Acoustic Wave Equation

Sjoerd de Ridder (most of the slides) & Biondo Biondi

January 16^{th} 2011

Table of Topics

- Basic Acoustic Equations
- Wave Equation
- Finite Differences
- Finite Difference Solution
- Pseudospectral Solution
- Stability and Accuracy
- Green's function
- Perturbation Representation
- Born Approximation

Basic linearized acoustic equations in lossless, isotropic, non flowing media

Linearized - Linear for small perturbation on a static state.
Lossless - Material parameters are independent of time.
Isotropic - Material response independent of direction.
Non flowing - No material derivative

Equation of motion

$$\rho \partial_t v_i + \partial_i p = f_i \tag{1}$$

(three equations for three components)

Acoustic stress-strain relationship

$$\rho \partial_t p + \partial_i v_i = q \tag{2}$$

(a pressure-rate strain-rate relation)

Fields

$$
\begin{array}{ll}
p = p(\mathbf{x}, t) & \text{pressure} \\
v_i = v_i(\mathbf{x}, t) & \text{i} - \text{component of velocity}
\end{array}
$$

Sources

$$
\begin{array}{ll}
q = q(\mathbf{x}, t) & \text{volume injection rate} \\
f_i = f_i & \text{i} - \text{component of external force}
\end{array}
$$

Medium Parameters

$$
\begin{array}{ll}
\kappa = \kappa(\mathbf{x}) & \text{compressibility} \\
\rho = \rho(\mathbf{x}) & \text{density}
\end{array}
$$

Wave Equation

Solve equations (1) and (2) for pressure

$$\rho\partial_i\rho^{-1}\partial_i p - \rho\kappa\partial_t^2 p = \rho\partial_i\rho^{-1} f_i - \rho\partial_t q, \tag{3}$$

or

$$\partial_i^2 p - \rho\kappa\partial_t^2 p = \rho\partial_i\rho^{-1} f_i - \rho\partial_t q + \rho^{-1}\partial_i\rho\partial_i p. \tag{4}$$

Thus in a constant density and sourceless medium

$$\partial_i^2 p - c^{-2}\partial_t^2 p = 0, \tag{5}$$

with wave velocity $c = c(\mathbf{x}) = \sqrt{\kappa\rho}$, $\kappa = \kappa(\mathbf{x})$, $\rho = \rho_0$.

Finite Differences

Derivation of finite difference stencils for $\frac{\partial F(s)}{\partial s}$

Expand $F(s+\Delta s)$ in Taylor series

$$F(s+\Delta s) = F(s) + \frac{1}{1!}\partial_s F(s)\Delta s + \sum_{i=2}^{\infty} \frac{1}{i!}\partial_s^i F(s)\left\{\Delta s\right\}^i \qquad (6)$$

Express $\frac{\partial F(s)}{\partial s}$ as a function of ...

$$\partial_s F(s) = \frac{1}{\Delta s}\left\{F(s+\Delta s) - F(s)\right\} - \sum_{i=2}^{\infty} \frac{1}{i!}\partial_s^i F(s)\left\{\Delta s\right\}^{i-1} \qquad (7)$$

this is a forward finite difference stencil.

Expand $F(s+\Delta s)$ and $F(s-\Delta s)$ in Taylor series

$$F(s+\Delta s) = F(s) + \frac{1}{1!}\partial_s F(s)\Delta s + \sum_{i=2}^{\infty} \frac{1}{i!}\partial_s^i F(s)\left\{\Delta s\right\}^i \tag{8}$$

$$F(s-\Delta s) = F(s) - \frac{1}{1!}\partial_s F(s)\Delta s + \sum_{i=2}^{\infty} \frac{1}{i!}\partial_s^i F(s)\left\{-\Delta s\right\}^i \tag{9}$$

Substract equations (9) from (8), express $\frac{\partial F(s)}{\partial s}$ as a function of ...

$$\partial_s F(s) = \frac{1}{2\Delta s}\left\{F(s+\Delta s) - F(s-\Delta s)\right\} - \sum_{i=1}^{\infty} \frac{1}{(1+2i)!}\partial_s^{1+2i} F(s)\left\{\Delta s\right\}^{2i} \tag{10}$$

this is a centered finite difference stencil.

or last, $\partial_s F(s)$ in a backward finite difference stencil from equation (9) as

$$\partial_s F(s) = \frac{1}{\Delta s}\left\{F(s) - F(s-\Delta s)\right\} - \sum_{i=2}^{\infty} \frac{1}{i!}\partial_s^i F(s)\left\{\Delta s\right\}^{i-1} \quad (11)$$

Derivation of finite difference stencils for $\partial_s^2 F(s)$

Add equation (9) and (8), express $\partial_s^2 F(s)$ as a function of ...

$$\partial_s^2 F(s) = \frac{1}{(\Delta s)^2}\left\{F(s-\Delta s) - 2F(s) + F(s+\Delta s)\right\} \quad + \sum_{i=1}^{\infty} \frac{1}{(2+2i)!}\partial_s^{2+2i} F(s)\left\{\Delta s\right\}^{2i} \qquad (12)$$

This is a centered finite difference. Forward and backward finite difference stencils for $\partial_s^2 F(s)$ can be obtained from combinations of Taylor series for $F(s+\Delta s)$ and $F(s+2\Delta s)$, or $F(s-\Delta s)$ and $F(s-2\Delta s)$ respectively.

Finite Difference Solution of WE

Wave equation, FD 2^{nd}-order in space

$\Delta h = \Delta y = \Delta h$

$$\nabla^2 P(x,t) - \frac{1}{c^2(x)}\partial_t^2 P(x,t) = \frac{1}{(\Delta h)^2}\begin{array}{|c|c|c|}\hline & +1 & \\ \hline +1 & -2 & +1 \\ \hline & +1 & \\ \hline\end{array}\ P_x(t) - \frac{1}{c^2(x)}\partial_t^2 P_x(t) \tag{13}$$

Laplacian, FD 4^{th}-order in space

$$\nabla^2 P(x,t) - \frac{1}{c^2(x)}\partial_t^2 P(x,t) = \frac{1}{12(\Delta h)^2}\begin{array}{|c|c|c|c|c|}\hline & & -1 & & \\ \hline & & +16 & & \\ \hline -1 & +16 & -30 & +16 & -1 \\ \hline & & +16 & & \\ \hline & & -1 & & \\ \hline\end{array}\ P_x(t) - \frac{1}{c^2(x)}\partial_t^2 P_x(t) \tag{14}$$

Laplacian, FD 4^{th}-order in space, isotropic

$$\nabla^2 P(x,t) = \frac{1}{\alpha\ 12(\Delta h)^2} \begin{array}{|c|c|c|c|c|} \hline & & -1 & & \\ \hline & & +16 & & \\ \hline -1 & +16 & -30 & +16 & -1 \\ \hline & & +16 & & \\ \hline & & -1 & & \\ \hline \end{array} P_x(t) +$$

$$\frac{1}{\beta\ 12(\Delta h)^2} \begin{array}{|c|c|c|c|c|} \hline -1 & & & & -1 \\ \hline & +16 & & +16 & \\ \hline & & -30 & & \\ \hline & +16 & & +16 & \\ \hline -1 & & & & -1 \\ \hline \end{array} P_x(t) \qquad (15)$$

$\beta = \frac{1-\alpha}{2}$, for example: $\alpha = 1 \rightarrow \beta = 0$, $\alpha = 1/2 \rightarrow \beta = 1/4$ or $\alpha = 2/3 \rightarrow \beta = 1/6$.

Wave equation, FD 2^{nd}-order time stepping

$$\partial_t^2 P(x,t) - c^2(x)\nabla^2 P(x,t) = \frac{1}{\Delta t^2} \boxed{1 \mid -2 \mid 1} \, P_t(x) - c^2(x)\nabla^2 P_t(x) \tag{16}$$

Solve for $P_{t+1}(x)$

$$P_{t+1}(x) = 2P_t(x) - P_{t-1}(x) + \Delta t^2 c^2(x)\nabla^2 P_t(x) \tag{17}$$

Wave equation, FD 4^{th}-order time stepping

Include the 4^{th}-order derivative from equation (12), by substituting the wave equation (Dablain, 1986), as

$$\partial_t^4 P(x,t) = \partial_t^2 \partial_t^2 P(x,t) = \partial_t^2 c^2(x)\nabla^2 P(x,t) \tag{18}$$

Pseudospectral (Fourier) methods

- **Laplacian computed using FFTs:**
  $c^2(x)\,\nabla^2 P_t(x) \approx c^2(x)\,\mathrm{FFT}^{-1}\left\{-\left|\vec{k}\right|^2 \mathrm{FFT}\left[P_t(x)\right]\right\}$

- **Wave equation, FD 2nd-order time stepping and pseudospectral Laplacian:**
  $P_{t+1}(x) =$
  $2P_t(x) - P_{t-1}(x) + \Delta t^2 c^2(x)\,\mathrm{FFT}^{-1}\left\{-\left|\vec{k}\right|^2 \mathrm{FFT}\left[P_t(x)\right]\right\}$

Stability and accuracy of explicit methods

- **Courant number :** $\mathrm{Cour} = \frac{c_{max}\Delta t}{\min(\Delta x, \Delta y, \Delta z)}$ where c_{max} is the maximum velocity.

- **Courant-Friedrichs-Lewy (CFL) condition :** $\mathrm{Cour} \leq 1$ it is a *necessary*, but not *sufficient* condition for a stable explicit extrapolator.

- **Numerical dispersion** causes $c_P \neq c$, where $c_P = \omega / |k|$ is the effective phase velocity of numerically propagated waves

Stability and accuracy analysis of pseudospectral methods

- **Substitute a generic plane wave solution**: $\exp\left[i\left(\vec{k}x+\omega t\right)\right]$
- **Dispersion relation:** $\omega = \dfrac{2\sin^{-1}\left(\pm\frac{c\Delta t|\vec{k}|}{2}\right)}{\Delta t}$
- **Phase velocity:** $c_P = \dfrac{\omega}{|\vec{k}|} = \dfrac{2\sin^{-1}\left(\pm\frac{c\Delta t|\vec{k}|}{2}\right)}{\Delta t|\vec{k}|}$
- **For stability it must be** $\dfrac{c\Delta t|\vec{k}|}{2} \leq 1$:
  - **1D:** Maximum k equal to Nyquist wavenumber $k_{\mathrm{Nyq}} = \pi/\Delta x$ stability requires $\mathrm{Cour} \leq 2/\pi \approx 0.636$
  - **2D:** $k_{\max} = \sqrt{2}k_{\mathrm{Nyq}}$ stability requires $\mathrm{Cour} \leq \sqrt{2}/\pi \approx 0.45$
  - **3D:** $k_{\max} = \sqrt{3}k_{\mathrm{Nyq}}$ stability requires $\mathrm{Cour} \leq 2/\sqrt{3}\pi \approx 0.367$

Stability and accuracy of 2^{nd}-order in time and space

- **Substitute a generic plane wave solution**:
  $\exp\left[i\left(\vec{k}x + \omega t\right)\right]$

- **Dispersion relation:** $\omega = \frac{2\sin^{-1}\left[\frac{c\Delta t}{\Delta x}\sqrt{\sin^2\left(\frac{k_x\Delta x}{2}\right)+\sin^2\left(\frac{k_z\Delta z}{2}\right)}\right]}{\Delta t}$

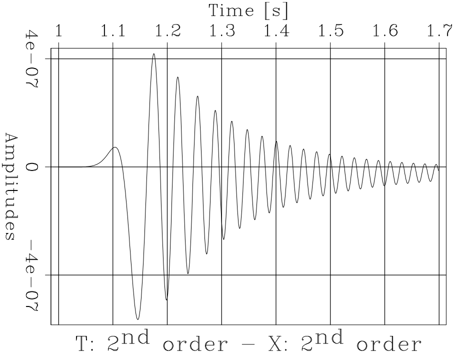
- **Phase velocity (worst case at $k_x = 0$ or $k_z = 0$):**
  $c_P = \frac{\omega}{k_x} = \frac{2\sin^{-1}\left[\frac{c\Delta t}{\Delta x}\sin\left(\frac{k_x\Delta x}{2}\right)\right]}{\Delta t k_x}$

- **For stability the argument of $\sin^{-1}$ must be between -1 and 1:**
  - **1D:** $\mathrm{Cour} \leq 1$
  - **2D:** Worst case at $k_x = k_z = k_{\mathrm{Nyq}}$: $\mathrm{Cour} \leq \sqrt{2}/2 \approx 0.707$

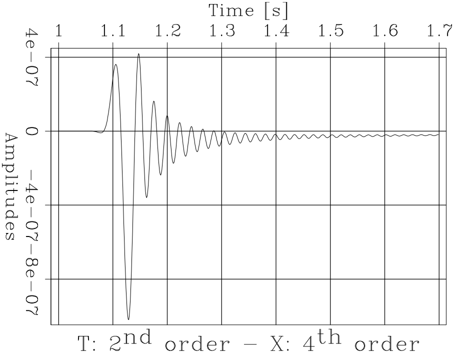
Observations

- **Stability**
  - Stability constraint becomes more stringent with higher dimensions
  - FD "more stable" than pseudospectral because errors in the spatial derivatives slows down high frequencies.
- **Dispersion**
  - Pseudospectral
    - High frequencies (wavenumbers) arrive before low frequencies (wavenumbers).
    - Dispersion gets worse as the Courant number increases.
  - FD
    - High frequencies (wavenumbers) "tend" to arrive after low frequencies (wavenumbers).
    - Dispersion gets better as the Courant number increases.

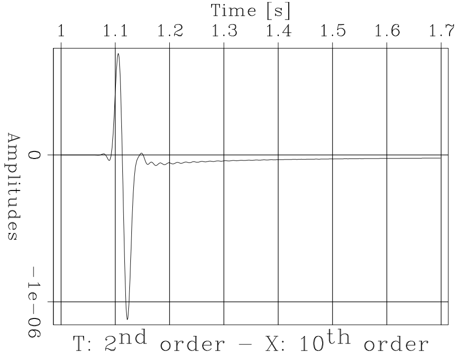
Frequency dispersion with finite-differences schemes



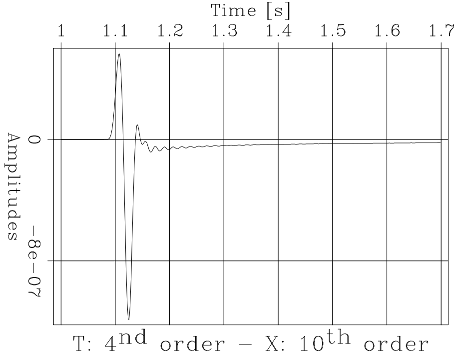
Frequency dispersion with finite-differences schemes



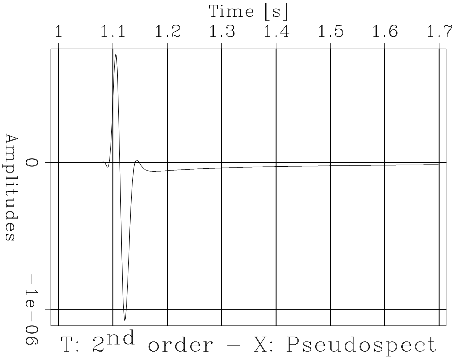
Frequency dispersion with finite-differences schemes



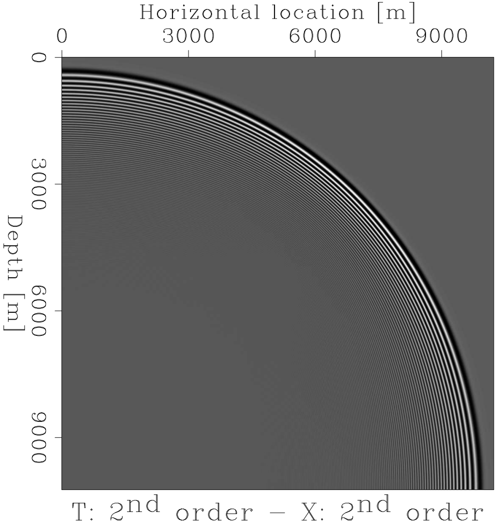
Frequency dispersion with finite-differences schemes



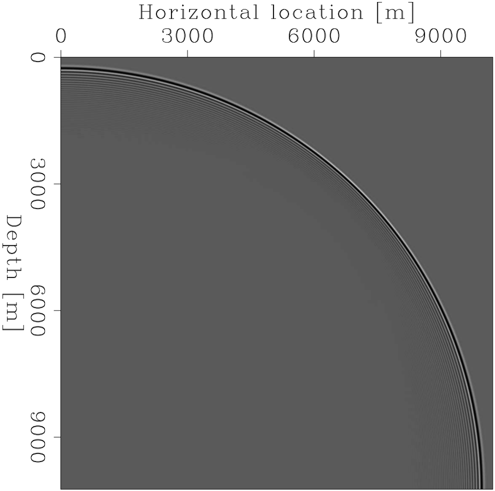
Frequency dispersion with pseudspectral Laplacian



Wavelength dispersion with finite-differences schemes

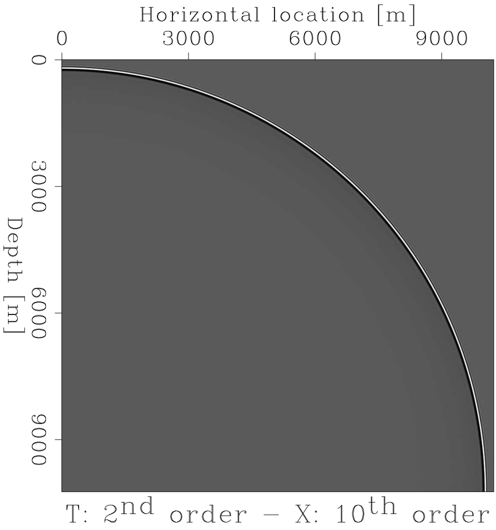


Wavelength dispersion with finite-differences schemes

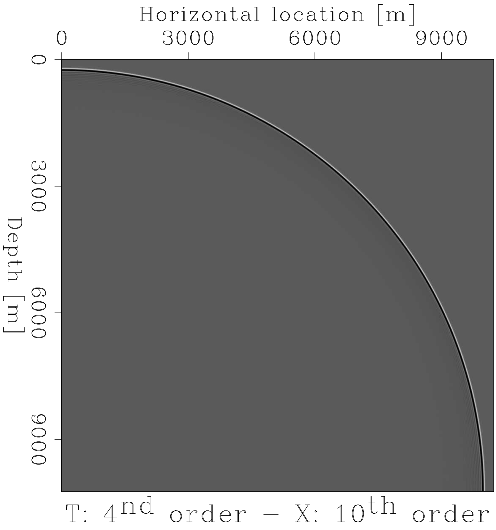


T: 2^{nd} order – X: 4^{th} order

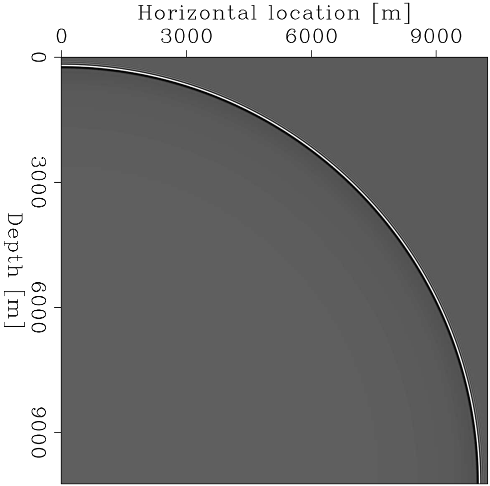
Wavelength dispersion with finite-differences schemes



Wavelength dispersion with finite-differences schemes



Wavelength dispersion with pseudspectral Laplacian



T: 2nd order – X: Pseudospect

Green's function

Introduce Green's function for a constant density and sourceless medium equation (5) by a point source term acting at $t = 0$ and $\mathbf{x} = \mathbf{x}_s$

$$\partial_i^2 G - c^{-2}\partial_t^2 G = -\delta(\mathbf{x} - \mathbf{x}_s)\delta(t), \tag{19}$$

where $G = G(\mathbf{x}, \mathbf{x}_s, t)$ is the Green's function.

The solution for pressure to another forcing function for example $s = s\mathbf{x}, t$ can be represented as

$$p(\mathbf{x}, t) = -\int \oint G(\mathbf{x}, \mathbf{x}', t - t')s(\mathbf{x}', t')\mathrm{d}\mathbf{x}'\mathrm{d}t' \tag{20}$$

Perturbation Representation

Represent the medium velocity as a background velocity and a perturbation

$$c^{-2}(\mathbf{x}) = c_b^{-2}(\mathbf{x})\left[1 + \alpha(\mathbf{x})\right] \qquad (21)$$

Substitution into equation (19) gives

$$\begin{aligned} \partial_i^2 G(\mathbf{x}, \mathbf{x}_s, t) - c_b^{-2}(\mathbf{x})\partial_t^2 G(\mathbf{x}, \mathbf{x}_s, t) = \\ -\delta(\mathbf{x} - \mathbf{x}_s)\delta(t) + \alpha(\mathbf{x})c_b^{-2}(\mathbf{x})\partial_t^2 G(\mathbf{x}, \mathbf{x}_s, t), \end{aligned} \qquad (22)$$

Introducing $G_b(\mathbf{x}, \mathbf{x}_s, t)$ as a solution to

$$\partial_i^2 G_b(\mathbf{x}, \mathbf{x}_s, t) - c_b^{-2}(\mathbf{x}) \partial_t^2 G_b(\mathbf{x}, \mathbf{x}_s, t) = -\delta(\mathbf{x} - \mathbf{x}_s)\delta(t), \tag{23}$$

we see that is we represent the full solution as a sum of the background solution plus a perturbed solution as

$$G(\mathbf{x}, \mathbf{x}_s, t) = G_b(\mathbf{x}, \mathbf{x}_s, t) + G_p(\mathbf{x}, \mathbf{x}_s, t). \tag{24}$$

Equation (22) can be thus written as

$$\partial_i^2 G_p(\mathbf{x}, \mathbf{x}_s, t) - c_b^{-2}(\mathbf{x}) \partial_t^2 G_p(\mathbf{x}, \mathbf{x}_s, t) = \alpha(\mathbf{x}) c_b^{-2}(\mathbf{x}) \partial_t^2 G(\mathbf{x}, \mathbf{x}_s, t). \tag{25}$$

Note the forcing function dependent on medium parameter α. Thus using a representation as (20) for $G_p(\mathbf{x}, \mathbf{x}_s, t)$ we find for $G(\mathbf{x}, \mathbf{x}_s, t)$

$$G(\mathbf{x}, \mathbf{x}_s, t) = G_b(\mathbf{x}, \mathbf{x}_s, t) - \int \oint G_b(\mathbf{x}, \mathbf{x}', t - t') \alpha(\mathbf{x}') c_b^{-2}(\mathbf{x}') \partial_t^2 G(\mathbf{x}', \mathbf{x}_s, t') \mathrm{d}\mathbf{x}' \mathrm{d}t' \tag{26}$$

Born Approximation

The Born approximation is made in the perturbation representation by substituting the total field under the integral for the background field.

$$G(\mathbf{x}, \mathbf{x}_s, t) = G_b(\mathbf{x}, \mathbf{x}_s, t) - \int \oint G_b(\mathbf{x}, \mathbf{x}', t - t')\alpha(\mathbf{x}')c_b^{-2}(\mathbf{x}')\partial_t^2 G_b(\mathbf{x}', \mathbf{x}_s, t')\mathrm{d}\mathbf{x}'\mathrm{d}t' \tag{27}$$

This is an explicit representation for $G(\mathbf{x}, \mathbf{x}_s, t)$.

The perturbation can represent a (single additional) scattered wavefield as

$$G_s(\mathbf{x}, \mathbf{x}_s, t) = d(\mathbf{x}, \mathbf{x}_s, t) = - \int \oint G_b(\mathbf{x}, \mathbf{x}', t - t')\alpha(\mathbf{x}')c_b^{-2}(\mathbf{x}')\partial_t^2 G_b(\mathbf{x}', \mathbf{x}_s, t')\mathrm{d}\mathbf{x}'\mathrm{d}t'. \tag{28}$$