axis laid out angularly along that circle. DC appears at (1,0), 1/2 Nyquist at (0,1), and Nyquist at (-1,0) in the complex Z-plane.

Figure 1 shows a trace of 256 time points with a 4 mil sampling rate and its power spectrum. The polynomial formed from the first 105 coefficients was factored, and the corresponding 104 roots plotted. The root-finding program now in use cannot handle polynomials of much higher degree without a marked decrease in accuracy.

Note that the first coefficient of the time series is small and that this did not ruin the even distribution of zeros about the origin that was expected. It is also worth noticing that the root distribution is especially even in the left half-plane of the complex Z-plane. This corresponds to frequencies between half-Nyquist and the Nyquist frequency which have almost been filtered out. Presumably the regularity of the root distribution in this region reflects the fact that the frequencies involved are completely made up of uncorrelated noise.

The full set of 256 time points was used to construct a minimum phase wavelet. The construction was done in the frequency domain so the result will probably not be identical to the wavelet that a predictive deconvolution algorithm would have constructed with the same data. The results are in figure 2. Only the first 105 time points are plotted - the remaining samples were almost all zero.

The root pattern corresponding to the polynomial formed out of the first 105 time points from the wavelet has some interesting features. The first is that all of its roots lie within the unit circle - a

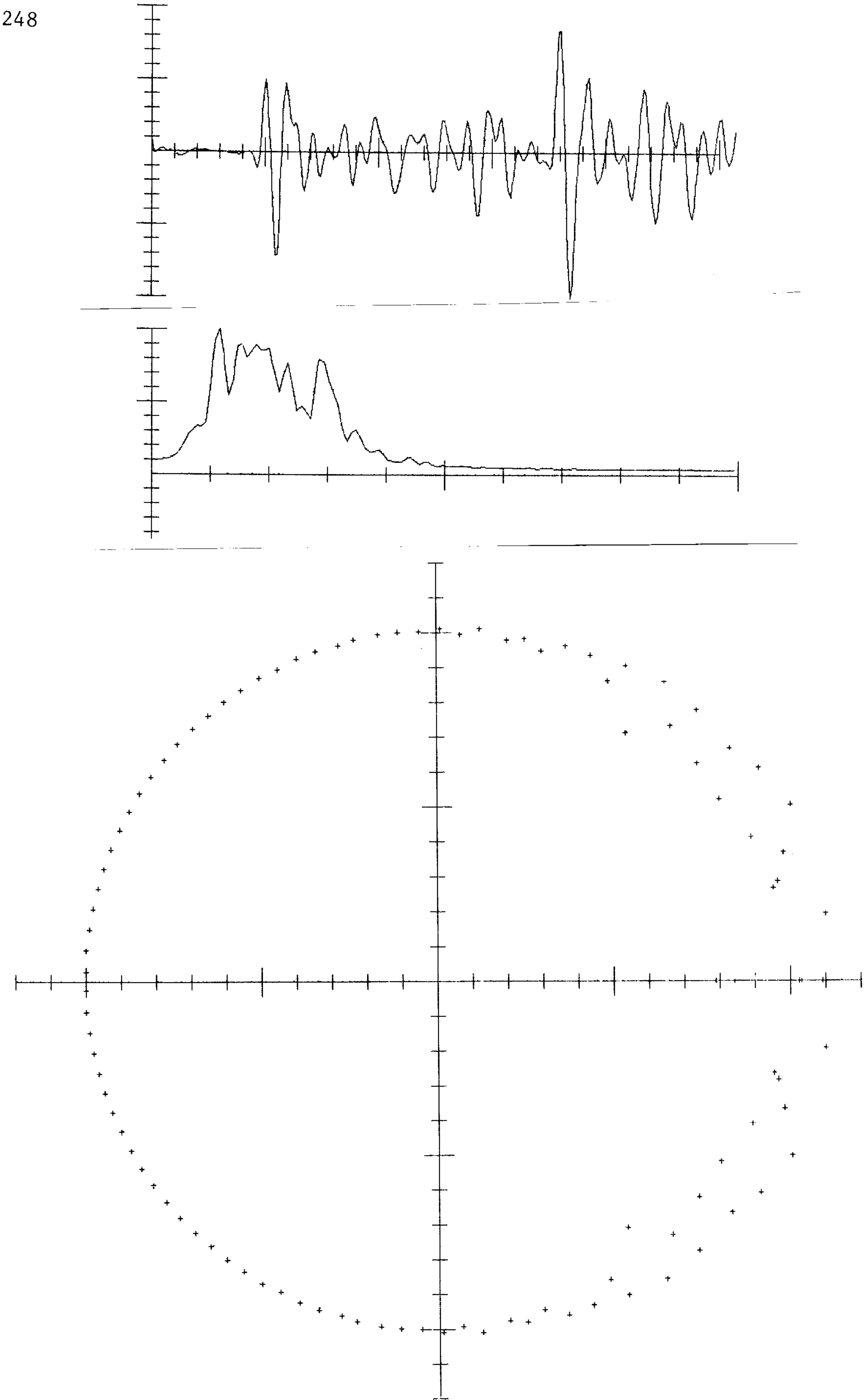
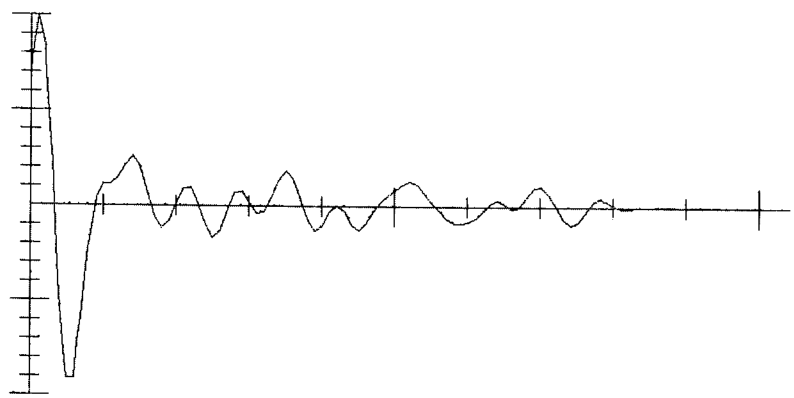


FIG. 1. Top: Seismic trace of 256 samples. Middle: Power spectrum. Bottom: Root distribution for the first 106 coefficients from the trace.



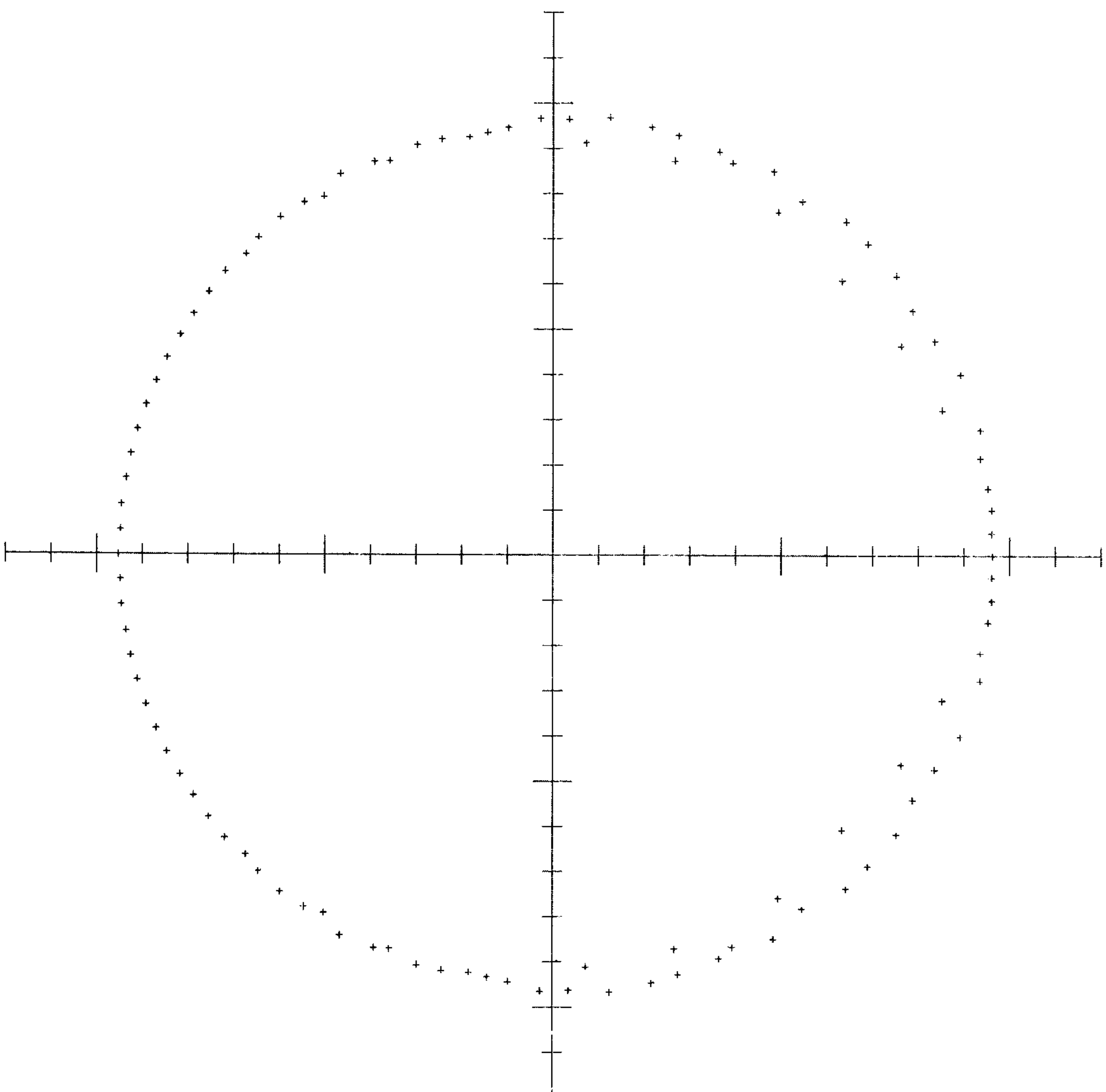


FIG. 2. Minimum phase wavelet and its root distribution. The roots are all inside the unit circle because the wavelet is minimum delay. The roots come in complex conjugate pairs because the wavelet is real.

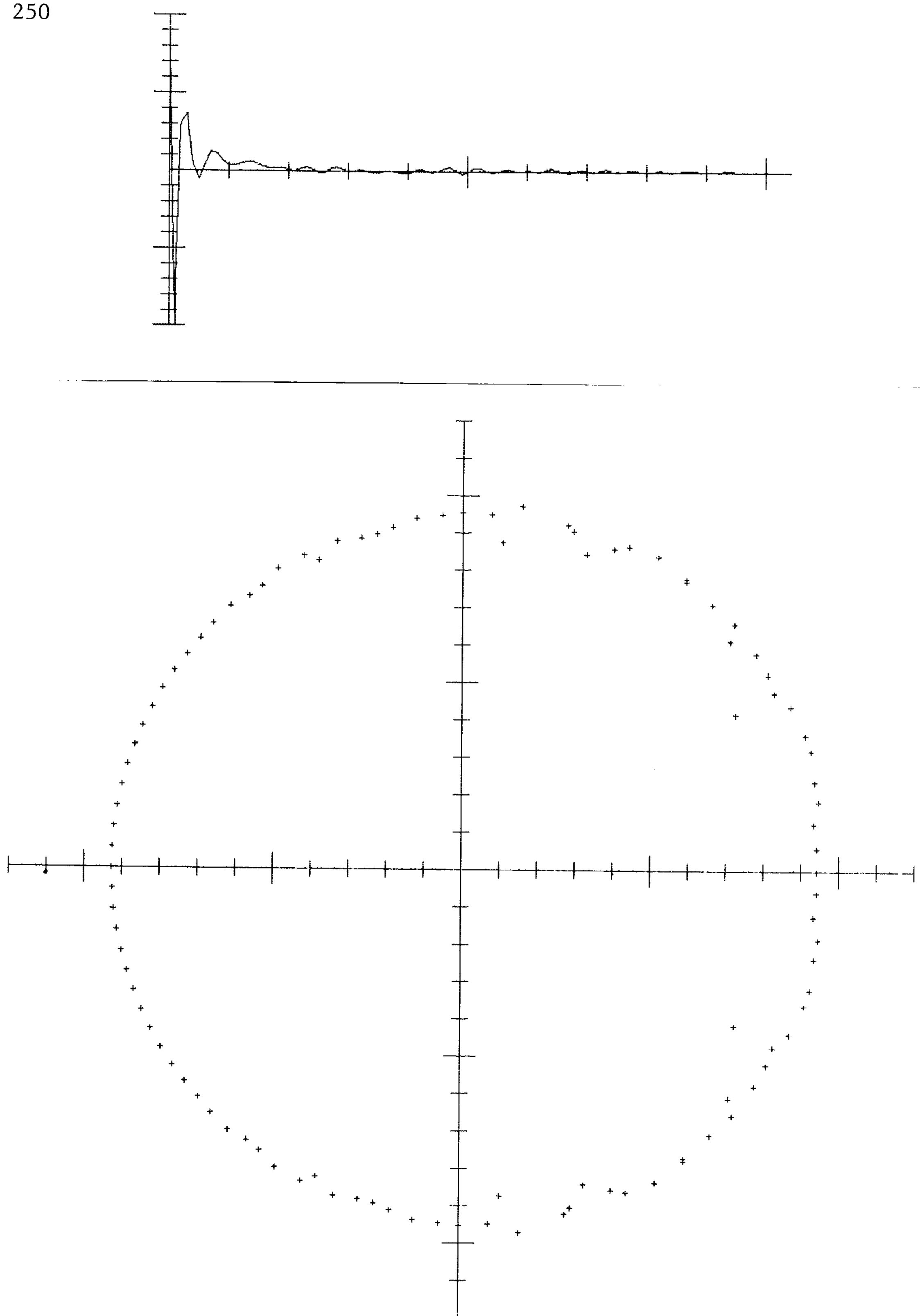


FIG. 3. Minimum delay inverse filter and root distribution.

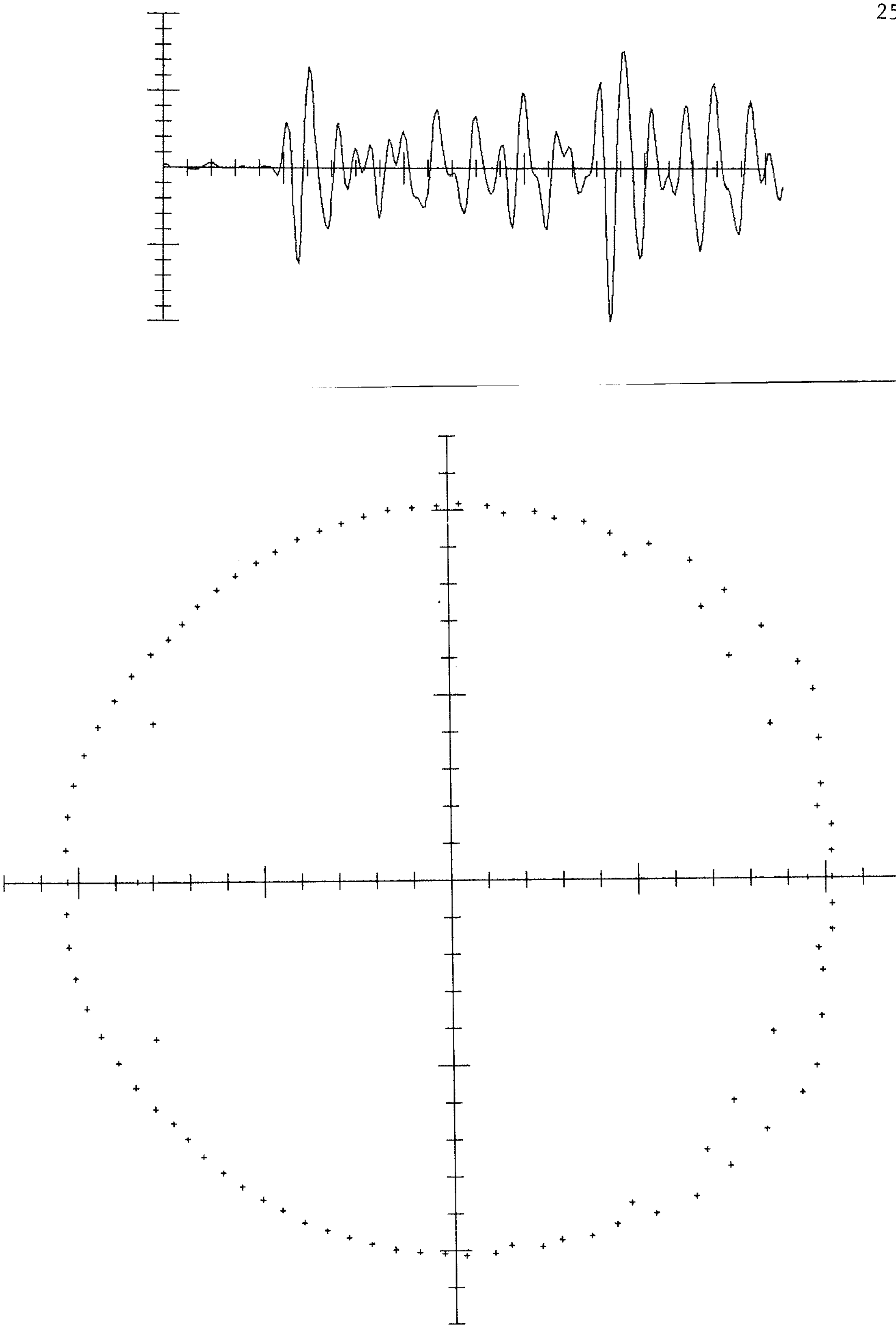


FIG. 4. Deconvolved seismic trace with 256 samples plotted. The root distribution is that of the polynomial formed from the first 96 coefficients of this time series.

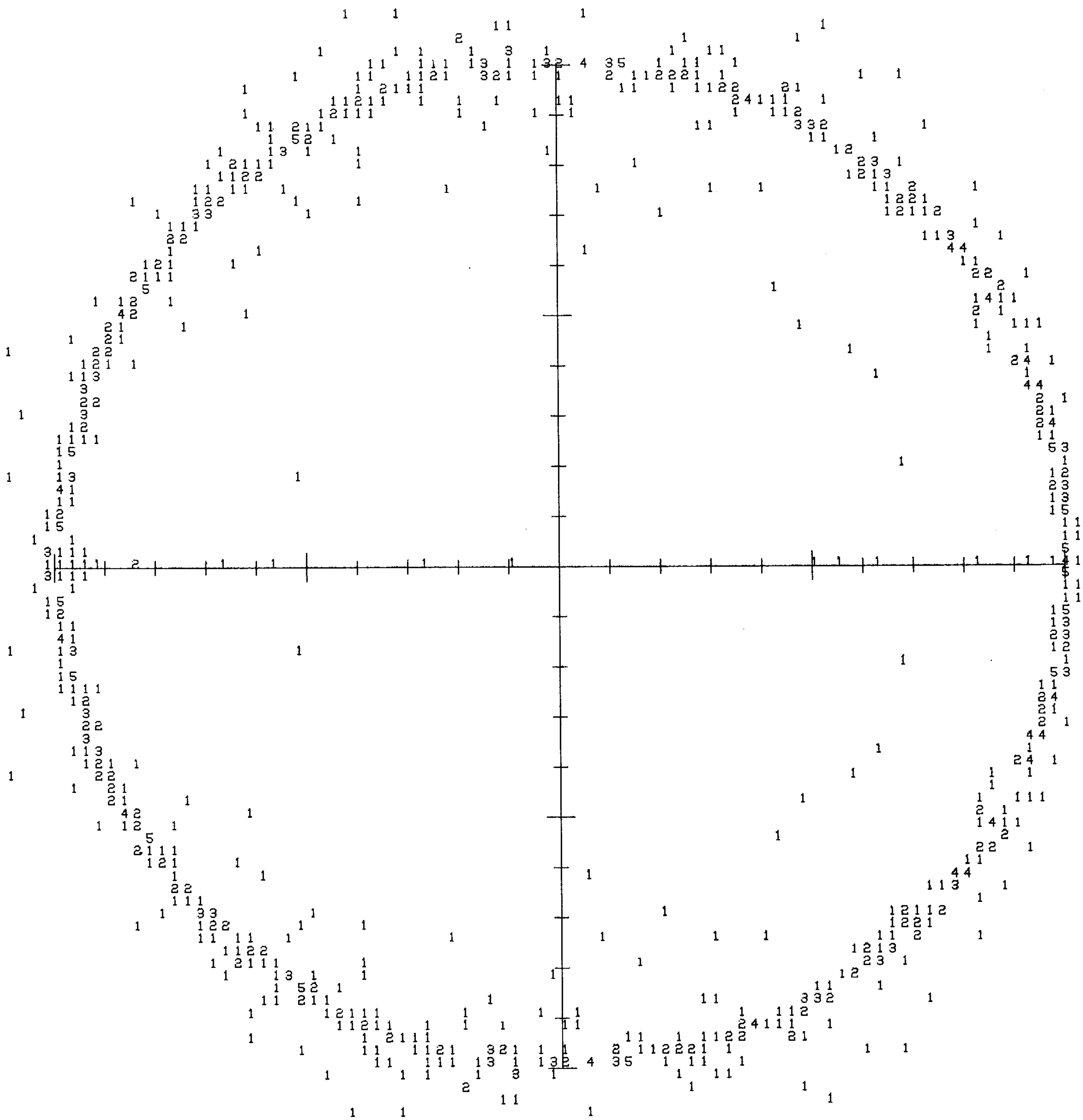


FIG. 5. An attempted root stack of 10 data segments from a common shot gather. Each number appears at the location of a small square in the complex plane and represents the number of roots which stack into that square bin.

corollary to the minimum delay property. The second is that the root patterns in figures 1 and 2 have points in common in the frequency range between 1/8 and 1/2 Nyquist. Finally, many of the waveform roots are not found in figure 1. This is because their real waveform is probably not minimum phase and the seismogram is too short to include all the wavelet roots. Roots at high frequencies and very low frequencies are the most likely to be missed.

The inverse wavelet corresponding to the time series of figure 2 was constructed in the frequency domain in such a way as to guarantee that it too will be minimum delay. This filter and its root pattern are in figure 3.

The convolution of this inverse wavelet with the input time series is in figure 4. It is hard to say whether there has been any improvement though some marked changes were made. The root pattern for the polynomial formed from the first 96 coefficients is also displayed. Many of the roots in the original seismic trace are still there - an encouraging sign. Only 96 coefficients were used because the program failed to accurately determine the roots near Nyquist when fed a 105-sample deconvolved time series.

A Root Stack Attempt

Since the wavelet is invariant from trace to trace within a common shot gather, its roots will occur in the factorizations of each of the traces of such a gather. On the other hand, the reflectivity series of one trace may bear little resemblance to that of a neighboring trace, so the roots of adjacent reflectivity series polynomials will probably not be the same. It follows from the convolution theorem that if one is given the roots from all of the traces of a common-shot gather, then the wavelet roots can be identified by looking for redundancies from trace to trace.

To identify such recurring roots it is convenient to sort the roots of the traces of a seismic gather into bins in the complex plane. This is because the redundancies will only be approximate since noise, algorithmic and additive, will perturb the root locations. The result of this sorting is called a root stack. Wavelet roots are associated with the bins which are filled to overflowing with roots.

The next figure is a stab at making a root stack. A long seismic trace was chopped into ten segments, each 101 samples long. Each segment was factored and the roots obtained sorted into bins in the complex plane. Each bin was a square with edges parallel to the axes and of width 0.025. The non-zero bin counts are plotted in figure 5, each count appearing at its bin location in the complex plane.

Though there are a number of peaks in the bin counts there are at least two reasons for suspecting them. First, the effect of all the truncations may be drastic - there is not much reason to suspect that much of the phase spectrum remains intact. Second, bin sorting is not smooth - so peaks between bins would tend to be divided into two pieces and disappear. An improved sorting scheme is possible by using a pyramidal weighting scheme to assign root counts to grid points. The disadvantage with this scheme is that the grid points will then have nonintegral numbers of roots associated with them.

Putting the Phase Back in the Minimum Phase Wavelet

Root stacking may never work, but root finding algorithms may still prove useful in deconvolution. In particular, it may be possible to take a minimum phase wavelet from a predictive deconvolution program, identify its roots, and shift root locations until an optimum phase spectrum is achieved.

Suppose we were given an autocorrelation $B(1/Z)B(Z)$ and we proceeded to examine its roots. The root pattern would be found to possess a fourfold symmetry. If Z_0 is a root of $B(1/Z)B(Z)$ then so is Z_0^* $.1/Z_0$, and $1/Z_0^*$. The minimum phase wavelet with this autocorrelation can be formed by gathering together all of the roots within the unit circle and doing the requisite polynomial multiplications.

If the physical waveform is not minimum phase, it simply means that we should substitute a root from outside the unit circle for one inside. Given a set of minimum delay wavelet roots there are only a finite number of possible root locations outside the unit circle to choose from - those obtained by taking the reciprocal of the minimum delay roots.

Thus, to adjust the phase spectrum of a minimum delay wavelet, one can simply proceed to exchange complex conjugate root pairs from within the unit circle for reciprocal pairs outside. After exchanging roots and recalculating a wavelet estimate, the new wavelet can be applied to a segment of data. Using some spikiness statistic, an optimum root pattern can be picked from among the possible candidates.

A Short Note on Algorithms

The root finding algorithm used in getting the results for this article is a modification of the three-stage, variable-shift Jenkins-Traub algorithm published in ACM/TOMS. The modifications involve a rescaling of the polynomial before each deflation and a few lines which identify rare divergent iterations. It takes the modified algorithm about one minute to find 100 roots. Execution time goes up roughly as the square of the number of roots found.

Steiglitz and Dickinson at Princeton report that they have an algorithm which factors 255 degree polynomials in 4.3 seconds on an IBM 360/91 machine. The speed, accuracy, and stability of this variant of Newton-Raphson is probably obtained by not deflating the polynomial whenever a zero is found (the author knows of another group that used Newton-Raphson with deflation, achieving results inferior to that obtained with the modified Jenkins-Traub algorithm). The preprint in which this algorithm is mentioned is probably for IEEE Trans. on Acoust. Speech and Signal Processing.

ACKNOWLEDGMENTS

The idea for root stacking belongs to Fabio Rocca who suggested it while he was working at Stanford. Swavek Deregowski was of great assistance with some of the numerical aspects of accurate root finding.

REFERENCES

Erdos, P. and Turan, P., 1950, On the distribution of roots of polynomials: Annals of Mathematics, v. 51, no. 1, January.

Henrici, P., Applied and computational complex analysis: New York, John Wiley & Sons.

Jenkins, M.A., 1969, Three-stage variable-shift iterations for the solution of polynomial equations with a posteriori error bounds for the zeros: Technical Report No. CS 138, August, Computer Science Dept., Stanford University.

Jenkins, M.A., 1975, Algorithm 493: zeroes of a real polynomial: ACM Transactions on Mathematical Software: v. 1, no. 2, June.

Steiglitz, K. and Dickinson, B., Computation of the complex cepstrum by factorization of the Z-transform, preprint.

Wilkinson, J.H., 1963, Rounding errors in algebraic processes: London, Her Majesty's Stationery Office.