Lecture Notes on

Nonlinear Inversion and Tomography:

I. Borehole Seismic Tomography

From a Series of Lectures by

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Originally Presented at

Earth Resources Laboratory
Massachusetts Institute of Technology

July 9–30, 1990

Revised and Expanded

October, 1991

Chapter 8

Other Nonlinear Inversion Problems

Although traveltime inversion has been the main thrust of these lecture notes, I want to make it clear that the ideas involving the feasibility constraints are very general. In fact, they apply to any inversion problem where the data are the minima of one of the variational problems of mathematical physics.

So in this final section of these notes, I present two other inversion problems that lead to convex feasible sets and then show the general structure needed to guarantee convex global feasibility. Finally, I present another example that leads to a nonconvex feasibility set and discuss the consequences of this difference for computing the solution to the inverse problem.

8.1 Electrical Impedance Tomography

Electrical impedance tomography [Dines and Lytle, 1981; Berryman and Kohn, 1990] attempts to image the electrical impedance (or just the conductivity) distribution inside a body using electrical measurements on its boundary. See Fig. 8.1. The method has been used successfully in both biomedical [Barber and Brown, 1986] and geophysical applications [Wexler, Fry, and Neuman, 1985; Daily, Lin, and Buscheck, 1987; Daily and Owen, 1991], but the analysis of optimal reconstruction algorithms is still progressing [Yorkey, Webster, and Tompkins, 1987; Kohn and McKenney, 1990]. The most common application is monitoring the influx or efflux of a conducting fluid (such as brine in a porous rock) through the body whose conductivity is being imaged. This method does not have high resolving power like radiological methods, but it is comparatively inexpensive and it therefore provides a valuable alternative when continuous monitoring is desired.

First, we review some facts about this problem that play an important role in the analysis that follows. Recall that the power dissipated into heat is [Jackson, 1962]

$$P = \int \mathbf{J}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x})\, d^3x, \tag{8.1}$$

Figure 8.1: Experimental setup for electrical impedance tomography.

where

$$\mathbf{J}(\mathbf{x}) = \sigma(\mathbf{x})\mathbf{E}(\mathbf{x}), \tag{8.2}$$

$$\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}), \tag{8.3}$$

and the current distribution satisfies

$$\nabla \cdot \mathbf{J}(\mathbf{x}) = 0 \tag{8.4}$$

away from all current sources. The quantities displayed are the current distribution $\mathbf{J}$, the isotropic conductivity σ, the electric field $\mathbf{E}$, and the potential Φ. Substituting (8.2) and (8.3) into (8.4) gives Poisson's equation

$$\nabla \cdot (\sigma\nabla\Phi) = 0. \tag{8.5}$$

Substituting (8.3) into (8.1) and using (8.4), we have

$$P = -\int \mathbf{J} \cdot \nabla\Phi \, d^3x = -\int \nabla \cdot (\Phi\mathbf{J}) \, d^3x. \tag{8.6}$$

Then, the divergence theorem shows that

$$P = -\int \Phi \mathbf{J} \cdot \hat{n}\, da, \tag{8.7}$$

where $\hat{n}$ is a unit outward normal vector and da is the infinitesimal surface area on the boundary. If current is injected through metallic electrodes, the potential takes a constant value Φ_k on the kth electrode of surface area a_k. If there are K electrodes, then (8.7) becomes

$$P = \sum_{k=1}^{K} \Phi_k I_k, \tag{8.8}$$

where

$$I_k = -\int_{a_k} \mathbf{J} \cdot \hat{n}\, da \tag{8.9}$$

is the total current injected ($I_k > 0$) or withdrawn ($I_k < 0$) at the kth electrode. Since these are the only sources and sinks, we also have the sumrule

$$\sum_{k=1}^{K} I_k = 0. \tag{8.10}$$

If there are only two injection electrodes, then (8.8) reduces to

$$P = (\Phi_1 - \Phi_2) I_1 = \Delta\Phi I, \tag{8.11}$$

so the power is the product of the measured potential difference $\Delta\Phi$ across the injection electrodes and the injected current I.

The data for electrical impedance tomography have most often been gathered by injecting a measured current between two electrodes while simultaneously measuring the voltage differences between pairs of other electrodes placed around the boundary of the body being imaged. This process is then repeated, injecting current between all possible (generally adjacent) pairs of electrodes, and recording the set of voltage differences for each injection pair i. This data set has normally not included the voltage difference across the injection electrodes, because these voltages cannot be measured as reliably. A substantial contact impedance develops at the interface between the body and the injection electrodes when large currents are present. This problem can be reduced by using large electrodes or small currents. In this lecture, we will assume that voltage differences (and therefore the powers dissipated) across the injection electrodes are known, but it is not necessary that they be known to high accuracy.

Dirichlet's principle [Courant, 1950; Courant and Hilbert, 1953] states that, given a conductivity distribution $\sigma(\mathbf{x})$ and a potential distribution $\Phi(\mathbf{x})$, the power dissipation p_i realized for the ith current injection configuration is the one that minimizes the integral $\int \sigma |\nabla\Phi|^2\, d^3x$ so that

$$p_i(\sigma) = \int \sigma(\mathbf{x}) |\nabla \Phi_i^*(\mathbf{x})|^2\, d^3x = \min_{\Phi_i} \int \sigma(\mathbf{x}) |\nabla \Phi_i(\mathbf{x})|^2\, d^3x. \tag{8.12}$$

The trial potential field for the ith injection pair is $\Phi_i(\mathbf{x})$, while the particular potential field that actually minimizes the the power is $\Phi_i^*(\mathbf{x})$, and this one also satisfies Poisson's equation $\nabla\cdot(\sigma\nabla\Phi_i^*) = 0$ within the body. Furthermore, if the effective power dissipation associated with the trial potential $\Phi_i(\mathbf{x})$ is defined as

$$\bar{p}_i^{(\Phi_i)}(\sigma) \equiv \int \sigma(\mathbf{x})|\nabla\Phi_i(\mathbf{x})|^2\, d^3x, \tag{8.13}$$

then the measured powers P_i must satisfy

$$P_i = p_i(\sigma^*) \le \bar{p}_i^{(\Phi_i)}(\sigma^*), \tag{8.14}$$

if $\sigma^*(\mathbf{x})$ is the true conductivity distribution. Note that if we vary the trial power dissipation (8.13) with respect to the trial potential, we find

$$2\int \sigma\nabla\Phi\cdot\nabla\delta\Phi\, d^3x = -2\int \nabla\cdot(\sigma\nabla\Phi)\delta\Phi\, d^3x = 0 \tag{8.15}$$

at a stationary point. We integrated once by parts to obtain (8.15). Since the volume variation $\delta\Phi$ is arbitrary, its coefficient inside the integral must vanish, so we just recover Poisson's equation, as expected.

Now we begin to see the analogy developing between the seismic traveltime tomography problem and the electrical impedance tomography problem. If we consider the following set of correspondences:

$$s(\mathbf{x}) \to \sigma(\mathbf{x}),$$

$$t_i(s) \to p_i(\sigma),$$

$$\tau_i^P(s) \to \bar{p}_i^{(\Phi_i)}(\sigma),$$

$$T_i \to P_i,$$

$$dl_i^P \to |\nabla\Phi_i(\mathbf{x})|^2\, d^3x,$$

$$dl_i^{P^*} \to |\nabla\Phi_i^*(\mathbf{x})|^2\, d^3x,$$

then we see that the analysis of convex functionals and feasibility sets presented for seismic traveltime tomography carries over directly to the electrical impedance tomography problem when it is formulated this way. For example, the scale invariance property holds for electrical impedance tomography, so multiplying σ by a scalar γ does not change the optimum potential distribution.

The feasibility constraints for electrical impedance tomography now take the form

$$\boxed{\mathbf{K}\hat{\sigma} \quad \ge \quad \mathbf{p},} \tag{8.16}$$

where $\hat{\sigma}^T = (\sigma_1, \ldots, \sigma_n)$, $\mathbf{p}^T = (p_1, \ldots, p_m)$, and the E-squared matrix is given by

$$K_{ij} = \int_{\text{cell}_j} |\nabla\Phi_i|^2 \, d^3x. \tag{8.17}$$

Least-squares methods may be applied to this problem in much the same fashion as in traveltime tomography [Kallman and Berryman, 1992].

A thorough analysis of the electrical impedance tomography problem would require another set of lectures. Lucky for you, I will not try to present them here. However, to excite your curiosity, I will mention another feature of the electrical impedance tomography problem not shared by the seismic tomography problem. So far we have discussed only Dirichlet's principle (8.12). In fact, there are two distinct variational principles for the conductivity problem: Dirichlet's principle and its dual, known as Thomson's principle [Thomson, 1848; Thomson, 1884; Maxwell, 1891; Courant and Hilbert, 1953] The second variational principle takes the form

$$P_i \leq \int |\mathbf{J}_i(\mathbf{x})|^2/\sigma(\mathbf{x}) \, d^3x, \tag{8.18}$$

where $\mathbf{J}_i(\mathbf{x})$ is a trial current distribution vector for the ith current injection pair that satisfies the continuity equation $\nabla \cdot \mathbf{J}_i = 0$. The trial current distribution $\mathbf{J}_i(\mathbf{x})$ and the trial gradient of the potential $\nabla\Phi_i(\mathbf{x})$ are generally unrelated except that, when the minimum of both variational functionals is attained, then $\mathbf{J}_i^*(\mathbf{x}) = -\sigma\nabla\Phi_i^*(\mathbf{x})$. Then, of course, the current equals the conductivity times the electric field.

The existence of dual variational principles is a general property whenever the primal variational principle is a true minimum principle. Fermat's principle is only a stationary (not a minimum) principle, and so traveltime tomography does not possess this dual property. (If we attempt to formulate a dual for Fermat's principle as we did in the lecture on linear and nonlinear programming, we find the content of the dual results are essentially trivial.) The existence of the dual variational principles for electrical impedance tomography is important because it means that there are two independent sets of feasibility constraints for the conductivity model $\sigma(\mathbf{x})$. Furthermore, as illustrated in Fig. 8.2, these two sets of constraints allow us (in some sense) to obtain upper and lower bounds on the region of the conductivity model space that contains the solution to the inversion problem. See Berryman and Kohn [1990] for more discussion of this point.

8.2 Inverse Eigenvalue Problems

Inverse eigenvalue problems arise in the earth sciences during attempts to deduce earth structure from knowledge of the modes of vibration of the earth [Dahlen, 1968; Wiggins, 1972; Jordan and Anderson, 1974; Hald, 1980; Hald, 1983; Anderson and Dziewonski, 1984; McLaughlin, 1986; Dziewonski and Woodhouse, 1987; Lay, Ahrens, Olson, Smyth, and Loper, 1990; Snieder, 1993].

Consider the typical forward problem associated with the inverse eigenvalue problem

$$-\nabla^2 u(\mathbf{x}) + q(\mathbf{x})u(\mathbf{x}) = \lambda u(\mathbf{x}) \tag{8.19}$$

Figure 8.2: Dirichlet's principle and Thomson's principle provide upper and lower bounds on the dual feasibility region for electrical impedance tomography.

on a finite domain with some boundary conditions on u. This is known as a Sturm-Liouville equation to mathematicians and as the Schroedinger equation to physicists. In quantum mechanics, the time-independent wave function is given by $u(\mathbf{x})$ and $q(\mathbf{x})$ is the potential. The eigenvalue is λ.

Now it is well-known that a Rayleigh-Ritz procedure may be used to approximate the eigenvalues λ [Courant and Hilbert, 1953]. In particular, the lowest eigenvalue is given in general by

$$\lambda_0 = \min_u \frac{\int (|\nabla u|^2 + qu^2)\, d^3x}{\int u^2\, d^3x}, \tag{8.20}$$

where admissible us satisfy the boundary conditions of (8.19) and have no other constraints, except being twice differentiable. The ratio on the right to be minimized is known as the Rayleigh quotient, and the denominator $\int u^2\, d^3x$ serves to normalize the wave function u.

Define the Rayleigh-quotient functional as

$$\Lambda(q, u_i) = \frac{\int (|\nabla u_i|^2 + qu_i^2)\, d^3x}{\int u_i^2\, d^3x}, \tag{8.21}$$

where u_i is a trial wave function subject to i constraints. Taking the variation of Λ with respect to u_i, we find that the stationary points of Λ satisfy

$$\frac{\int[-\nabla^2 u_i + qu_i - \Lambda(q,u_i)u_i]\delta u_i\, d^3x}{\int u_i^2\, d^3x} = 0. \tag{8.22}$$

We integrated once by parts to obtain (8.22) using the fact that the variations of δu vanish on the boundary. So, since the variations δu within the domain may be arbitrary, the term in brackets must vanish and the stationary points of the Rayleigh quotient therefore occur for u_is that satisfy (8.19) with $\lambda_i = \Lambda(q, u_i)$.

Clearly, we may define feasibility constraints for this problem in a manner analogous to that for the traveltime tomography problem and for the electrical impedance tomography problem. If the eigenvalues λ_i are our data, then for the correct potential q^* we must have

$$\lambda_i \equiv \Lambda(q^*, u_i^*[q^*]) \le \Lambda(q^*, u_i), \tag{8.23}$$

where $u_i^*[q]$ is the eigenfunction associated with eigenvalue λ_i of the potential q. Thus, feasible qs satisfy

$$\boxed{\lambda_i \quad \le \quad \Lambda(q, u_i)} \tag{8.24}$$

for all admissible u_is.

To show that this problem leads to a convex feasibility set, consider two potentials that satisfy the feasibility constraints for some fixed choice of u_i. Then,

$$\lambda_i \le \Lambda(q_1, u_i) \quad \text{and} \quad \lambda_i \le \Lambda(q_2, u_i) \tag{8.25}$$

and

$$\lambda_i \le \epsilon\Lambda(q_1, u_i) + (1-\epsilon)\Lambda(q_2, u_i) \tag{8.26}$$

$$= \frac{\int(|\nabla u_i|^2 + [\epsilon q_1 + (1-\epsilon)q_2]u_i^2)\, d^3x}{\int u_i^2\, d^3x} \tag{8.27}$$

$$= \Lambda(q_\epsilon, u_i), \tag{8.28}$$

where the convex combination $q_\epsilon \equiv \epsilon q_1 + (1-\epsilon)q_2$. Thus, local (fixed u_is) feasibility follows simply from the linearity of the Rayleigh quotient (except for the shift at the origin) with respect to the potential q. Global feasibility follows from the variational properties of Λ with respect to u_i. (See the next section for the proof.)

Note that there is no scale invariance property for Λ similar to the one for the traveltime functional. However, it is true that wave functions are invariant to a constant shift in the potential, since it is easy to see that

$$\Lambda(q+\gamma, u) = \Lambda(q, u) + \gamma. \tag{8.29}$$

In our analysis, we can also make use of other members of the invariance group of (8.19) [Ames, 1972].

This inverse eigenvalue problem can be reformulated in terms of a different set of variational functionals. In particular, one such set of generalized Rayleigh-Ritz quotients has been constructed by Berryman [1988]; however, these functionals have a more complicated dependence on the potential q. Without linearity or shifted linearity in q, we cannot prove the convexity of the feasibility set and the structure of the inversion problem becomes less certain and possibly more complex.

8.3 General Structure for Convex Inversion Problems

The feasibility analysis presented in these lectures applies to a wide class of inverse problems that can be formulated so the data are minima of an appropriate variational problem. To see the general structure, consider a set of functionals $\Gamma_i(q,u)$ of two variables q and u. Then, if each functional is linear in one variable so that

$$\Gamma_i(aq_1 + bq_2, u) = a\Gamma_i(q_1,u) + b\Gamma_i(q_2,u), \tag{8.30}$$

and if the data γ_i bound $\Gamma_i(q_1,u)$ and $\Gamma_i(q_2,u)$ from below for any second argument u, then

$$\gamma_i \le \Gamma_i(q_1,u) \quad \text{and} \quad \gamma_i \le \Gamma_i(q_2,u) \quad \text{for all} \quad i = 1,\dots,m, \tag{8.31}$$

and we have

$$\gamma_i \le \lambda\Gamma_i(q_1,u) + (1-\lambda)\Gamma_i(q_2,u) = \Gamma_i(\lambda q_1 + (1-\lambda)q_2, u). \tag{8.32}$$

Therefore, Γ_i evaluated at the convex combination $q_\lambda = \lambda q_1 + (1-\lambda)q_2$ is also bounded below by the data. Thus, linearity for fixed u is sufficient to prove that feasible qs for the linear problem form a convex set. We call this the *local convex feasibility* property.

Then, when we consider variations of the second argument and assume that the data are minima of the variational functional over all possible us, we have

$$\gamma_i \equiv \Gamma_i(q^*, u^*[q^*]) \le \Gamma_i(q^*, u), \tag{8.33}$$

where $u^*[q]$ is the particular function that minimizes the the functional Γ_i when q is the first argument. Then, we have

$$\gamma_i \le \Gamma_i(q_1, u^*[q_1]) \le \Gamma_i(q_1, u^*[\cdot]), \tag{8.34}$$

$$\gamma_i \le \Gamma_i(q_2, u^*[q_2]) \le \Gamma_i(q_2, u^*[\cdot]), \tag{8.35}$$

where $u^*[\cdot]$ is the correct (minimizing) u for some yet to be specified q. Combining (8.34) and (8.35) using the linearity property of Γ_i for its first argument, we have

$$\begin{aligned}
\gamma_i &\le \lambda\Gamma_i(q_1, u^*[q_1]) + (1-\lambda)\Gamma_i(q_2, u^*[q_2]) && (8.36)\\
&\le \lambda\Gamma_i(q_1, u^*[\cdot]) + (1-\lambda)\Gamma_i(q_2, u^*[\cdot]) && (8.37)\\
&= \Gamma_i(q_\lambda, u^*[\cdot]), && (8.38)
\end{aligned}$$

where $q_\lambda = \lambda q_1 + (1-\lambda)q_2$ is again the convex combination of q_1 and q_2. Now we are free to choose the $\cdot$ to be any permissible q, so we choose it for convenience to be q_λ. Then, we have the final result that

$$\boxed{\gamma_i \quad \leq \quad \Gamma_i(q_\lambda, u^*[q_\lambda]).} \tag{8.39}$$

The conclusion from (8.32) is that there are *local convex feasibility sets* and from (8.39) that there is a *global convex feasibility set* for the full nonlinear inversion problem, just as in the case for traveltime tomography.

The only properties used in the derivation were the linearity of the variational functional Γ_i for fixed u and the concavity of the functional that results from its variational nature.

The preceding proof is appropriate for Fermat's, Dirichlet's, and Thomson's principles. However, the proof must be modified for the inverse eigenvalue problem because the Rayleigh quotient is a shifted linear functional of the potential q. We can fix this minor difficulty by considering

$$\Delta\Lambda(q, u_i) = \Lambda(q, u_i) - \Lambda(0, u_i) = \frac{\int q u^2\, d^3x}{\int u^2\, d^3x}, \tag{8.40}$$

which is linear in q. If

$$\lambda_i \leq \Lambda(0, u_i) + \Delta\Lambda(q_1, u_i), \tag{8.41}$$

$$\lambda_i \leq \Lambda(0, u_i) + \Delta\Lambda(q_2, u_i), \tag{8.42}$$

then we carry through the analysis as before and conclude that

$$\lambda_i \leq \Lambda(0, u_i) + \Delta\Lambda(\epsilon q_1 + (1-\epsilon)q_2, u_i) = \Lambda(q_\epsilon, u_i). \tag{8.43}$$

This proves the *local convex feasibility* property for problems with variational functionals linear in the first argument except for a constant. The proof of *global convex feasibility* follows the proof already presented step by step and will be left as an exercise.

Problem

PROBLEM 8.3.1 *Prove global convex feasibility for the inverse eigenvalue problem.*

8.4 Nonconvex Inversion Problems with Feasibility Constraints

Although we expect the idea of using feasibility constraints in inversion problems with variational structure to be a very general method, it may not always be true that the variational functional is a concave functional of its arguments. If not, then the resulting nonlinear programming problem will not be convex.

As an example, consider the electrical impedance tomography problem again, but this time for complex (still isotropic) conductivity $\sigma = \sigma_R + i\sigma_I$. The dissipative part of σ is the real part σ_R, while the reactive part (proportional to the dielectric constant) is the imaginary part σ_I.

The current is proportional to the conductivity and the electric field, but now all quantities are complex so

$$\mathbf{J} = \sigma \mathbf{E} \tag{8.44}$$

becomes

$$\mathbf{j}_R + i\mathbf{j}_I = (\sigma_R + i\sigma_I)(\mathbf{e}_R + i\mathbf{e}_I). \tag{8.45}$$

The power dissipation for this problem is given by

$$P = \frac{1}{2}\int (\mathbf{J}\cdot\mathbf{E}^* + \mathbf{J}^*\cdot\mathbf{E})\, d^3x \tag{8.46}$$

$$= \int (\mathbf{j}_R\cdot\mathbf{e}_R + \mathbf{j}_I\cdot\mathbf{e}_I)\, d^3x \tag{8.47}$$

$$= \int \sigma_R(\mathbf{e}_R\cdot\mathbf{e}_R + \mathbf{e}_I\cdot\mathbf{e}_I)\, d^3x. \tag{8.48}$$

Rewriting (8.45) in matrix notation we have

$$\begin{pmatrix} \mathbf{j}_R \\ \mathbf{j}_I \end{pmatrix} = \begin{pmatrix} \sigma_R & -\sigma_I \\ \sigma_I & \sigma_R \end{pmatrix} \begin{pmatrix} \mathbf{e}_R \\ \mathbf{e}_I \end{pmatrix}. \tag{8.49}$$

Now we want to reformulate this problem as a variational principle in order to apply the ideas of feasibility constraints, but to do so we need a positive scalar functional. The power dissipation is a good choice again, but (8.49) is inconvenient for this purpose since the matrix is not positive definite [Milton, 1990; Cherkaev and Gibiansky, 1994]. Performing a Legendre transform on (8.49), we find that an alternative equation is

$$\begin{pmatrix} \mathbf{j}_R \\ \mathbf{e}_I \end{pmatrix} = \begin{pmatrix} \sigma_R + \frac{\sigma_I^2}{\sigma_R} & -\frac{\sigma_I}{\sigma_R} \\ -\frac{\sigma_I}{\sigma_R} & \frac{1}{\sigma_R} \end{pmatrix} \begin{pmatrix} \mathbf{e}_R \\ \mathbf{j}_I \end{pmatrix} \equiv \mathbf{\Sigma} \begin{pmatrix} \mathbf{e}_R \\ \mathbf{j}_I \end{pmatrix}. \tag{8.50}$$

Then, the matrix $\mathbf{\Sigma}$ is positive definite (for $\sigma_R > 0$), since

$$\mathbf{\Sigma} \begin{pmatrix} \mathbf{e}_R \\ \mathbf{j}_I \end{pmatrix} = \lambda \begin{pmatrix} \sigma_R & 0 \\ 0 & 1/\sigma_R \end{pmatrix} \begin{pmatrix} \mathbf{e}_R \\ \mathbf{j}_I \end{pmatrix} \tag{8.51}$$

implies that

$$\lambda + \frac{1}{\lambda} = 2 + \frac{\sigma_I^2}{\sigma_R^2}, \tag{8.52}$$

which guarantees that the eigenvalues λ and $1/\lambda$ are positive.

So now the power is given by

$$P = \int (\mathbf{j}_R \cdot \mathbf{e}_R + \mathbf{j}_I \cdot \mathbf{e}_I)\, d^3x \tag{8.53}$$

$$= \int (\,\mathbf{e}_R \quad \mathbf{j}_I\,)\, \mathbf{\Sigma} \begin{pmatrix} \mathbf{e}_R \\ \mathbf{j}_I \end{pmatrix} d^3x \tag{8.54}$$

$$= \int [\sigma_R |\mathbf{e}_R|^2 + \frac{1}{\sigma_R} |\mathbf{j}_I - \sigma_I \mathbf{e}_R|^2]\, d^3x. \tag{8.55}$$

This is the final expression for the power. In this form, we have a valid variational principle. Also, note that the term $\mathbf{j}_I - \sigma_I \mathbf{e}_R = \sigma_R \mathbf{e}_I$ so the second term in the final expression for P is just $\sigma_R |\mathbf{e}_I|^2$.

To check the conditions for stationarity of this integral, we find that, if we vary with respect to $\mathbf{e}_R$, then

$$2 \int [\sigma_R \mathbf{e}_R - \frac{\sigma_I}{\sigma_R} (\mathbf{j}_I - \sigma_I \mathbf{e}_R)] \cdot \delta \mathbf{e}_R\, d^3x = 0. \tag{8.56}$$

If we vary with respect to $\mathbf{j}_I$, we find that

$$2 \int [\frac{1}{\sigma_R} (\mathbf{j}_I - \sigma_I \mathbf{e}_R)] \cdot \delta \mathbf{j}_I\, d^3x = 0. \tag{8.57}$$

Since the electric field is the gradient of a potential, (8.56) implies that

$$\nabla \cdot [\sigma_R \mathbf{e}_R - \frac{\sigma_I}{\sigma_R} (\mathbf{j}_I - \sigma_I \mathbf{e}_R)] = 0. \tag{8.58}$$

Similarly, since the current distribution is divergence free, (8.57) implies that

$$\frac{1}{\sigma_R} (\mathbf{j}_I - \sigma_I \mathbf{e}_R) = -\nabla \Phi \tag{8.59}$$

for some scalar potential function Φ. Thus, the expression in (8.59) acts like an electric field (in fact, it is $\mathbf{e}_I$) at the stationary point, while the quantity whose divergence is zero in (8.58) acts like a current distribution (in fact, it is $\mathbf{j}_R$). This completes the proof that (8.55) is a legitimate variational principle for the complex conductivity problem.

We can still talk about feasibility constraints for this problem, since

$$P_i \equiv \bar{p}_i(\sigma_R^*, \sigma_I^*, \mathbf{e}_R^*, \mathbf{j}_I^*) \leq \bar{p}_i(\sigma_R^*, \sigma_I^*, \mathbf{e}_R, \mathbf{j}_I) \tag{8.60}$$

with the trial power dissipation given by

$$\bar{p}_i(\sigma_R, \sigma_I, \mathbf{e}_R, \mathbf{j}_I) = \int [\sigma_R |\mathbf{e}_R|^2 + \frac{1}{\sigma_R} |\mathbf{j}_I - \sigma_I \mathbf{e}_R|^2]\, d^3x. \tag{8.61}$$

The starred quantities in (8.60) are the true ones for the experimental configuration. If we can find σs that violate the constraints implied by (8.60), then those σs are infeasible and the rest form the feasible set. However, $\bar{p}$ is not linear in its dependence on σ, so we cannot prove that this functional is concave. [1] Therefore, we lack a proof of the convexity of the feasible set.

For fixed σ_I, $\mathbf{j}_I$, and $\mathbf{e}_R$, the minimum of (8.61) is achieved, for a model of constant conductivity cells, when the real conductivity in the jth cell is given by

$$\sigma_R^2 = \frac{\int_{\text{cell}_j} |\mathbf{j}_I - \sigma_I \mathbf{e}_R|^2 \, d^3x}{\int_{\text{cell}_j} |\mathbf{e}_R|^2 \, d^3x}. \tag{8.62}$$

This minimum value is

$$\min_{\sigma_R} \bar{p}_i = 2 \sum_{j=1}^{n} [\int_{\text{cell}_j} |\mathbf{e}_R|^2 \, d^3x \int_{\text{cell}_j} |\mathbf{j}_I - \sigma_I \mathbf{e}_R|^2 \, d^3x]^{\frac{1}{2}}. \tag{8.63}$$

Since the imaginary part of the conductivity may still be viewed as a variable, we can further minimize (8.63) by finding the minimum with respect to σ_I. This minimum occurs when

$$\sigma_I = \frac{\int_{\text{cell}_j} \mathbf{j}_I \cdot \mathbf{e}_R \, d^3x}{\int_{\text{cell}_j} \mathbf{e}_R \cdot \mathbf{e}_R \, d^3x} \tag{8.64}$$

for the imaginary part of the conductivity in the jth cell. Substituting into (8.63), we have the minimum power

$$\min_{\sigma_R, \sigma_I} \bar{p}_i = 2 \sum_{j=1}^{n} [\int_{\text{cell}_j} \mathbf{e}_R \cdot \mathbf{e}_R \, d^3x \int_{\text{cell}_j} \mathbf{j}_I \cdot \mathbf{j}_I \, d^3x - (\int_{\text{cell}_j} \mathbf{j}_I \cdot \mathbf{e}_R \, d^3x)^2]^{\frac{1}{2}}. \tag{8.65}$$

It follows from the Schwartz inequality for integrals that

$$(\int \mathbf{a} \cdot \mathbf{b} \, d^3x)^2 \leq \int \mathbf{a} \cdot \mathbf{a} \, d^3x \int \mathbf{b} \cdot \mathbf{b} \, d^3x \tag{8.66}$$

with equality applying *only* when $\mathbf{b}$ is proportional to $\mathbf{a}$, that each bracket in (8.65) is positive unless there is an exact solution such that

$$\mathbf{j}_I = \gamma \mathbf{e}_R, \tag{8.67}$$

for some scalar γ.

If the nonlinear programming problem is nonconvex but feasibility constraints are still applicable, what are the consequences for numerical solution of the inversion problem? For convex feasibility sets, the convex combination of any two points on the feasibility boundary is also feasible and therefore either lies in the interior or on the boundary of the feasible set. This property implies a certain degree of smoothness for the boundary itself. Clearly, if the

[1] Looking at (8.54) we see that the power is a linear functional of the matrix elements of $\mathbf{\Sigma}$. However, this apparent linearity unfortunately does not help the analysis, because a physical constraint on the matrix elements is that $\det \mathbf{\Sigma} \equiv 1$. It is not difficult to show that the convex combination of two matrices with unit determinant does not preserve this property. So the nonlinearity cannot be avoided by the trick of considering convex combinations of the matrix elements.

feasible set is nonconvex, then the convex combination of two points on the boundary may or may not lie in the feasible set; thus, the boundary itself may be jagged. Since the solution of the inversion problem still lies on the boundary (just as it did in the convex case), the lack of smoothness of the boundary may have important computational consequences: the boundary is still expected to be continuous, of course, but sharp local jumps could occur that might make convergence of an iterative method difficult to achieve.

As an iterative scheme progresses, the absolute minimum of the trial power (8.65) decreases towards zero. Thus, the feasibility constraints become *more* important for this problem as the scheme progresses to convergence.