Geophysical Prospecting 34, 735–742, 1986.

ON THE UPWARD CONTINUATION OF POTENTIAL FIELD DATA BETWEEN IRREGULAR SURFACES*

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ABSTRACT

IVAN, M. 1986, On the Upward Continuation of Potential Field Data Between Irregular Surfaces, Geophysical Prospecting 34, 735–742.

The potential field and its derivatives at points above an irregular surface can be approximately obtained from the sampled potential field data acquired on that surface.

A method of minimizing the truncation effect, which appears when gravity and magnetic maps are processed with the aid of surface integrals, is derived.

The results are compared with those of the most relevant similar methods by using a theoretical, but realistic, model.

INTRODUCTION

The upward continuation of the gravity field acquired on an arbitrary surface has been extensively studied. Various methods have been proposed to obtain the field values at points of the upper half-space as a function of its values on the topographic relief (Dirichlet problem).

The practical importance of this problem is related to the disturbing effects which are due to the fact that the Bouguer anomaly values are located at varying elevations of their respective points of measurement (Naudy and Neumann 1965, Ervin 1977). The necessity of minimizing the terrain effect exists also for magnetic field anomalies, where even an expression analogous to the Bouguer anomaly is usually difficult to compute. The preliminary processing of the gravity and magnetic data by an upward continuation is then an essential requirement, particularly in a region of very irregular topographic relief.

A method based on the potential field theory is presented. The results are compared with those obtained by other similar techniques (Syberg 1972, Bhattacharyya and Chan 1977).

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* Received July 1984, revision accepted July 1985.

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The minimization of the truncation effect due to the finite extent of real gravity and magnetic maps is also discussed.

APPROXIMATE GENERALIZATION OF THE DIRICHLET INTEGRAL

With respect to an arbitrary reference plane, the topographic surface S and a horizontal plane P above S have the equations:

$$z = h(x, y) \tag{1}$$

and

$$z = h_0, \tag{2}$$

respectively.

The exact solution of the upward continuation problem from the plane P at a point $M(\bar{x}, \bar{y}, H)$ above that plane is represented by the Dirichlet integral (Grant and West 1965),

$$g(\bar{x}, \bar{y}, H) = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} \frac{g(x, y, h_0)(H - h_0)\,\mathrm{d}x\,\mathrm{d}y}{[(\bar{x} - x)^2 + (\bar{y} - y)^2 + (H - h_0)^2]^{3/2}}. \tag{3}$$

An arbitrary point C is selected between the surface S and P, then a sphere of center C and very large radius is built (see fig. 1). The surface of the spherical zone between S and P is denoted by L. The closed volume bounded by S, L and P is denoted by V. The distance from M to an arbitrary point $N(x, y, z)$ of V is

$$R(x, y, z) = \sqrt{(x - \bar{x})^2 + (y - \bar{y})^2 + (z - H)^2}. \tag{4}$$

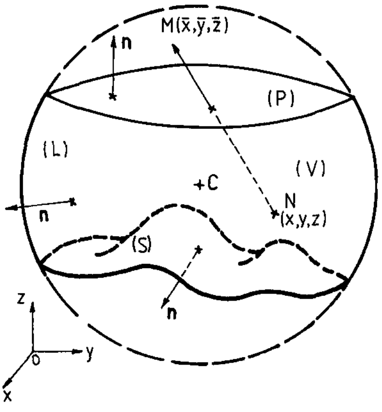

Fig. 1. Illustration of the Dirichlet problem.

The function $1/R$ is harmonic in V. The gravity field g at the point N and the ratio $1/R$ satisfy the conditions of the first Green identity (Kellog 1953),

$$\iint_L gn \cdot \nabla(1/R)\,\mathrm{d}s + \iint_P gn \cdot \nabla(1/R)\,\mathrm{d}s + \iint_S gn \cdot \nabla(1/R)\,\mathrm{d}s = \iiint_V \nabla g \cdot \nabla(1/R)\,\mathrm{d}v, \tag{5}$$

where n is the outer normal to V.

When the radius of the above sphere approaches infinity the first integral of (5) vanishes and the second one is just the Dirichlet integral (3). It follows that

$$g(\bar{x}, \bar{y}, H) = -\frac{1}{2\pi}\iint_S gn \cdot \nabla(1/R)\,\mathrm{d}s + \frac{1}{2\pi}\iiint_V \nabla g \cdot \nabla(1/R)\,\mathrm{d}v. \tag{6}$$

Neglecting the second integral of (6) gives

$$g(\bar{x}, \bar{y}, H) \approx \frac{1}{2\pi}\iint_S g(x, y, h(x, y)) \times \frac{(x-\bar{x})\dfrac{\partial h}{\partial x} + (y-\bar{y})\dfrac{\partial h}{\partial y} + H - h(x, y)}{[(x-\bar{x})^2 + (y-\bar{y})^2 + (H - h(x, y))^2]^{3/2}}\,\mathrm{d}x\,dy. \tag{7}$$

This formula will give good results, at least when

1. the topographical surface is not very irregular (i.e., the value of the volume V in (6) is small);
2. the upward continuation point M has a great elevation above the topographic surface or the source body is deep enough (i.e., when the inner product $\nabla g \cdot \nabla(1/R)$ in (6) is small).

Formula (7) can also be used for the upward continuation of magnetic anomalies. Because the magnetic gradients are smaller than those of the gravity field, the results of the upward continuation in this case are expected to be more accurate.

The differentiation of (7) with respect to H leads to the vertical gradient of the field at the point M

$$-\left.\frac{\partial g}{\partial H}\right|_M \approx \frac{1}{2\pi}\iint_S g(x, y, h(x, y)) \times \frac{3(H - h(x, y))\left[(x-\bar{x})\dfrac{\partial h}{\partial x} + (y-\bar{y})\dfrac{\partial h}{\partial y} + H - h(x, y)\right]}{[(x-\bar{x})^2 + (y-\bar{y})^2 + (H - h(x, y))^2]^3}\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2\pi}\iint_S \frac{g(x, y, h(x, y))\,\mathrm{d}x\,\mathrm{d}y}{[(x-\bar{x})^2 + (y-\bar{y})^2 + (H - h(x, y))^2]^{3/2}}. \tag{8}$$

The horizontal derivatives of the field can be obtained in the same way.

THE TRUNCATION EFFECT

Formula (7) requires the evaluation of an infinite double integral. Actually, the available data domain is a rectangular map of dimensions $2X$ and $2Y$, respectively. The usual approximation is to consider the integrand to be equal to zero for all points outside the data domain. One then has

$$\iint_{-\infty}^{+\infty} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_{-X}^{X} \int_{-Y}^{Y} f(x, y)\,\mathrm{d}x\,\mathrm{d}y. \tag{9}$$

A better approximation is to assume the gravity field to be equal to the field of a certain homogeneous sphere for all points outside the data domain (Grant and West 1965). This sphere has a mass equal to the mass of the source body and its center located at the mass center of that body. In this case one can obtain

$$\iint_{-\infty}^{+\infty} [f(x, y) - f_s(x, y)]\,\mathrm{d}x\,\mathrm{d}y = \int_{-X}^{X} \int_{-Y}^{Y} [f(x, y) - f_s(x, y)]\,\mathrm{d}x\,\mathrm{d}y, \tag{10}$$

where f_s is the integrand corresponding to the gravity field of the above sphere. On the topographic surface, this field is

$$g_s(x, y, h(x, y)) = \frac{M(h(x, y) - h_c)}{[(x - x_c)^2 + (y - y_c)^2 + (h(x, y) - h_c)^2]^{3/2}}, \tag{11}$$

where M is the mass multiplied by the gravitational constant and (x_c, y_c, h_c) are the coordinates of the mass center.

It remains to find the unknowns (M, x_c, y_c, h_c) using a least-squares minimum condition between the values of the field obtained with (11) and the available data on the boundary of the gravity map, i.e.,

$$\sum_{i=1}^{N} \left[\frac{M(h_i - h_c)}{[(x_i - x_c)^2 + (y_i - y_c)^2 + (h_i - h_c)^2]^{3/2}} - g_i \right]^2 = \text{minimum}, \tag{12}$$

where g_i is the anomalous value at the data point with coordinates $(x_i, y_i, h_i = h(x_i, y_i))$, and N is the number of points used for the computation.

Differentiating (12) with respect to the unknowns, a nonlinear system is obtained which can be solved by using a suitable iterative method (e.g., Marquardt 1963).

Now, (7) is also valid after replacing g with $g - g_s$. Consequently, the upward continued field can be computed by using the relation

$$g(\bar{x}, \bar{y}, H) \approx g_s(\bar{x}, \bar{y}, H) + \frac{1}{2\pi} \int_{-X}^{X} \int_{-Y}^{Y} [g(x, y, h(x, y)) - g_s(x, y, h(x, y))] \times \frac{(x - \bar{x}) \dfrac{\partial h}{\partial x} + (y - \bar{y}) \dfrac{\partial h}{\partial y} + H - h(x, y)}{[(x - \bar{x})^2 + (y - \bar{y})^2 + (H - h(x, y))^2]^{3/2}}\,\mathrm{d}x\,\mathrm{d}y. \tag{13}$$

The procedure can be extended without difficulty to the magnetic field.

NUMERICAL EVALUATION OF THE INTEGRAL AND DERIVATIVES

Usually, the derivatives of the function $h = h(x, y)$ have to be computed from the values of h in a rectangular grid. A corresponding evaluation of a double integral has also to be performed. Syberg (1972) avoided computing ordinary double integrals by considering his final relations to be convolution integrals and by using spectral computations. Unfortunately, this is valid only when the topographic relief is a horizontal plane. Bhattacharyya and Chan (1977) evaluated the horizontal derivatives by using bicubic splines. The double integrals are evaluated with a window of 41 by 41 points.

In this paper a window of 5 by 5 points has been used to evaluate the integral on each square of 3 by 3 points (see fig. 2). First, the data domain is divided into such squares. For each of these squares the integral is evaluated by using the formula

$$\int_{-s}^{s}\int_{-s}^{s} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = (16/9f_{33} + 17/45(f_{23} + f_{32} + f_{34} + f_{43}) + 1/5(f_{22} + f_{42} + f_{24} + f_{44}) + 1/90(f_{12} + f_{21} + f_{14} + f_{41} + f_{25} + f_{52} + f_{45} + f_{54}))s^2, \quad (14)$$

where s is the sampling interval. Finally, the integrals for all the squares are summed.

The formula for the derivative with respect to x is

$$\left.\frac{\partial h}{\partial x}\right|_{33} = \frac{h_{44} - h_{22} + h_{42} - h_{24} + h_{43} - h_{23}}{4s} - \frac{h_{45} - h_{21} + h_{54} - h_{12} + h_{41} - h_{25} + h_{52} - h_{14}}{24s}. \quad (15)$$

The relation for the derivative with respect to y is analogous.

Calculations are simplified if previously the elevations are divided by the sampling interval. Formulae (14) and (15) have been obtained by using a technique similar to that of Rosenbach (1953).

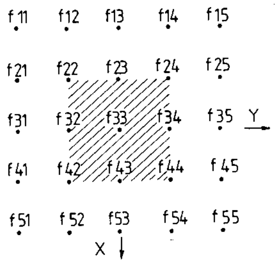

Fig. 2. Matrix used to evaluate the double integral and the partial derivatives.

TESTS OF THE METHOD ON MODEL DATA

The gravity source is represented by three vertical mass points. (An equivalent source distribution is shown in fig. 3.) The gravity field of this body was calculated at the points of a square grid, with a spacing of 25 m, on the relief of fig. 4 and on a horizontal plane placed at the height $H = 2075$ m. The results of these calculations are shown in figs 5 and 6.

The 33 by 33 field data matrix and the 37 by 37 elevation data matrix were processed in order to obtain the 33 by 33 gravity field matrix at $H = 2075$ m. First, the unknown parameters M, x_c, y_c, h_c were obtained. The convergence of the method is fast (especially for good initial values) and the results give a good fit to (12). The center of the sphere of (11) is located at about one sampling interval from the mass center of the source body. The mass of that sphere is only 60% of the expected mass, mainly because the upper part of the body is placed above the boundary points of the gravity map used for computations. If no additional information about the body is available, for intricate geological structures, only the edge effect can be minimized by the described technique.

The results obtained using Syberg's (1972) method and minimizing the truncation effect are shown in fig. 7. If this effect is disregarded the values of the upward continued field are between 0.24 mGal and 0.28 mGal. The computation required was 1.5 CPU min on an IBM-370.

The results obtained using the method of Bhattacharyya and Chan (1977) are shown in figs 8 and 9. The integrals and the horizontal derivatives have been evaluated by using (14) and (15), respectively. Four complete iterations required 398 CPU min but the method failed to converge.

The results obtained with the method proposed here are shown in fig. 10. The computer time required was 47 CPU min in each case. Better results are obtained if wider grids are used (a 43 by 43 gravity matrix and a 47 by 47 topographic matrix). In this case, the field of (11) gives a maximum deviation of ±0.09 mGal from the

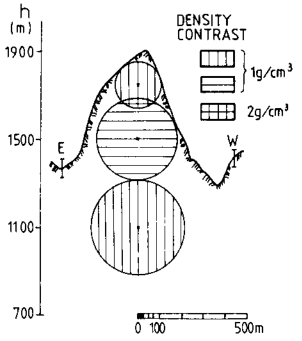

Fig. 3

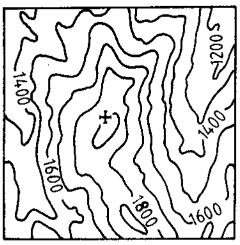

Fig. 4

Fig. 3. East–West topographic profile with source body locations.

Fig. 4. Map of the topographic surface. Contour interval 100 m. + the projection of the mass points.

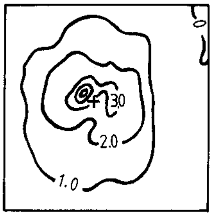

Fig. 5

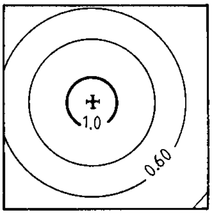

Fig. 6

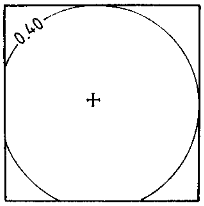

Fig. 7

Fig. 5. Map of the gravity field on the topographic surface. Contour interval 1 mGal.

Fig. 6. Map of the gravity field on the plane at height 2075 m. Contour interval 0.20 mGal.

Fig. 7. Upward continued field obtained using the Syberg method. The truncation effect is minimized. Contour interval 0.20 mGal.

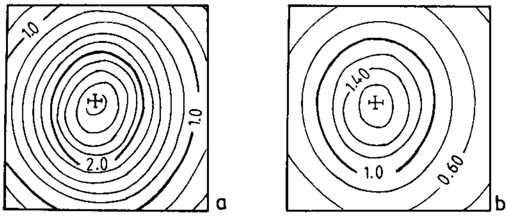

Fig. 8. Upward continued field obtained using the Bhattacharyya and Chan method. Zero order approximation. (a) Truncation effect is neglected; (b) truncation effect is minimized. Contour interval 0.20 mGal.

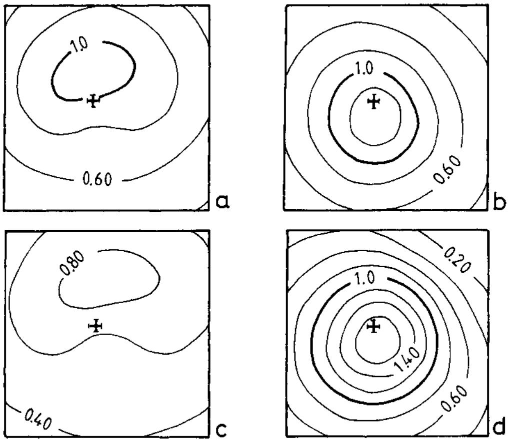

Fig. 9. Upward continued field obtained by using the Bhattacharyya and Chan method. The truncation effect is minimized. (a) First order approximation; (b) second order approximation; (c) third order approximation; (d) fourth order approximation. Contour interval 0.20 mGal.

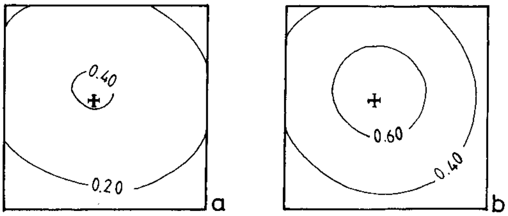

Fig. 10. Upward continued field obtained by using the proposed method. (a) Truncation effect is neglected; (b) truncation effect is minimized. Contour interval 0.20 mGal.

boundary values used for computations in (12). Even when the edge effect is minimized the method of Bhattacharyya and Chan (1977) fails to converge, mainly because the topographic surface is very rugged. The new values of the upward-continued field obtained by using this method show a maximum deviation of 0.06 mGal from the field of figs 8(b) and 9(a, b).

CONCLUSIONS

The proposed method gives good results at reasonably low cost. No matter what method is used, the edge effect has an important impact on the upward continuation problem. When real gravity or magnetic maps are processed it is sometimes difficult to extend the size of an anomalous field because of contiguous fields. In such cases only qualitative results can be expected.

The procedure derived to minimize the edge effect could also be used in spectral analysis of gravity and magnetic maps.

The results obtained in this paper are easily modified for profile data. The sphere used for minimizing the edge effect will consequently be replaced by a horizontal homogeneous infinite cylinder.

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