Role of Porosity in Estimates of Composite Elastic Constants

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ABSTRACT

The presence of small amounts of residual void space (on the order of one percent by volume) is sufficient to make estimates of composite elastic constants (obtained while ignoring this porosity) violate the Hashin-Shtrikman bounds in many cases. Experimental values falling below the incorrectly estimated lower bound is the normal direction of violation, but erroneous neglect of porosity in "pure" samples can also lead to experimental values rising above the incorrectly computed upper bound. Examples are presented to illustrate the nature of the error and methods of correcting it for data on composite ceramics. The result is improved agreement between the Hashin-Shtrikman bounds, effective medium theory estimates, and experimental data.

1 INTRODUCTION

Efforts to design new materials through fabrication of composites would become easier, and therefore more routine, if a theory were available to make accurate predictions of the resulting material properties of the composite. Probably the most useful general results to date for these purposes are the Hashin-Shtrikman bounds (Hashin and Shtrikman, 1962; 1963), giving upper and lower bounds (for example) on elastic constants when the constituents' moduli and relative volume fractions are known. Even though the microstructure can make a significant difference in the material properties in some cases, the Hashin-Shtrikman bounds do not depend on the details of the microstructure. Therefore, these bounds have the advantage that they require relatively little information for their use, but the disadvantage that when microstructure is important the bounds will generally lie too far apart to be useful for practical purposes. It is therefore desirable to find some other theory or theories that make use of the microstructure (perhaps in a crude way) to obtain more accurate estimates than the HS bounds in these situations.

Although such a long-term goal for a theory of composites seems very reasonable at first, our efforts in this direction are thwarted immediately by the common observation that real data often actually fall outside the Hashin-Shtrikman bounds (Garg et al., 1973). How can this be? What hope is there for a comprehensive theory if the data do not obey even the simplest and most general of the theoretical results?

It is fairly common to observe violations of the Hashin-Shtrikman bounds when real data are plotted together with the curves that should bound the allowed region of the parameter space.

Typical deviations observed take the form of data falling below the lower HS bound. This happens so often that it seems clear there is some bias — either in the data or in the analysis of the data. It is the purpose of the present paper to point out that the most likely source of bias is the presence of porosity in samples that are treated — for purposes of analysis — as if they have no porosity. However, even small amounts of porosity change the analysis drastically. The Hashin-Shtrikman bounds show that samples containing porosity have well-defined and useful upper bounds, but that the lower bounds on elastic moduli are trivial (zero). Thus, we can immediately understand the generally observed bias if porosity is present, because modulus data can never fall below zero.

It also happens sometimes (though rarely) that some data rise above the upper Hashin-Shtrikman bound. How can this happen? Again the presence of porosity in those samples used to characterize the moduli of the constituents would easily explain this discrepancy. If there is porosity in the nominally "pure" samples, the measured moduli are correspondingly smaller than they should be and corrections should be made to the data (Chung, 1971; 1974) before using them to compute Hashin-Shtrikman bounds. If we observe data rising above the bounds, we should be suspicious that the values of the pure constituents' moduli we are using are actually in error, being biased to lower than realistic values.

We will illustrate these problems by doing a series of calculations on several different data sets. We consider the Hashin-Shtrikman bounds and some common theories that are known to give results consistent with HS bounds. In Section 2, we review three theories of composite elastic constants: the Kuster-Toksöz method, a self-consistent generalization of the Kuster-Toksöz method, and a differential method. For spherical inclusions, the basic Kuster-Toksöz method gives the same estimates for elastic constants as the Hashin-Shtrikman bounds, while the other two methods always (for any inclusion shape) give results lying between these bounds. In Section 3, we show how to translate the bounds and estimates on bulk and shear moduli into bounds and estimates on Young's modulus and Poisson's ratio. In Section 4, we analyze a series of data sets.

Although the discussion of bounds in this paper is limited to the Hashin-Shtrikman bounds, it is known that better (tighter) bounds than the HS bounds, also independent of microgeometry, exist (Berryman and Milton, 1988). However, limited space prevents us from discussing such issues here. In any case, the improvements are generally small and probably not of great interest for most comparisons to real data.

2 APPROXIMATIONS FOR K^* AND G^*

In this section, we present a brief review of three theories of composite elastic constants. A more comprehensive review of the history of this subject than can be given here has been presented previously by Berryman (1992).

The author prefers to derive the common theories of composites using a scattering-theory approach, but all these formulas may be derived instead using a purely static analysis. One reason for preferring the dynamic derivation is that much of the data on composite elastic properties is obtained from ultrasonic measurements; the analysis is then closer conceptually to the methods actually used in a majority of the experiments.

The three single-scattering approximations for effective elastic constants we consider in this paper are: the Kuster-Toksöz approximation (KT), the self-consistent approximation (SC), and the differential method (DM). To provide a simple review of the nature of these approximations and to compare and contrast these methods, we start by quoting results for elastic scattering from isolated spherical inclusions of radius a imbedded in an infinite host. The host medium has bulk modulus K and shear modulus G. The spherical inclusion has bulk modulus K' and shear modulus G'.

If an incident compressional wave has the form

$$\mathbf{u} = \hat{z}(A_0/ik)\exp i(kz - \omega t), \tag{1}$$

where k is the wavenumber, ω is the angular frequency, and A_0 is the amplitude of the dilatation $(\nabla \cdot \mathbf{u})$, then the radial component of the scattered wave field is

$$u_r = (ik)^{-1} \exp i(kr - \omega t)\left[B_0 - B_1 \cos\theta - B_2(3\cos 2\theta + 1)/4\right]. \tag{2}$$

These scattering coefficients B_i for $i = 0, \ldots, 2$ have been computed by Ying and Truell (1956) and by Yamakawa (1962). The first scattering coefficient is

$$B_0 = -\frac{k^3 a^3 A_0}{3}\frac{K' - K}{K' + \frac{4}{3}G}. \tag{3}$$

The third coefficient is

$$B_2 = \frac{k^3 a^3 A_0}{3}\frac{10G}{3(K + 2G)}\frac{G' - G}{G' + F}, \tag{4}$$

where

$$F = (G/6)(9K + 8G)/(K + 2G). \tag{5}$$

The coefficient B_1 depends on material densities. Based on this coefficient, the methods used here always show that the effective density is just the average density (Berryman, 1980).

We will use the convention that effective constants are distinguished by the $*$ superscript (*e.g.*, K^* and G^*).

2.1 Kuster-Toksöz and Hashin-Shtrikman

The Kuster-Toksöz approximation for elastic constants was introduced by Kuster and Toksöz (1974). The KT approximation is justified by the following thought experiment: If the host medium is type-1, then the scattering coefficient for a spherical inclusion of radius a of the composite composed of type-1 and type-2 with volume fractions $v^{(1)}$ and $v^{(2)} = 1 - v^{(1)}$ is porportional to

$$B_0 \propto a^3 \frac{K^* - K^{(1)}}{K^* + \frac{4}{3}G^{(1)}}. \tag{6}$$

If the composite is actually composed of n spherical inclusions of type-2 of radii a_j for $j = 1, \ldots, n$ imbedded within the sphere of radius a, then (including only the single-scattering effects) the scattering coefficient must also satisfy

$$B_0 \propto \sum_{j=1}^{n} a_j^3 \frac{K^{(2)} - K^{(1)}}{K^{(2)} + \frac{4}{3}G^{(1)}}, \tag{7}$$

which is correct to $O(k^3)$. Higher order $O(k^5)$ terms due to the offset of the small scatterers from the origin may be properly neglected in this approximation. Equating (6) and (7) and noting that the volume fraction $v^{(2)}$ is given by

$$v^{(2)} = \frac{\sum_{j=1}^{n} a_j^3}{a^3}, \tag{8}$$

the resulting expression for the Kuster-Toksöz approximation to the bulk modulus K^*_{KT} is

$$\frac{K^*_{KT} - K^{(1)}}{K^*_{KT} + \frac{4}{3}G^{(1)}} = v^{(2)} \frac{K^{(2)} - K^{(1)}}{K^{(2)} + \frac{4}{3}G^{(1)}}. \tag{9}$$

It is not hard to see that (9) can be generalized for multiple inclusions to

$$\frac{K^*_{KT} - K^{(1)}}{K^*_{KT} + \frac{4}{3}G^{(1)}} = \left\langle \frac{K(\mathbf{x}) - K^{(1)}}{K(\mathbf{x}) + \frac{4}{3}G^{(1)}} \right\rangle, \tag{10}$$

since the volume average $\langle\cdot\rangle$ over the spatial coordinate $\mathbf{x}$ produces the same weighting as in (9), while there is no contribution to (10) from regions where $K(\mathbf{x}) = K^{(1)}$. Since the right hand side of (10) is just the average of the scattering coefficient, we see now why this approximation is sometimes called the "average T-matrix approximation." Equation (10) can be rearranged into the convenient form

$$\frac{1}{K^*_{KT} + \frac{4}{3}G^{(1)}} = \left\langle \frac{1}{K(\mathbf{x}) + \frac{4}{3}G^{(1)}} \right\rangle. \tag{11}$$

A similar computation based on B_2 gives the KT approximation for the shear modulus

$$\frac{1}{G^*_{KT} + F^{(1)}} = \left\langle \frac{1}{G(\mathbf{x}) + F^{(1)}} \right\rangle, \tag{12}$$

where $F^{(1)}$ is found from (5) by adding superscripts to all constants. The formulas (11) and (12) are not coupled and therefore give explicit equations for the effective constants.

For spherical scatterers, it should be noted that, when the shear modulus of the host $G^{(1)}$ is the smallest (largest) of all the shear moduli of all constituents, the KT approximation produces the same estimate of the bulk modulus as the Hashin-Shtrikman lower (upper) bound (Hashin and Shtrikman, 1962; 1963). Thus, in our later comparisons, we generally use the label HS for these results, but it should be keep in mind that they are the same as the KT estimates for spheres.

2.2 Self-consistent approximation

Although there are several methods that are called "self-consistent" [see Budiansky (1965), Hill (1965) and Christensen (1979)] — and often for similar reasons, such as the necessity to iterate implicitly coupled equations to find the desired solutions — the particular self-consistent approach we favor here is the same one described by Gubernatis and Krumhansl (1975) and by Berryman (1980). It will become clear that this SC approximation is just a self-consistent generalization of the KT approximation.

Our self-consistent approximation is justified by a slightly different thought experiment: If the host medium is the composite itself (*i.e.*, type-*), then the scattering coefficient for scattering from a spherical inclusion of type-i material is just

$$B_0 \propto a_i^3 \frac{K^{(i)} - K^*}{K^{(i)} + \frac{4}{3}G^*}. \tag{13}$$

The value of the composite bulk modulus may be found by treating K^* as a tunable quantity. We average the single-scattering contributions at infinity and adjust K^* until the sum vanishes:

$$\sum_{i=1}^{m} v^{(i)} \frac{K^{(i)} - K^*_{SC}}{K^{(i)} + \frac{4}{3}G^*_{SC}} = 0, \tag{14}$$

assuming m constituents whose volume fractions satisfy $\sum_{i=1}^{m} v^{(i)} = 1$. This equation can also be rearranged and written in terms of volume averages

$$\frac{1}{K^*_{SC} + \frac{4}{3}G^*_{SC}} = \left\langle \frac{1}{K(\mathbf{x}) + \frac{4}{3}G^*_{SC}} \right\rangle. \tag{15}$$

Notice that this formula also depends on knowledge of the effective value of the shear modulus G^*_{SC}. This value is determined self-consistently using the analogous computation based on B_2 as described by Berryman (1980). The result is

$$\frac{1}{G^*_{SC} + F^*_{SC}} = \left\langle \frac{1}{G(\mathbf{x}) + F^*_{SC}} \right\rangle, \tag{16}$$

where F is again defined in (5). Note that (15) and (16) are coupled and therefore provide only implicit formulas for the effective constants. Normally these equations are solved by iteration.

The SC approximation produces a single formula in which all components are treated equally. Such a symmetric formula is appropriate when there is no one constituent acting as host to all the others, as in complex aggregates such as rocks. Other approximations of this type may be obtained by using the single-scattering coefficients for nonspherical (often ellipsoidal) scatterers. Another similar, but nevertheless distinct, self-consistent approach introduced by Hill (1965) and Budiansky (1965) for K^* and G^* produces formulas identical to (15) and (16) when the inclusions are spherical; however, for other shapes of inclusions, their approach is known to produce different results. Milton (1985) has shown that the SC approximation presented here is realizable (*i.e.*, in principle a model can be found that has the same value of the effective constants as that computed from the SC approximation), whereas the self-consistent method of Hill (1965) and Budiansky (1965) is not realizable for all inclusion shapes. Thus, in many circumstances involving nonspherical inclusions, it may be the present SC approximation that is the preferred method of generating estimates.

2.3 Differential method

A thorough review of the differential method to be introduced here was presented by Cleary et al. (1980).

The differential method is constructed using another *gedanken* experiment: If there are only two constituents whose volume fractions are $x = v^{(1)}$ and $y = v^{(2)} = 1 - x$, then suppose we know the value of the effective bulk modulus $K^*_{DM}(y)$ at one value of y. Treating $K^*_{DM}(y)$ as the host medium and $K^*_{DM}(y+dy)$ as the effective constant after a small proportion $dy/(1-y)$ of the host has been replaced by spherical inclusions of type-2, we find

$$\frac{K^*_{DM}(y+dy) - K^*_{DM}(y)}{K^*_{DM}(y+dy) + \frac{4}{3}G^*_{DM}(y)} = \frac{dy}{(1-y)} \frac{K^{(2)} - K^*_{DM}(y)}{K^{(2)} + \frac{4}{3}G^*_{DM}(y)}. \tag{17}$$

Since the host contains the volume fraction x of type-1 and y of type-2, on average a fraction $dy/(1-y)$ of the host must be replaced by type-2 in order to change the overall fraction of type-2 to $y + dy$. Taking the limit $dy \to 0$ gives the result

$$(1-y)\frac{d}{dy}\left[K^*_{DM}(y)\right] = \frac{K^{(2)} - K^*_{DM}(y)}{K^{(2)} + \frac{4}{3}G^*_{DM}(y)} \left[K^*_{DM}(y) + \frac{4}{3}G^*_{DM}(y)\right], \tag{18}$$

where the initial host is pure type-1 so $K^*_{DM}(0) = K^{(1)}$. The corresponding formula for the shear modulus is

$$(1-y)\frac{d}{dy}\left[G^*_{DM}(y)\right] = \frac{G^{(2)} - G^*_{DM}(y)}{G^{(2)} + F^*_{DM}(y)} \left[G^*_{DM}(y) + F^*_{DM}(y)\right], \tag{19}$$

with F defined in (5). Note that (18) and (19) are coupled and must therefore be integrated simultaneously.

Norris (1985) and Avellaneda (1987), among others, have shown that the DM is realizable just as Milton (1985) showed that the SC approximation is realizable.

Like the KT approximation and the HS bounds, the DM formulas are not symmetrical in the constituents. Equations (18) and (19) have been derived assuming the host is of type-1 and the inclusion of type-2. A second distinct DM estimate is found by interchanging the roles of type-1 and type-2. The imbedding process guarantees that the host material remains connected in the composite. This feature makes the DM a very appealing choice in situations where smaller volume fractions of inclusions are dispersed in a host medium. At larger volume fractions where the inclusions begin to form a percolating cluster of contiguous particles, the DM may not be as appropriate as the SC approximation.

If the composite of interest contains more than two constituents, it is more difficult to include the additional components in the DM than in KT or SC approximations. The essential difference is that the result for the DM estimate depends on the path taken, *i.e.*, the order in which the constituents are added to the composite. Norris (1985) and Berge *et al.* (1992) have presented methods of generalizing (18) to multiple constituents.

2.4 Hill's exact formula as a validity check

Hill (1963) has derived an exact formula for the effective bulk modulus of a composite when the shear modulus of all constituents is the same $G^{(1)} = \ldots = G^{(m)} = G^*$. His result is

$$\frac{1}{K^* + \frac{4}{3}G^*} = \left\langle \frac{1}{K(\mathbf{x}) + \frac{4}{3}G^*} \right\rangle. \tag{20}$$

By inspection we see that the KT approximation (11)–(12) and the SC approximation (15)–(16) each reduce correctly to (20) when the Gs are all equal.

When the shear moduli are constant, (19) shows that $G^*(y) = G^*$ is constant for the DM estimate. Thus, (18) may be rewritten in this limit as

$$(1-y)\frac{d}{dy}\left[\frac{1}{K^*_{DM}(y)+\frac{4}{3}G^*}\right] = \frac{1}{K^{(2)}+\frac{4}{3}G^*} - \frac{1}{K^*_{DM}(y)+\frac{4}{3}G^*}. \tag{21}$$

To check the agreement of the DM (21) with Hill's result, take the y derivative of (20) in the case of only two constituents. Then, we find

$$\frac{d}{dy}\left[\frac{1}{K^*(y)+\frac{4}{3}G^*}\right] = \frac{1}{K^{(2)}+\frac{4}{3}G^*} - \frac{1}{K^{(1)}+\frac{4}{3}G^*}, \tag{22}$$

whereas substituting (20) into the right hand side of (21) (*i.e.*, $K^* \to K^*_{DM}$) produces exactly the same expression as (22). Norris (1985) has also established the agreement of DM with Hill's result. So all three approximations become exact if the shear modulus of the composite is constant throughout.

This discussion highlights the major difference between the DM and SC approximations. The DM takes the fluctuations in the shear modulus into account approximately when calculating the estimate K^*, while the KT and SC approximations essentially ignore effects due to such fluctuations.

The results presented so far are based on scattering coefficients for spherical inclusions, thus limiting their range of applicability. Scattering coefficients for various nonspherical inclusions are known, and have been reviewed by Berryman (1980). It is straightforward to generalize all the formulas presented here for problems with nonspherical inclusions.

TABLE 1. Conversion formulas for the various elastic constants.

Bulk modulus K	*Shear modulus* G	*Young's modulus* E	*Poisson's ratio* ν
$K = \dfrac{E}{3(1-2\nu)}$	$G = \dfrac{E}{2(1+\nu)}$	$\dfrac{1}{E} = \dfrac{1}{9K} + \dfrac{1}{3G}$	$\nu = \dfrac{3K-2G}{2(3K+G)}$

3 (K, G) VERSUS (E, ν)

The discussion so far has centered on the elastic constants K and G — the bulk and shear moduli. Experimental data are often presented instead in terms of E and ν — Young's modulus and Poisson's ratio, respectively. The relations among these constants are well-known, but we summarize them for reference in Table 1.

Consider the (K, G)-plane (see Figure 1). The Hashin-Shtrikman bounds define a rectangle in this plane with the corners given by the points (K^-, G^-), (K^+, G^-), (K^+, G^+), and (K^-, G^+). Each of these corners corresponds to a distinct limiting value of either E or ν. For example, since Young's modulus is determined from the expression

$$\frac{1}{E} = \frac{1}{9K} + \frac{1}{3G}, \tag{23}$$

we see that E is a monotonically increasing function of both K and G. Thus, the diagonal corners (K^-, G^-) and (K^+, G^+) of the Hashin-Shtrikman rectangle determine the lower and upper HS bounds E_{HS}^- and E_{HS}^+ on Young's modulus. Similarly, since

$$G = \frac{3(1-2\nu)}{2(1+\nu)} K, \tag{24}$$

and since the coefficient of K in (24) is a monotonically decreasing function of ν for all physical values in the range $(-1, 1/2)$, we find easily that minimum and maximum values of Poisson's ratio ν_{HS}^- and ν_{HS}^+ occur respectively at the corners (K^-, G^+) and (K^+, G^-). [Also, see Hashin (1962) and Zimmerman (1992).] Although these bounds are rigorous, better bounds in the (E, ν)-plane are given by the outlines of the quadrilateral region shown in Figure 1; the full range of possible pairs (E, ν), as determined by the Hashin-Shtrikman bounds, lies inside this quadrilateral. The four dotted lines shown in Figure 1 are

$$\nu = \frac{1}{2} - \frac{E}{6K^+}, \quad \nu = \frac{1}{2} - \frac{E}{6K^-}, \tag{25}$$

$$\nu = -1 + \frac{E}{2G^+}, \quad \text{and} \quad \nu = -1 + \frac{E}{2G^-}, \tag{26}$$

which follow easily from the relations in Table 1.

If the composite contains porosity, then the lower Hashin-Shtrikman bounds on the bulk and shear moduli become trivial (zero), so the HS rectangle is bounded by the K and G axes as shown in Figure 2. Similarly, when transformed into the (E, ν)-plane, we find that the only nontrivial universal bound remaining is E_{HS}^+, since the overall bounds on Poisson's ratio are the same as the physical limits $\nu_{HS}^- = -1$ and $\nu_{HS}^+ = 1/2$. The full range of possible pairs (E, ν) is now determined by the triangular region shown in Figure 2. For our tables, the value $\nu(K_{HS}^+, G_{HS}^+)$ is listed, not because it is a bound (it is not), but because it is the value of ν corresponding to the point of highest possible $E = E_{HS}^+$.

Finally, note that data are also sometimes presented in terms of (E, G) pairs. Then, it follows from the preceding results that the HS rectangle in the (K, G)-plane transforms to another rectangle in the (E, G)-plane.

4 EXAMPLES

4.1 Porous glass

Walsh *et al.* (1965) made measurements on the compressiblity $(1/K)$ of a porous glass over a wide range of porosities. The samples they used were fabricated from P-311 glass powder

(54.4% SiO_2, 14.4% B_2O_3, 14.1% CaO, 10% Al_2O_3, 6.5% Na_2O, and 0.7% K_2O). The powder and binder were die-pressed and then sintered in alumina crucibles. Depending on the thermal history of the samples, they obtained porous glass with porosities ranging from 0.70 to near zero. Porosity measurements were stated to be accurate ± 0.01. Linear compressibility of the samples was measured using electrical resistance strain gages attached to their surfaces while immersed in a high-pressure medium. Samples were tested for reversibility (having the same behavior for either increasing or decreasing pressure), which was satisfied in all but the most porous samples. They showed that their results agreed well with the theoretical predictions of Mackensie (1950). It turns out that Mackensie's result for the bulk modulus of a porous solid is also identical to the upper Hashin-Shtrikman bound for this problem. Thus, Walsh *et al.* (1965) actually showed that their porous glass satisfies the HS bounds for a wide range of porosities and, furthermore, that the values they found essentially track the upper bound.

Results of the theoretical calculations are shown in Table 2 and Figure 3. To be consistent with the microgeometry of these porous glasses for the SC approximation, we have treated the glass as if it were shaped like needles randomly dispersed in the void at the highest porosities; the voids are treated as spherical inclusions in the glass at the lowest porosities. (If instead we had chosen to treat the glass as spheres, the SC approximation would have vanished at porosities of 50 % and greater. However, this choice is a very unrealistic representation of the true microstructure of the glass at high porosities.) Since the DM approximation treats the glass as host medium at all values of porosity, we assume only that the voids are spherical for this estimate. We see that both theories (SC and DM) do well at predicting the measured values out to about 25% porosity. For higher porosities, both theories overestimate the influence of the voids while the Hashin-Shtrikman bound (equivalent to the KT theory for this problem) does somewhat better at estimating the measured values over the whole range of porosities. Zimmerman (1991) gives an analysis of this data similar to the one presented here. Ferrari and Filipponi (1991) show that an application of Mori and Tanaka theory for composite materials (Benveniste, 1987) to this same data set gives somewhat better agreement with the data than any of the theories considered here. Berge *et al.* (1992) also consider some variations of the theory that produce improved agreement.

Table 2. Values of isothermal bulk moduli of porous P-311 glass measured at room temperature compared to theoretical estimates. Bulk and shear moduli of the pure glass were measured to be $K = 46.3$ and $G = 30.5$ GPa, respectively. All data from Walsh, Brace, and England (1965).

Porosity ϕ	*Experimental* K (GPa)	*Hashin-Shtrikman* K_{HS}^{+} (GPa)	*Spherical Voids* K_{DM} (GPa)	*Needles-Spheres* K_{SC} (GPa)
0.00	46.1	46.3	46.3	46.3
0.00	45.9	46.3	46.3	46.3
0.05	41.3	41.6	41.5	41.4
0.11	36.2	36.6	36.1	35.6
0.13	37.0	35.1	34.4	33.7
0.25	23.8	27.0	25.2	22.8
0.33	21.0	22.5	19.9	16.4
0.36	18.6	21.0	18.1	14.2
0.39	17.9	19.6	16.4	12.3
0.44	15.2	17.3	13.7	9.4
0.46	13.5	15.5	12.7	8.5
0.50	12.0	14.8	10.9	6.7
0.70	6.7	7.7	3.8	2.1

4.2 Reinforced glass

Hasselman and Fulrath (1965a;1965b) have performed a series of experiments measuring Young's modulus for a sodium borosilicate glass reinforced with different solid inclusions.

First, they tested D glass (16% Na_2O, 14% B_2O_3, and 70% SiO_2) containing dispersions of angular alumina (Al_2O_3) particles. The composites were prepared by vacuum hot-pressing mixtures of the two components in powder form into graphite dies at 725° C for 10 minutes. An image taken of a sample cross-section shows the dispersed alumina particles are quite jagged and nonspherical in shape. Young's modulus was measured using a flexural resonance technique. Using the experimental values of Young's modulus and Poisson's ratio for the host glass and alumina inclusions, we obtain the results shown in Table 3 and Figure 4. We find that the experimental results easily obey the Hashin-Shtrikman bounds, except at the highest inclusion volume fraction (55%) where it was observed that the consolidated sample had a porosity of about 9.6%. The experimental results track the lower HS bound approximately, while they are very well estimated by the DM for spheres but not so well by the SC approximation for spheres. It is encouraging that the agreement is so good for the DM estimates even though the particles are not very spherical in shape, suggesting that modeling efforts may proceed (at least for this problem) with only partial information about the microgeometry. In a separate calculation accounting for the porosity of the one sample, we obtain the results presented in Table 4. Now we find this datum does satisfy the appropriate Hashin-Shtrikman bounds. Also, the two SC approximations considered show that the measured value corresponds on average to porosity with shape intermediate between spherical voids and penny-shaped cracks.

Second, Hasselman and Fulrath (1965b) tested the same D glass with dispersions of tungsten. The techniques used for sample preparation and for measuring E were the same as before. An image of one sample cross-section showed this time that the tungsten particles were all nearly spherical in shape. Using measured values of Young's modulus and Poisson's ratio for the constituents, we obtain the results in Table 5 and Figure 5. The experimental results again track the lower Hashin-Shtrikman bound, but this time we must consider the experimental errors in order to obtain complete agreement. So, in this example as well as the preceding one, the presence of porosity is *not* required to explain the results (except for the one example with high volume fraction of inclusions considered in Table 4). The self-consistent and differential methods both provide reasonable estimates of the measured values, but the DM again produces the best predictions overall.

Our main conclusion from these two examples — based on the data of Hasselman and Fulrath (1965a; 1965b) — is that the DM gives estimates in good agreement with measured data whenever there is no porosity present and whenever it is appropriate to treat one phase as host and the other as inclusion. Also, the shapes of the included particles seem to be of secondary importance, so that a theory based on spherical inclusions may be adequate for a broad range of composites.

TABLE 3. Values of Young's modulus E and Poisson's ratio ν for a sodium borosilicate glass containing dispersions of angular alumina particles measured at room temperature compared to theoretical estimates. Data from Hasselman and Fulrath (1965a). Estimates are computed assuming no porosity.

Volume fraction *of* Al_2O_3	*Experimental* E (GPa)	*Hashin-Shtrikman* E^+_{HS}/E^-_{HS} (GPa)	*Spheres* E_{SC} (GPa)	*Spheres* E_{DM} (GPa)
0.00	80.5	80.5	80.5	80.5
0.10	92.4	101.1/92.0	92.7	92.2
0.20	107.1	123.6/105.5	108.0	106.7
0.30	122.8	147.9/121.3	127.4	123.9
0.40	144.7	174.7/139.8	151.8	145.0
0.45	158.5	189.1/150.3	166.1	157.2
0.50	167.8	204.1/162.0	182.0	170.5
0.55	137.2	220.0/175.0	199.1	185.3
1.00	411.0	411.0	411.0	411.0
	ν	ν^+_{HS}/ν^-_{HS}	ν_{SC}	ν_{DM}
0.00	0.194	0.194	0.194	0.194
0.10	—	0.225/0.176	0.198	0.198
0.20	—	0.245/0.164	0.203	0.202
0.30	—	0.259/0.158	0.208	0.207
0.40	—	0.268/0.157	0.214	0.212
0.45	—	0.272/0.158	0.218	0.215
0.50	—	0.274/0.161	0.221	0.218
0.55	—	0.276/0.164	0.225	0.221
1.00	0.257	0.257	0.257	0.257

TABLE 4. Values of Young's modulus E and Poisson's ratio ν for a sodium borosilicate glass containing dispersions of angular alumina particles (and some porosity) measured at room temperature compared to theoretical estimates. Data from Hasselman and Fulrath (1965a).

Solids fraction *of* Al_2O_3	*Porosity* ϕ	*Experimental* E (GPa)	*Hashin-Shtrikman* E^+_{HS} (GPa)	*Spheres* E_{SC} (GPa)	*Sphere-Sphere-Crack* E_{SC} (GPa)
0.55	0.096	137.2	189.1	154.9	114.5
		ν	$\nu(K^+_{HS}, G^+_{HS})$	ν_{SC}	ν_{SC}
0.55	0.096	—	0.230	0.220	0.164

TABLE 5. Values of Young's modulus E and Poisson's ratio ν for a sodium borosilicate glass containing dispersions of spherical tungsten particles measured at room temperature compared to theoretical estimates. Data from Hasselman and Fulrath (1965b). Estimates are computed assuming no porosity.

Volume fraction of Tungsten	*Experimental* E (GPa)	*Hashin-Shtrikman* E^+_{HS}/E^-_{HS} (GPa)	*Spheres* E_{SC} (GPa)	*Spheres* E_{DM} (GPa)
0.00	80.5	80.5	80.5	80.5
0.10	90.9 ± 1.4	98.0/91.3	91.7	91.5
0.20	105.5 ± 1.1	116.9/103.7	105.6	104.7
0.30	118.0 ± 0.2	137.6/118.1	122.7	120.3
0.40	137.5 ± 0.3	160.1/134.7	143.8	138.7
0.50	159.9 ± 0.5	184.8/154.5	168.9	160.8
1.00	355.0	355.0	355.0	355.0
	ν	ν^+_{HS}/ν^-_{HS}	ν_{SC}	ν_{DM}
0.00	0.197	0.197	0.197	0.197
0.10	—	0.214/0.180	0.197	0.197
0.20	—	0.226/0.168	0.198	0.198
0.30	—	0.233/0.159	0.198	0.198
0.40	—	0.238/0.154	0.198	0.198
0.50	—	0.239/0.153	0.198	0.198
1.00	0.198	0.198	0.198	0.198

4.3 Porous silicon nitride

Fate (1975) has performed a series of experiments measuring elastic constants of polycrystalline silicon nitride (Si_3N_4).

Instead of fabricating samples, Fate obtained polycrystalline samples from several commercial suppliers such as Alcan, Ceradyne, Ford, Haselden, and Norton. Sample fabrication was accomplished by hot pressing for the higher densities and by slip casting for the lower densities. Fate used pulse-echo measurements to find the ultrasonic compressional- and shear-wave speeds in the samples at frequencies around 10 MHz. The elastic constant data are believed to be accurate to ±3%, but errors may be somewhat larger for the lowest density samples. The data are shown in Table 6 and Figure 6. Fate showed approximate agreement with Budiansky's theory (Budiansky, 1965) in the original paper, but, for this problem, Budiansky's theory is just the same as our SC approximation for spherical inclusions. For comparison, the SC estimates for spherical particles and needle-shaped pores are also listed. The overall agreement with the data is improved somewhat using this estimate. The DM estimate for spherical inclusions was also computed (but is not shown here) and again found to overestimate the importance of the voids in the overall properties of the composite for porosities greater than 15 %.

There is one anomaly in this data set at $\phi = 0.025$. The measured value of E is larger than the value for the sample at $\phi = 0.0$, suggesting as we discussed earlier either that the true value of Young's modulus for the nonporous sample has been underestimated, or that the actual value of the porosity for that sample was underestimated. The data of Fisher et al. (1991) and Fisher et al. (1992) show that the moduli of their pure samples of Si_3N_4 were $E = 316.5$ and $G = 126.1$ GPa, which are higher than the values Fate measured by about 8 %. The samples of Fisher et al. were known to contain about 3 % MgO, which has $E = 307.2$ and $G = 129.3$ GPa. Whether the observed differences in properties of the "pure" Si_3N_4 are due to the presence of impurities or to some bias in the measurements, we can only speculate.

Table 6. Values of adiabatic elastic moduli of porous polycrystalline silicon nitride measured at room temperature compared to theoretical estimates. Young's modulus and the shear modulus of the pure Si_3N_4 were measured to be $E = 289.0$ and $G = 118.2$ GPa, respectively. All data from Fate [1975].

Porosity ϕ	*Experimental* E (GPa)	*Hashin-Shtrikman* E^+_{HS} (GPa)	*Sphere-Sphere* E_{SC} (GPa)	*Sphere-Needle* E_{SC} (GPa)
0.000	289.0	289.0	289.0	289.0
0.025	292.1	274.9	274.4	272.2
0.028	259.3	273.1	272.7	270.2
0.041	244.4	266.2	265.3	261.5
0.151	172.3	213.1	201.6	189.3
0.214	142.9	187.0	165.1	149.3
0.226	131.8	182.4	158.2	141.7
0.255	128.2	171.3	141.5	123.6
ϕ	G (GPa)	G^+_{HS} (GPa)	G_{SC} (GPa)	G_{SC} (GPa)
0.000	118.2	118.2	118.2	118.2
0.025	117.6	112.5	112.3	111.4
0.028	111.0	111.8	111.6	110.6
0.041	100.7	109.0	108.6	107.1
0.151	72.4	87.4	82.8	77.8
0.214	61.1	76.8	67.9	61.5
0.226	53.4	74.9	65.1	58.4
0.255	54.8	70.4	58.3	51.0

4.4 Ceramic composites with SiC-whisker reinforcement

Fisher et al. (1991) and Fisher et al. (1992) have performed a series of experiments on elastic properties of Al_2O_3/SiC and Si_3N_4/SiC whisker composites.

The Al_2O_3/SiC samples were formed from a powder of Al_2O_3 mixed with randomly dispersed SiC whiskers having approximate dimensions 1 μm diameter and 30 μm length. These mixtures were hot pressed into samples $30 \times 25 \times 5$ mm thick. The Si_3N_4/SiC samples were similarly fabricated, but also contained 3% MgO, which is known to have bulk modulus $K = 164.0$ and shear modulus $G = 129.3$ GPa. Subsequent SEM studies showed the whiskers were preferentially aligned normal to the direction of the press. Thus, these samples were essentially transversely isotropic; however, the measurements show that the magnitude of the anisotropy was never more than about 2% in the Al_2O_3 or 4% in the Si_3N_4, and generally much less. We will ignore the anisotropy in this study.

Moduli were determined by measuring ultrasonic compressional and shear velocities of propagation through the samples for frequencies in the range 10-50 MHz. Errors were estimated to be $\pm 0.02\%$ in the velocity measurements. This translates into an error of about $\pm 0.1\%$ in the measured bulk moduli and about 0.02% in the shear moduli, neglecting errors in the density measurements.

The experimental results are compared to the Hashin-Shtrikman bounds in Table 7. Because the contrast in the elastic moduli of the host and inclusions is so small, the bounds are very close together and the data do not lie between the bounds. In all cases, the data fall below the lower HS bounds on both moduli. In the original papers, the data were compared instead with the bounds of Paul (1960) which are not nearly as tight as the HS bounds, and even then the agreement was only marginal. The discrepancies are not always large in magnitude and therefore might be interpreted as due to unresolved experimental error, but nevertheless the discrepancies are also not random — a definite bias is present. All the data for mixtures fall below the HS bounds — suggesting that porosity is present in the hot-pressed samples and that this is the most likely source of the discrepancy. Repeating these calculations using the density information published by Fisher et al. (1992), we performed three-component HS and SC calculations. The results are shown in Table 8. Now, in all cases but one, the data satisfy the bounds without taking experimental errors (± 0.02 GPa) into account. Also, note that the HS bounds are no longer monotonic functions of the solids fraction of SiC, in agreement with the data. We use needles to approximate the geometry of the SiC whiskers and penny-shaped cracks for the assumed void shape in the SC calculations. The agreement is reasonable, with the SC approximation catching the proper trends if not the actual values of the elastic constants. Remaining discrepancies may be attributed to neglected anisotropy of the samples.

Table 7. Comparison of adiabatic bulk and shear moduli of Al_2O_3/SiC and Si_3N_4/SiC measured at 25° C to theoretical values. Bulk and shear moduli of the pure SiC were measured to be $K = 244$ and $G = 189$ GPa, respectively. All data from Fisher, Manghnani, and Routbort (1991) or Fisher, Manghnani, and Wang (1992). Values computed assuming no porosity is present.

Volume Fraction of SiC	*Experimental* K (GPa)	*Hashin-Shtrikman* K^+_{HS}/K^-_{HS} (GPa)	*Experimental* G (GPa)	*Hashin-Shtrikman* G^+_{HS}/G^-_{HS} (GPa)
in Al_2O_3				
0.000	254.6	254.6	161.4	161.4
0.061	249.8	253.9/253.9	161.1	163.0/163.0
0.180	250.0	252.7/252.7	165.5	166.1/166.0
0.293	243.3	251.4/251.4	167.6	169.1/169.0
in Si_3N_4				
0.000	215.4	215.4	126.1	126.1
0.049	214.9	216.7/216.7	122.5	128.7/128.6
0.098	211.9	218.1/218.1	128.8	131.4/131.2
0.197	212.6	220.8/220.7	133.6	136.8/136.5

Table 8. Values of adiabatic bulk and shear moduli of of the composites Al_2O_3/SiC and Si_3N_4/SiC measured at 25° C compared to theoretical estimates. Bulk and shear moduli of the pure SiC were measured to be $K = 244$ and $G = 189$ GPa, respectively. All data from Fisher, Manghnani, and Routbort (1991) or Fisher, Manghnani, and Wang (1992).

Solids Fraction	*Porosity*	*Experiment*	*Hashin-Shtrikman*	*Sphere-Needle-Crack*
of SiC *in* Al_2O_3	ϕ	K (GPa)	K^+_{HS} (GPa)	K_{SC} (GPa)
0.000	.0000	254.6	254.6	254.6
0.061	.0008	249.8	253.5	252.4
0.180	.0000	250.0	252.7	252.7
0.293	.0053	243.3	248.8	241.9
	ϕ	G (GPa)	G^+_{HS} (GPa)	G_{SC} (GPa)
0.000	.0000	161.4	161.4	161.4
0.061	.0008	161.1	162.7	162.5
0.180	.0000	165.5	166.1	166.1
0.293	.0053	167.6	167.4	165.7
of SiC *in* Si_3N_4	ϕ	K (GPa)	K^+_{HS} (GPa)	K_{SC} (GPa)
0.000	.0000	215.4	215.4	215.4
0.049	.0025	214.9	215.7	212.4
0.098	.0089	211.9	214.5	203.4
0.197	.0098	212.6	216.7	204.8
	ϕ	G (GPa)	G^+_{HS} (GPa)	G_{SC} (GPa)
0.000	.0000	126.1	126.1	126.1
0.049	.0025	122.5	128.1	127.5
0.098	.0089	128.8	129.3	126.9
0.197	.0098	133.6	134.5	131.7

4.5 Alloys of cobalt reinforced with tungsten carbide

Nishimatsu and Gurland (1960) have performed deformation studies of binary alloys containing tungsten carbide and cobalt. These alloys represent aggregates of brittle and comparatively rigid particles (WC) imbedded in a plastic matrix (Co), and could therefore serve as a useful test of various theories of composite behavior. Strain was measured as a function of tensile stress. Unfortunately, only measurements of Young's modulus E were made for these mixtures, whereas measurements of both E and Poisson's ratio ν are needed to test the theories properly. Nevertheless, this situation is a fairly common one, so we will do what we can with the available data.

These alloys were prepared from powders of WC and Co. Nishimatsu and Gurland (1960) describe the sample preparation by saying "The powder mixtures were ball milled in acetone for 72 hours, compacted to shape and sintered in hydrogen ...." The densities of the resulting alloys were measured and found to be somewhat lower than the theoretical density expected based on the volume fractions of the two constituents and their known densities ($\rho_{WC} = 15.7$ g/cc and $\rho_{Co} = 8.9$ g/cc). We will interpret the differences in measured and theoretical density as an indication of the presence of small amounts of voids and/or cracks. Thus, when Nishimatsu and Gurland (1960) find that the density of the 50-50 alloy is 99.4% of the theoretical maximum density, we infer that the sample volume contains 0.6% porosity. In fact, the "pure" cobalt sample they fabricated was itself found to have only 97% of the theoretical density, suggesting that the powder sintering process left substantial void space in this instance. Their paper does not say whether the values quoted for E and ν for the 100% cobalt sample have been corrected for this porosity, but we will assume that such a correction was made. If not, then all of our calculated results for estimates of E must be corrected upwards approximately $3\% \times x$, where x is the volume fraction of Co in the particular alloy. Since all the results in a given row of the tables would require approximately the same correction, our overall conclusions are not expected to be changed.

Hashin-Shtrikman bounds were computed twice for these alloys: First, we computed upper and lower bounds on E and ν assuming that the porosity can be neglected. Table 9 and Figure 7(a) contain the results of this computation, while Figure 8 illustrates the Hashin-Shtrikman rectangle in the (K, G)-plane together with its transformation to a quadrilateral in the (E, ν)-plane for the 50-50 sample. Some of the measured values (3 out of 7) fall below the HS lower bound. Of the three measurements that appear to violate Hashin-Shtrikman, two are quite small violations (1.5% or less) and therefore might be attributed to experimental error; however, the third example (at $x = 0.22$) violates the bounds by about 3%, and thus appears to require some other explanation. For comparison, we also show the results of the differential method (DM) assuming the cobalt is the host medium and tungsten carbide is introduced in the form of spherical inclusions. The DM estimates track the experimental measurements within about 2 or 3 % except at $x = 0.22$ where the deviation is closer to 6 %. Second, we computed the HS upper bounds on E assuming that the small measured porosities are present. The results of these three-component calculations are contained in Table 10 and Figure 7(b). Now we find no violations of the bounds. Shown for comparison are the estimates obtained using the three-component self-consistent approximation.

TABLE 9. Values of isothermal Young's modulus E and Poisson's ratio ν for the $(WC)_{1-x}Co_x$ alloy system measured at room temperature compared to theoretical estimates. Subscripts in chemical formulas refer to volume fractions. Data from Nishimatsu and Gurland (1960). Estimates are computed here assuming no porosity.

Alloy	*Experimental* E (GPa)	*Hashin-Shtrikman* E^+_{HS}/E^-_{HS} (GPa)	*Spheres* E_{DM} (GPa)
Co	207.0	207.0	207.0
$(WC)_{0.10}Co_{0.90}$	228.0	241.0/231.2	231.7
$(WC)_{0.35}Co_{0.65}$	317.0	336.5/305.9	310.6
"	331.0	336.5/305.9	310.6
$(WC)_{0.50}Co_{0.50}$	372.0	403.9/363.8	372.8
$(WC)_{0.63}Co_{0.37}$	424.0	468.9/425.4	438.5
$(WC)_{0.78}Co_{0.22}$	500.0	553.9/514.9	530.5
$(WC)_{0.90}Co_{0.10}$	607.0	631.1/606.6	618.5
WC	703.0	703.0	703.0
	ν	ν^+_{HS}/ν^-_{HS}	ν_{DM}
Co	0.30	0.30	0.30
$(WC)_{0.10}Co_{0.90}$	—	0.297/0.285	0.293
$(WC)_{0.35}Co_{0.65}$	—	0.285/0.255	0.274
"	—	0.285/0.255	0.274
$(WC)_{0.50}Co_{0.50}$	—	0.276/0.241	0.262
$(WC)_{0.63}Co_{0.37}$	—	0.266/0.232	0.251
$(WC)_{0.78}Co_{0.22}$	—	0.251/0.225	0.239
$(WC)_{0.90}Co_{0.10}$	—	0.236/0.221	0.229
WC	0.22	0.22	0.22

TABLE 10. Values of isothermal Young's modulus E and Poisson's ratio ν for the $(WC)_{1-x}Co_x$ alloy system measured at room temperature compared to theoretical estimates. Subscripts in chemical formulas refer to solids fractions. Data from Nishimatsu and Gurland (1960). Porosity included as penny-shaped cracks in SC calculation.

Alloy	*Porosity* ϕ	*Experimental* E (GPa)	*Hashin-Shtrikman* E_{HS}^{+} (GPa)	*Sphere-Sphere-Crack* E_{SC} (GPa)
Co	—	207.0	207.0	207.0
$(WC)_{0.10}Co_{0.90}$	0.014	228.0	236.3	217.4
$(WC)_{0.35}Co_{0.65}$	0.004	317.0	334.8	309.8
"	0.004	331.0	334.8	309.8
$(WC)_{0.50}Co_{0.50}$	0.006	372.0	400.1	372.6
$(WC)_{0.63}Co_{0.37}$	0.002	424.0	467.4	449.1
$(WC)_{0.78}Co_{0.22}$	0.006	500.0	547.9	531.9
$(WC)_{0.90}Co_{0.10}$	0.008	607.0	621.7	606.6
WC	—	703.0	703.0	703.0
	ϕ	ν	$\nu(K_{HS}^{+}, G_{HS}^{+})$	ν_{SC}
Co	—	0.30	0.30	0.30
$(WC)_{0.10}Co_{0.90}$	0.014	—	0.288	0.279
$(WC)_{0.35}Co_{0.65}$	0.004	—	0.265	0.268
"	0.004	—	0.265	0.268
$(WC)_{0.50}Co_{0.50}$	0.006	—	0.253	0.254
$(WC)_{0.63}Co_{0.37}$	0.002	—	0.244	0.246
$(WC)_{0.78}Co_{0.22}$	0.006	—	0.234	0.231
$(WC)_{0.90}Co_{0.10}$	0.008	—	0.226	0.221
WC	—	0.22	0.22	0.22

5 CONCLUSIONS

We have found that different theories are needed to obtain good agreement with experiment, depending on the details of the composite microstructure. This is as it should be. Especially when porosity is present and the Hashin-Shtrikman bounds are far apart (the lower bounds being zero), the microstructure should play a crucial role in determining the composite elastic constants (Berge *et al.*, 1992; Berge *et al.*, 1993; Berryman and Berge, 1993). The derivation of each theoretical estimate tends to focus on a particular type of microgeometry. Our results have shown that when the composite microstructure corresponds approximately to that of the theory, the results are generally in good agreement with the data — as long as we account for the porosity in the calculations.

The differential method (DM) gives excellent predictions of the elastic constants — in good agreement with data — whenever the porosity is negligible. When porosity is not negligible, it is more difficult to use the DM because it is inherently a two-component approach. To allow for two solid constituents and also include porosity requires a three-component method. Either the Kuster-Toksöz (equivalent to Hashin-Shtrikman for spherical inclusions) method or the self-consistent approximation is easily generalized to multiple components. The three-component methods have been found to give improved agreement with experimental results, especially compared to predictions made while neglecting porosity altogether. Agreement could have been improved further by treating the average inclusion shape as a parameter; all three methods considered here can easily be generalized to account for arbitrary ellipsoidal inclusions (Berryman, 1980).

One common criticism of the SC and DM estimates is that they are somewhat more difficult to compute than the KT or HS estimates. With modern computing capabilities, this criticism does not seem serious in most cases. However, if it is desired to try to invert the elastic data to analyze the microstructure of a composite (as is sometimes attempted for data on rocks), it is clearly advantageous to have an analytical formula, rather than a computational one. Although the KT and HS estimates can be used for such inversion purposes, their agreement with the data on porous samples is seldom good enough to justify this approach. In future work, it is therefore worthwhile to consider the possibility of finding new analytical formulas for use in inversion studies.

Acknowledgments

I thank Patricia A. Berge, Dae-Hyun Chung, Sabodh K. Garg, Robert V. Kohn, and Graeme W. Milton for helpful conversations. This work was performed under the auspices of the U. S. Department of Energy by the Lawrence Livermore National Laboratory under contract No. W-7405-ENG-48 and supported specifically by the Geosciences Research Program of the DOE Office of Energy Research within the Office of Basic Energy Sciences, Division of Engineering and Geosciences.

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Figure 1. Schematic illustration of the transformation of a Hashin-Shtrikman rectangle for (K,G)-bounds to a quadrilateral for (E,ν)-bounds. Corners of the (K,G) rectangle correspond to the bounding values in (E,ν).

Figure 2. Schematic as in Figure 1, but for porous materials where $K^-_{HS} = 0 = G^-_{HS}$. Hashin-Shtrikman rectangles in (K,G) become triangles in (E,ν).

Figure 3. Isothermal bulk modulus K of porous P-311 glass measured at room temperature compared to the Hashin-Shtrikman upper bound (solid line) and differential method (dashed line) assuming spherical voids. Data from Walsh *et al.* (1965).

Figure 4. Young's modulus E for a sodium borosilicate glass containing dispersions of alumina (Al_2O_3) particles assuming no porosity, although it is known that the sample at 0.55 volume fraction of Al_2O_3 contained 0.096 fraction of voids (see Tables 3 and 4). Data from Hasselman and Fulrath (1965a).

Figure 5. Young's modulus E for a sodium borosilicate glass containing dispersions of spherical tungsten (W) particles assuming no porosity present (see Table 5). Data from Hasselman and Fulrath (1965b).

Figure 6. (a) Young's modulus E and (b) shear modulus G of porous silicon nitride (Si_3N_4). Hashin-Shtrikman upper bound is the solid line. The self-consistent estimate assuming spherical particles and needle-shaped pores is the dot-dash line. Data from Fate (1975).

Figure 7. Young's modulus for tungsten-carbide (WC) in cobalt (Co). (a) Assuming no porosity, some of the data points fall below the lower Hashin-Shtrikman bound. (b) Assuming porosity as shown in Table 10, data points all lie below the upper Hashin-Shtrikman bound and are reasonably well-approximated by the self-consistent estimate assuming pores are in the shape of penny-shaped cracks. Data from Nishimatsu and Gurland (1960).

Figure 8. Real data example of the transformation from (K,G)-bounds to (E,ν)-bounds. Data from Nishimatsu and Gurland (1960) for the $(WC)_{0.50}(Co)_{0.50}$ sample.