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Critique of Two Explicit Schemes for Estimating Elastic Properties of Multiphase Composites

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Abstract

Explicit schemes of Mori-Tanaka and Kuster-Toksöz for estimating elastic properties of multiphase composites are compared and contrasted. Both methods are known to have a limited range of validity. Reexamination of the theories and comparisons to experiment leads to the conclusion that these explicit schemes should only be used to estimate properties of systems in which the host material occupies about 70–80% or more of the total volume. By introducing the concept of a reference material, we find a mutually consistent pair of equations for effective stiffness and compliance from which many of the standard approximation schemes may be easily derived. A unified analysis of the various approximation schemes results.

1 Introduction

Methods for computing estimates of elastic constants for composites are either implicit or explicit. Implicit methods are generally computed by iteration [for example, self-consistent methods such as those described by Gubernatis and Krumhansl (1975) or Berryman (1980)] or by integration [for example, differential effective medium methods such as those discussed by Cleary *et al.* (1980) or Norris (1985)]. On the other hand, explicit methods are easier to evaluate because they supply *formulas* for estimates of constants, not requiring any further computation beyond direct substitution.

Two of the more common implicit methods — the self-consistent (SC) method and the differential effective medium (DEM) method mentioned in the preceding paragraph — are known to be realizable (Milton, 1985; Norris, 1985; Avellaneda, 1987), *i.e.*, they compute constants that correspond to values realized for specific microgeometries (Berryman and Berge, 1993). Realizability of an estimator implies that the estimator will *always* satisfy rigorous bounds on the constants [such as Hashin-Shtrikman (1963), Walpole (1969), and Willis (1977) bounds]. It is important to recognize that several methods termed "self-consistent" have appeared in the literature, but these are not all equivalent (Berryman, 1980; Milton, 1985); some are known to be realizable and some are known not to be realizable. Since all self-consistent methods are about equally difficult to compute, realizability becomes a useful criterion for choosing among these alternatives.

Ease of implementation is one key criterion for choosing an explicit scheme for estimating effective elastic constants. Various types of explicit schemes have been introduced by Mori and Tanaka (1973), Kuster and Toksöz (1974), Hori and Nemat-Nasser (1993), Nemat-Nasser *et al.* (1993), and Ju and Chen (1994a,b), among others. Of these methods, the two that are most easily implemented and the two that have been most thoroughly studied are those of Mori-Tanaka and Kuster-Toksöz. Both of these methods are known not to be realizable (at least not for all possible microstructures), and in particular both methods are known to violate some of the rigorous bounds (Berryman, 1980; Norris, 1989; Christensen, 1990; Ferrari, 1991) in limiting cases with extreme microgeometries (such as high concentrations of thin disk inclusions). Although there are many similarities between these two methods, the Mori-Tanaka approach has been used almost exclusively by the engineering community, while the Kuster-Toksöz method has been with nearly equal exclusivity by the geophysics community. One purpose of the present paper is to draw attention to the similarities and differences of these two methods, which are both quite easy to implement.

Despite some lack of rigor, there are circumstances in which explicit schemes such as Mori-Tanaka and Kuster-Toksöz are both useful and necessary. In one situation frequently encountered in geophysical and rock mechanics studies, it is desirable to invert formulas for effective constants to try to estimate other physical properties of a composite or rock — properties such as the crack aspect ratio distribution (Toksöz *et al.*, 1976; Cheng and Toksöz, 1979; Zimmerman, 1991b) or the porosity (Wilkens *et al.*, 1991) — from measurements of seismic velocities or elastic constants. While such an inversion procedure is quite feasible with such simple explicit estimation schemes, it is practically intractable either with any implicit scheme (Berge *et al.*, 1992), or with more complex explicit schemes. The main limitations on the use of explicit schemes such Kuster-Toksöz for inversion are the nonuniqueness of the results obtained (a common difficulty in geophysical inversion problems) and the inherent inaccuracies of the approximate estimators themselves. Our present purpose is not to study the inverse problem, but rather to analyze these two commonly used explicit schemes: Mori-Tanaka (MT) and Kuster-Toksöz (KT). We do this with a view to determining whether any relations exist between them and also which — if either — of these methods might be more accurate and therefore more useful for such inversion studies. Range of validity of the two methods will also be discussed.

To achieve these goals, we first introduce a unified approach for obtaining these approximation schemes and use this method to present different derivations of the SC, MT, and KT estimates. For example, whereas the KT approximation has generally been derived using a long-wavelength approximation within scattering theory (Kuster-Toksöz, 1974; Berryman, 1980; Berryman, 1992), we show how to obtain this approximation within a quasistatic analysis [but also see Korringa *et al.* (1979)].

2 Background and general analysis

The constituents of the composites under consideration are linear elastic, isotropic materials whose average stresses σ_i and average strains e_i are related by the constitutive relations

$$\sigma_i = \mathbf{L}_i e_i \quad \text{and} \quad e_i = \mathbf{M}_i \sigma_i. \tag{1}$$

The subscript i refers to the i-th constituent, of which we assume there are N. The components of the fourth ranked stiffness tensor $\mathbf{L}_i$ are defined by

$$(\mathbf{L}_i)_{mnpq} = \lambda_i \delta_{mn}\delta_{pq} + \mu_i(\delta_{mp}\delta_{nq} + \delta_{np}\delta_{mq}), \tag{2}$$

where λ_i and μ_i are the Lamé parameters of the i-th constituent. The indices m, n, p, q take the values 1,2,3, corresponding to Cartesian axes $x_1 = x$, $x_2 = y$, and $x_3 = z$. The bulk modulus $K_i = \lambda_i + \frac{2}{3}\mu_i$, while the shear modulus is μ_i. Following Hill (1963) [also see Gubernatis and Krumhansl (1975)], we treat $\mathbf{L}_i$ as a six-by-six matrix, with σ_i and e_i being treated as six component vectors. Then, the compliance tensor $\mathbf{M}_i$, similarly viewed as a six-by-six matrix, satisfies

$$\mathbf{M}_i\mathbf{L}_i = \mathbf{I} = \mathbf{L}_i\mathbf{M}_i, \tag{3}$$

where $\mathbf{I}$ is the 6×6 identity matrix and $\mathbf{M}_i$ is the matrix inverse of $\mathbf{L}_i$.

We assume for simplicity that the overall behavior of the composite is linear elastic and isotropic, and that the effective constitutive laws relating the average composite stress σ and strain e are given by

$$\sigma = \mathbf{L}^* e \quad \text{and} \quad e = \mathbf{M}^* \sigma, \tag{4}$$

with

$$\mathbf{M}^*\mathbf{L}^* = \mathbf{I} = \mathbf{L}^*\mathbf{M}^*. \tag{5}$$

Our problem is to find ways of relating the effective tensors $\mathbf{L}^*$ and $\mathbf{M}^*$ to the properties of the components contained in the constituents' tensors $\mathbf{L}_i$ and $\mathbf{M}_i$.

2.1 General results

The volume fraction of the i-th constituent is v_i and we assume that the constituents of the composite are space filling so that $\sum v_i = 1$. Then, the average strain and average stress are given by the volume averages

$$e = \sum v_i e_i \quad \text{and} \quad \sigma = \sum v_i \sigma_i. \tag{6}$$

It is an elementary exercise to show that (1), (4), and (6) imply

$$\sum v_i(\mathbf{L}_i - \mathbf{L}^*)e_i = 0 \tag{7}$$

and

$$\sum v_i(\mathbf{M}_i - \mathbf{M}^*)\sigma_i = 0. \tag{8}$$

Following Hill (1963), we then recall that, since all the e_is must be linear functions of e and since all the σ_is must be linear functions of σ, we can write

$$e_i = \mathbf{A}^{*i} e \quad \text{and} \quad \sigma_i = \mathbf{B}^{*i}\sigma, \tag{9}$$

where $\mathbf{A}^{*i}$ and $\mathbf{B}^{*i}$ are unknown, dimensionless fourth ranked tensors (again to be viewed operationally as six-by-six matrices) depending on the properties and microgeometry of the composite and its constituents. From (6), these tensors must also satisfy the normalization conditions

$$\sum v_i \mathbf{A}^{*i} = \mathbf{I} = \sum v_i \mathbf{B}^{*i}. \tag{10}$$

From (1), (4), and (9) follow the consistency conditions

$$\mathbf{B}^{*i} = \mathbf{L}_i \mathbf{A}^{*i} \mathbf{M}^*. \tag{11}$$

Substituting (9) into (7) and (8), it follows (since e and σ may be considered arbitrary) that

$$\sum v_i(\mathbf{L}_i - \mathbf{L}^*)\mathbf{A}^{*i} = 0 \tag{12}$$

and

$$\sum v_i(\mathbf{M}_i - \mathbf{M}^*)\mathbf{B}^{*i} = 0. \tag{13}$$

While no approximation has been made in arriving at these equations, they are nevertheless of only limited usefulness since the tensors $\mathbf{A}^{*i}$ and $\mathbf{B}^{*i}$ are unknown. If we could find approximations to $\mathbf{A}^{*i}$ and $\mathbf{B}^{*i}$ based on their definitions (9), then such approximations would lead immediately to approximations for $\mathbf{L}^*$ and $\mathbf{M}^*$ using (12) and (13).

2.2 Wu tensors and single inclusion problems

Eshelby (1957) calculated the strain field in an elastic ellipsoidal inclusion imbedded in an elastic host material subjected to a displacement with uniform strain field at infinity. The important general result of Eshelby is that for all ellipsoidal inclusions the resulting strain in the inclusion is homogeneous. Host and inclusion strains are therefore related algebraically. For arbitrary spheroidal inclusions, Wu (1966) subsequently found the components of the tensor $\mathbf{T}^{hi}$ relating the strain in the host e_h to the strain in the inclusion e_i according to

$$e_i = \mathbf{T}^{hi} e_h. \tag{14}$$

If the composite of interest is isotropic, no particular orientation of the spheroidal inclusions is preferred. Then, it is not the full tensor $\mathbf{T}$ that is of most interest but rather its isotropic average. This average is obtained by considering

$$P^{hi} \equiv \frac{1}{3}(\mathbf{T}^{hi})_{mmpp} \quad \text{and} \quad Q^{hi} \equiv \frac{1}{5}\left[(\mathbf{T}^{hi})_{mnmn} - \frac{1}{3}(\mathbf{T}^{hi})_{mmpp}\right], \tag{15}$$

where the summation convention is implied for repeated subscripts. We treat $\mathbf{T}$ from here on in the paper as an isotropic tensor of the form

$$(\mathbf{T})_{mnpq} = \frac{1}{3}(P-Q)\delta_{mn}\delta_{pq} + \frac{1}{2}Q(\delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np}). \tag{16}$$

Specific results on P and Q for inclusions shaped like spheres, needles, and penny-shaped cracks are given in Table 1. Formulas for more general prolate and oblate spheroidal inclusions were presented by Berryman (1980), based on the work of Wu (1966) and Kuster and Toksöz (1974).

2.3 General equations using a reference material

Now consider a reference material labeled r (with stiffness and compliance tensors $\mathbf{L}_r$ and $\mathbf{M}_r$) which may or may not be one of the actual constituents of the composite under study. If there is one material in the composite that acts as host to the others, we may choose the reference material to be the host material h. However, in general, we want to leave the choice of reference material open for later consideration.

Define tensors relating (possibly fictitious) strains e_r and stresses σ_r to the actual average strains and stresses in constituent i by the linear relations

$$e_i = \mathbf{G}^{ri} e_r \quad \text{and} \quad \sigma_i = \mathbf{H}^{ri}\sigma_r. \tag{17}$$

The consistency relation between the new tensors is given by

$$\mathbf{H}^{ri} = \mathbf{L}_i \mathbf{G}^{ri} \mathbf{M}_r. \tag{18}$$

Note that, in the case where the reference material is chosen to be the composite material itself (*i.e.*, $r \to *$), it follows from (9) that the new tensors satisfy $\mathbf{G}^{*i} \equiv \mathbf{A}^{*i}$ and $\mathbf{H}^{*i} \equiv \mathbf{B}^{*i}$. These new tensors are therefore simply formal generalizations of $\mathbf{A}^{*i}$ and $\mathbf{B}^{*i}$, suitable for the wide variety of circumstances that may arise in our modeling efforts.

Again substituting (17) into (7) and (8) yields the equations

$$(\mathbf{L}^* - \mathbf{L}_r)\sum v_i \mathbf{G}^{ri} e_r = \sum v_i (\mathbf{L}_i - \mathbf{L}_r)\mathbf{G}^{ri} e_r \tag{19}$$

and

$$(\mathbf{M}^* - \mathbf{M}_r)\sum v_i \mathbf{H}^{ri} \sigma_r = \sum v_i (\mathbf{M}_i - \mathbf{M}_r)\mathbf{H}^{ri} \sigma_r. \tag{20}$$

These equations are not in simplest form (there are redundant terms proportional to either $\mathbf{L}_r$ or $\mathbf{M}_r$ on both side of the equations) for reasons that will become apparent. Nevertheless, these two equations are exact (*i.e.*, no approximations have been made) and mutually consistent. Consistency may be checked by deleting the redundant terms from (20), multiplying the result by $\mathbf{L}^*$, and then including the redundant terms found in (19). The result is exactly (19).

It is sometimes claimed that certain theories are superior to others because the compliance tensor $\mathbf{M}^*$ can be found using an equation analogous to the one used to find $\mathbf{L}^*$. However, since all the theories considered in this paper may be derived from the pair of equations (19) and (20) which are themselves consistent in this sense, we have found that all the theories considered possess this desirable property. Thus, this criterion appears much too weak to provide compelling choices among the theories.

The advantage of the introduction of the reference material will become apparent as we derive various approximate expressions from (19) and (20) using Eshelby's and Wu's results for single ellipsoidal inclusions and isotropic averages of ellipsoidal inclusions, respectively. These equations also enable us to present a unified approach to derivation of the various approximation schemes considered.

3 Approximation schemes

Starting with (19) and (20), we now derive various approximation schemes using Wu's tensor $\mathbf{T}$ to replace the unknown tensor $\mathbf{G}$. We will see that the art of designing an effective medium theory is founded on various clever choices of reference materials.

3.1 A realizable self-consistent scheme

The first approximation scheme we consider is an implicit method called the self-consistent (SC) scheme (Gubernatis and Krumhansl, 1975; Korringa *et al.*, 1979; Berryman, 1980). This scheme should be distinguished from another scheme by Wu (1966) that is also called a self-consistent scheme, but generally differs from the present one. Our self-consistent scheme is sometimes given the technical name "coherent potential approximation" or "CPA" because of its conceptual similarity to certain methods used to determine electronic bandstructures of disordered alloys. For completeness and to clarify these distinctions, Wu's scheme is also derived in Appendix A. Aalso for completeness, another type of implicit scheme (the differential scheme) is also derived in Appendix B.

Our approximation may be obtained by supposing the reference material is a medium with the same stiffnesses and compliances as the composite of interest. Thus, $r \to *$ in (19) and (20), so the left hand side of each equation vanishes. To obtain a tractable approximation, we suppose that $\mathbf{G}^{*i}$ is well approximated by Wu's tensor $\mathbf{T}^{*i}$. Then, considering e_r and σ_r arbitrary, the self-consistent approximation is given by

$$\sum v_i(\mathbf{L}_i - \mathbf{L}^*_{SC})\mathbf{T}^{*i} = 0. \tag{21}$$

The corresponding equation for the compliance tensor is

$$\sum v_i(\mathbf{M}_i - \mathbf{M}^*_{SC})\mathbf{W}^{*i} = 0, \tag{22}$$

where $\mathbf{W}^{*i} \equiv \mathbf{L}_i\mathbf{T}^{*i}\mathbf{M}^*_{SC}$. Both of these equations have the same general form as the exact equations (12) and (13), but nevertheless the results obtained are generally approximate since for most microstructures the exact factors $\mathbf{A}^{*i}$ and $\mathbf{B}^{*i}$ are only approximated by the factors $\mathbf{T}^{*i}$ and $\mathbf{W}^{*i}$, respectively.

We find easily that, by multiplying (22) on both the left and right by $\mathbf{L}^*_{SC}$, it reduces to (21) and is therefore consistent with it in this sense. This shows that it is sufficient to solve either equation and then use the uniqueness of the matrix inverse to find the solution of the other. We will continue to point out the existence of this property for the other theories as they are derived, but will not dwell on the proofs, since these proofs are completely analogous to the one just given in all cases.

3.2 Mori-Tanaka scheme

The first explicit scheme we consider is that of Mori and Tanaka (1973) as described by Weng (1984), Benveniste (1987), and Nemat-Nasser and Hori (1993). Comparisons between one version of the self-consistent approximation (*not* the one we introduced in the preceding subsection) and the Mori-Tanaka scheme have been presented by various authors including Dvorak and Benveniste (1992), Hori and Nemat-Nasser (1993), and Nemat-Nasser *et al.* (1993). Benveniste (1987), Weng (1990), and Ju and Chen (1994a,b) show that Mori-Tanaka satisfies the Hashin-Shtrikman bounds for two component composites.

To obtain the Mori-Tanaka approximation from (19) and (20), we assume that the composite has a host material with inclusions and choose the host to serve as reference material, so $r \to h$. Then, approximating $\mathbf{G}^{hi} \simeq \mathbf{T}^{hi}$, we find

$$\sum v_i(\mathbf{L}_i - \mathbf{L}^*_{MT})\mathbf{T}^{hi} = 0, \tag{23}$$

in simplest form. The corresponding equation for the compliance is

$$\sum v_i(\mathbf{M}_i - \mathbf{M}^*_{MT})\mathbf{W}^{hi} = 0, \tag{24}$$

where $\mathbf{W}^{hi} \equiv \mathbf{L}_i\mathbf{T}^{hi}\mathbf{M}_h$. Again by multiplying (24) on the right by $\mathbf{L}_h$ and on the left by $\mathbf{L}^*_{MT}$ to check consistency, we find that (23) is recovered, as shown previously by Weng (1984) and Benveniste (1987). Alternatively, the equation for $\mathbf{L}^*_{MT}$ can be written as

$$(\mathbf{L}^*_{MT} - \mathbf{L}_h)\sum v_i\mathbf{T}^{hi} = \sum v_i(\mathbf{L}_i - \mathbf{L}_h)\mathbf{T}^{hi}, \tag{25}$$

where we have retained the redundant terms $\mathbf{L}_h \sum v_i\mathbf{T}^{hi}$ on both sides of the equation. This form of the equation for $\mathbf{L}^*_{MT}$ is convenient for comparison with the KT explicit scheme to be considered next.

3.3 Kuster-Toksöz scheme

The second explicit scheme we consider is that due to Kuster and Toksöz (1974). To obtain this approximation, we first recall, from the definitions (17) of the tensors relating the reference material stresses and strains to those of the constituents, that the average strain and average stress are formally related to reference material quantities by

$$e = \sum v_i \mathbf{G}^{ri} e_r \quad \text{and} \quad \sigma = \sum v_i \mathbf{H}^{ri} \sigma_r. \tag{26}$$

So (19) and (20) may be rewritten as

$$(\mathbf{L}^* - \mathbf{L}_r)e = \sum v_i (\mathbf{L}_i - \mathbf{L}_r)\mathbf{G}^{ri} e_r \tag{27}$$

and

$$(\mathbf{M}^* - \mathbf{M}_r)\sigma = \sum v_i (\mathbf{M}_i - \mathbf{M}_r)\mathbf{H}^{ri} \sigma_r. \tag{28}$$

Now, we consider the relationship between the average strain e in the composite and the strain e_r in the reference material, supposing in particular that our composite is imbedded in the reference material so

$$e = \mathbf{G}^{r*} e_r \quad \text{and} \quad \sigma = \mathbf{H}^{r*} \sigma_r. \tag{29}$$

Note that it follows from (26) and (29) that

$$\mathbf{G}^{r*} = \sum v_i \mathbf{G}^{ri} \quad \text{and} \quad \mathbf{H}^{r*} = \sum v_i \mathbf{H}^{ri}. \tag{30}$$

Substituting (29) into (27) and (28), taking the reference material to be the host $r \to h$, and making the usual approximation that $\mathbf{G} \simeq \mathbf{T}$ gives the KT approximation

$$(\mathbf{L}^*_{KT} - \mathbf{L}_h)\mathbf{T}^{h*} = \sum v_i (\mathbf{L}_i - \mathbf{L}_h)\mathbf{T}^{hi}. \tag{31}$$

The corresponding equation for the compliance is

$$(\mathbf{M}^*_{KT} - \mathbf{M}_h)\mathbf{W}^{h*} = \sum v_i (\mathbf{M}_i - \mathbf{M}_h)\mathbf{W}^{hi}, \tag{32}$$

where $\mathbf{W}^{h*} \equiv \mathbf{L}^*_{MT}\mathbf{T}^{h*}\mathbf{M}_h$ and $\mathbf{W}^{hi} \equiv \mathbf{L}_i\mathbf{T}^{hi}\mathbf{M}_h$. Again multiplying (32) on both the left and right by $\mathbf{L}_h$, (31) is recovered.

Equations (31) and (32) may at first appear to be implicit rather than explicit formulas. However, Kuster and Toksöz (1974) made the further assumption that tensor $\mathbf{T}^{h*}$ on the left hand side of (31) is always the tensor for spherical inclusions. Then, (31) reduces to the two equations

$$(K^*_{KT} - K_h)\frac{K_h + \frac{4}{3}\mu_h}{K^*_{KT} + \frac{4}{3}\mu_h} = \sum v_i (K_i - K_h) P^{hi} \tag{33}$$

and (note that ζ is defined in Table 1)

$$(\mu^*_{KT} - \mu_h)\frac{\mu_h + \zeta_h}{\mu^*_{KT} + \zeta_h} = \sum v_i (\mu_i - \mu_h) Q^{hi}, \tag{34}$$

which are both easily rearranged into explicit formulas for K^*_{KT} and μ^*_{KT}. By making other assumptions about the form of $\mathbf{T}^{h*}$, we could obtain a new class of implicit schemes, but that is beyond the scope of the present effort.

Kuster and Toksöz (1974) consider their approach to be a low concentration approximation based on a "noninteraction" assumption. They translate this assumption into a quantitative condition stating that the aspect ratio of spheroidal inclusions should not exceed the corresponding volume fraction. Such a condition clearly places severe constraints on models containing oblate spheroids and penny-shaped cracks. These conditions can also cause problems with the applications when a broad range of inclusion shapes is used, while trying to satisfy the constraint that $\sum v_i/\alpha_i < 1$.

4 Comparing MT and KT schemes

Since the right hand sides of (25) and (31) are exactly the same if the host medium h is taken to be the same, an equation relating these two approximations is

$$(\mathbf{L}^*_{MT} - \mathbf{L}_h) \sum v_i \mathbf{T}^{hi} \equiv (\mathbf{L}^*_{KT} - \mathbf{L}_h)\mathbf{T}^{h*}, \tag{35}$$

showing that these two explicit approximations are very closely related but not generally identical. To achieve equality between the two approximations (*i.e.*, $\mathbf{L}^*_{MT} = \mathbf{L}^*_{KT}$), it is both necessary and sufficient that

$$\mathbf{T}^{h*} = \sum v_i \mathbf{T}^{hi} \tag{36}$$

— which is certainly not satisfied in general, but is true for spherical inclusions. To check satisfaction of (36), we find, using the Ps and Qs for *spheres* from Table 1, that (36) implies

$$\frac{K_h + \frac{4}{3}\mu_h}{K^* + \frac{4}{3}\mu_h} = \sum v_i \frac{K_h + \frac{4}{3}\mu_h}{K_i + \frac{4}{3}\mu_h} \tag{37}$$

for the bulk modulus, and

$$\frac{\mu_h + \zeta_h}{\mu^* + \zeta_h} = \sum v_i \frac{\mu_h + \zeta_h}{\mu_i + \zeta_h} \tag{38}$$

for the shear modulus. When the common factors depending only on host properties in the numerators are divided out of these two equations, the remaining formulas are exactly the results of the MT and KT theories for spherical inclusions. Since $\mathbf{T}^{hi} = \mathbf{I}$ when the host and inclusion properties are the same regardless of the assumed host-particle shape, the results (36), (37), and (38) are also valid for both MT and KT whenever all the inclusions except the host are sphere shaped. More often, it is found that equation (36) is just an approximation to (30) that is poorly satisfied.

5 Realizability

If a model corresponds to a realizable microgeometry, it must be possible to obtain the same results using either the quasistatic analysis presented here or using a scattering analogy (Berryman, 1992). Thus, the existence of a scattering analog is a necessary condition for realizability but not sufficient to guarantee it. For example, the KT approximation scheme was originally derived using a scattering analogy, but nevertheless this scheme has been shown not to be realizable for various choices of inclusion shape (Berryman, 1980).

These results are summarized in Table 2. Note that the nonrealizable models can nevertheless give exact results for some inclusion shapes (such as spherical inclusions) even though the microstructure implicit in their derivation is not realizable for all inclusion shapes.

6 Examples

The useful range of validity for explicit schemes depends on the material being modeled. For mixtures of solids having similar elastic moduli, the Kuster-Toksöz model can be applied to any mixture, regardless of relative volumes. For mixtures of fluids and solids, the microstructure becomes important. Explicit and implicit schemes produce similar results for porosities below about 10–15% (e.g., Cheng, 1978; Berge *et al.*, 1992). The Kuster-Toksöz scheme can also be used to model materials that behave as hosts containing isolated scatterers, whenever multiple scattering effects are not significant. This category includes rocks having relatively low porosities and materials having an identifiable background component that remains intact even at high porosities. Wilkens *et al.* (1991) found that velocities of oceanic basalts could be estimated using the Kuster-Toksöz model for rocks having porosities below about 20%. This scheme has also been used to model velocities for sandstones having porosities of about 16–25% (Cheng, 1978; Cheng and Toksöz, 1979). Such modeling of rock properties requires assumptions about the distribution of pore and crack shapes. There may be too many adjustable parameters, unless the modeling is constrained by independent information such as scanning electron microscope images of pores and cracks. We use materials having simpler microgeometries than rocks in order to examine differences among the various approximation schemes.

Porous materials have high contrasts in their constituents' properties and therefore serve as extreme examples to highlight practical differences among the various approximation schemes. These examples are also motivated by experience with rocks and earth materials which tend to be random assemblages of several minerals together with imbedded voids and/or fluids. The examples also include comparisons of data obtained from both static and dynamic measurements. In geophysics and rock mechanics, the most readily available data on elastic properties are usually seismic or ultrasonic velocity measurements [*e.g.*, Cheng and Toksöz (1979) and Wilkens *et al.* (1991)]. For solids, the differences between static and dynamic moduli are normally negligible at room temperature.

6.1 Spherical inclusions

Porous glass foam: Walsh *et al.* (1965) made measurements on the compressiblity ($1/K$) of a suite of glass samples with porosities ranging from 70% to near zero. Porosity measurements were stated to be accurate to $\pm$ 1%.

The glass samples were foam, with microgeometry such that large open dry pores were dispersed in a continuous glass background that thinned to a network of needle-like filaments at the highest porosities. Linear compressibility of the samples was measured by Walsh *et al.* (1965) using electrical resistance strain gages attached to sample surfaces in a high-pressure medium. The two explicit methods under consideration (MT and KT) produce the same bulk modulus estimates for composites with spherical inclusions and the solid curve in Figure 1 shows that these estimates are quite good at void concentrations below about 20%, but they provide upper bounds rather than estimates for the higher concentrations. Results of MT and KT modeling using nonspherical inclusions will be discussed in the next section. Cheng (1978) showed that KT provides good estimates for bulk moduli of these foam samples while a self-consistent method was useful at low porosity. Ferrari and Filipponi (1991) and Zimmerman (1991a) have also demonstrated that the MT method produces good estimates of the compressibilities of these foam samples.

Sintered glass beads: A suite of synthetic sandstone samples was made by fusing glass beads to achieve porosities of ~1–40%, with uncertainties of about 0.6–1% for the porosities. Ultrasonic P and S wave velocities for the dry and saturated cases were measured for these samples (Berge *et al.*, 1995), at atmospheric pressure and room temperature. Berge *et al.* (1993) showed that the SC approximation produced good estimates of the observed velocities of these porous glass samples because the granular structure of these synthetic sandstones is compatible with this theory. In Figures 2 and 3, dynamic bulk and shear moduli determined from the observed velocities are compared with calculated bulk and shear moduli from the KT and MT schemes, for the dry (Figure 2) and saturated (Figure 3) cases. The solid curves were computed assuming spherical pores and spherical grains for all samples. KT and MT results are identical for spherical inclusions, but these results do not agree with observed bulk and shear moduli for samples with porosities greater than about 20%. Improving the fit would require using other inclusion shapes, as discussed in the next section.

6.2 Nonspherical inclusions

If we want to model a material by assuming different shapes for the inclusions and host materials, we may do so using the Mori-Tanaka and Kuster-Toksöz schemes. Again considering the data of Walsh *et al.* (1965) for glass foam, suppose we were to model the foam using spherical glass particles and needle-shaped voids. The results of such KT and MT calculations are presented in Figure 1, shown as dashed curves. The agreement is somewhat better than that for the spherical inclusions model, although the relationship between this model and the microstructure of the physical composite seems unclear. In fact, this should be a better model for the sintered glass beads, than for the foam. It seems therefore that a more physically realistic model of the glass foam should be just the opposite: spherical voids and needle shaped glass inclusions. However, when we do this computation for MT and KT, we find that the results in both cases are identical to the results for spherical voids and spherical glass inclusions illustrated by the solid curve in Figure 1 and therefore this model gives no improvement in the estimates. We consider this to be a failure of these theories, because it shows *they are not flexible enough to capture differences in the microgeometry of the host.* For comparison, we also show in Figure 1 the result obtained using the KT and MT theories assuming sphere-shaped glass and penny-shaped voids. The dotted lines in Figure 1 show that both the MT and the KT theories greatly overestimate bulk moduli of the foam samples if penny-shaped inclusions are used to represent the void space. The KT theory predicts a vanishing bulk modulus at a porosity near 85%.

Considering the data of Berge *et al.* (1995) again, a reasonable model of a finite permeability granular sample is this: spherical grains and needle-shaped voids. This result computed using the KT and MT theories is plotted using dashed curves in Figures 2 and 3. The agreement with the data is comparable to that achieved by the theory assuming spherical grains and spherical voids. Bulk and shear moduli estimates fit the data reasonably well for porosities below about 20% and in some cases for porosities near 30%, but the theories greatly overestimate the moduli for higher porosity samples. It is possible that using some combination of spheres and needles to represent the void space would improve estimates of bulk and shear moduli for samples having porosities up to about 30%, using either the KT or the MT theory. But it is clear that no combination of spherical or needle-shaped inclusions will fit the data for higher porosity samples. The dotted curves in Figures 2 and 3 show KT and MT estimates of bulk and shear moduli using penny-shaped cracks to represent the porosity. The synthetic sandstone samples are known to be free of significant concentrations of flat cracks (Berge *et al.*, 1995), and these modeling results clearly do not fit the data for low to moderate porosity samples. The MT theory fits observed bulk moduli for samples having porosities near 35-40% but overestimates shear moduli. The KT theory using penny shaped cracks underestimates bulk moduli but fits shear moduli for high porosity samples. Neither theory can fit both bulk and shear moduli and these results assume unrealistically high concentrations of flat cracks in the synthetic sandstones. These theories fail to represent certain behavior of the observed moduli. For dry granular samples, the bulk modulus vanishes near the random packing point (around 40% porosity), and the shear modulus is approximately the same in the wet and dry cases. The MT theory never allows the bulk modulus to vanish; the KT theory has it vanishing at the wrong porosity. MT and KT theories using penny-shaped inclusions predict large differences in shear moduli in the wet and dry cases. Regardless of inclusion shape, we conclude that the MT and KT theories both fail to model the synthetic sandstones at porosities above about 30%.

In addition to modeling the porous glasses discussed above, we also modeled the porous silicon nitride (Si_3N_4) samples described by Fate (1975). Fate obtained polycrystalline samples from various commercial suppliers and used pulse-echo measurements to find ultrasonic compressional- and shear-wave velocities. Berryman (1994) has shown how the data compare to results of implicit schemes such as the SC theory, and that the data of Fisher *et al.* (1991, 1992) for pure samples of Si_3N_4 are more appropriate than assuming zero-porosity values for bulk and shear moduli using Fate's lowest porosity sample. Since the maximum porosity for these samples was only about 25%, the KT and MT theories may be expected to perform well for modeling this material. Figure 4 shows dynamic moduli calculated from Fate's (1975) velocity data compared to KT and MT theory results using various shapes to represent the pore space. The curves were calculated assuming zero-porosity values of bulk and shear moduli from Fisher *et al.* (1991, 1992), as shown by the squares plotted at zero porosity and having the higher values for bulk and shear moduli. The squares plotted at zero porosity and having lower bulk and shear moduli values are from Fate (1975), and obviously have some finite porosity present since the moduli are lower than values for Fate's sample having a porosity near 2.5%. Although there is a lot of scatter in the data for low porosity samples, we see that KT and MT theoretical estimates calculated using spherical or needle-shaped voids greatly overestimate bulk and shear moduli for all but the lowest porosity samples. This result would not be improved by using Fate's values for zero-porosity properties in the calculations. If penny-shaped voids are used, both the KT and MT theories provide useful estimates for shear moduli for all porosities up to about 25%. The MT theory provides better bulk modulus estimates than the KT theory for porosities above about 5%. These theories are adequate for modeling this material if penny-shaped inclusions are used to represent the void space.

7 Conclusions

Mori-Tanaka and Kuster-Toksöz give the same results for composites with spherical inclusions as illustrated in Figure 1. In fact, host microgeometry cannot contribute to the results for either Mori-Tanaka and Kuster-Toksöz estimates since $\mathbf{T}^{hi} = \mathbf{I}$ for any ellipsoidally shaped particle if $i = h$. Although this result is reasonable for low concentrations of inclusions in a host medium, it clearly limits the usefulness of these theories. In particular, we have no control over the limiting behavior (*i.e.*, we cannot specify the limiting host geometry) in either theory as the limit of vanishing host volume fraction is approached. Viewing these approximations as interpolation schemes, both theories have no means of adjusting the slope of the elastic constant trajectories as the host volume fraction vanishes. This failure has some serious consequences: For example, Norris (1989) has shown that MT can give results violating the Hashin-Shtrikman bounds for multiphase composites. Similarly, Ferrari (1991) has shown that for anisotropic composites the MT scheme produces the absurd result that the composite can still depend on the host elastic constants even in the limit of vanishing host volume fraction. Berryman (1980) shows that the KT scheme violates the Hashin-Shtrikman bounds when the inclusions are either disks at any finite concentration or needles at volume fractions greater than about 60%.

We conclude that KT and MT have limited ranges of usefulness and in particular clearly should not be used when the nominal host material does not have the dominant volume fraction. A reasonable recommendation for the use of these schemes then is to limit calculations to situations where the host occupies at least 70-80% of the overall volume, as we have done in this paper. Results obtained when these methods were applied to rocks (Cheng, 1978; Cheng and Toksöz, 1979; Berge *et al.*, 1992) indicate that at least 70–80% volume fraction of the host is often necessary.

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Appendix A. Wu's Scheme

Wu (1966) developed another type of self-consistent estimator, distinct from the approximation (21) and (22) that we call the SC theory (also known as the CPA). Wu's formulas can also be derived from (19) and (20). First, remove the redundant terms from these equations and then recall (26). The resulting equations are

$$\mathbf{L}^* e = \sum v_i \mathbf{L}_i \mathbf{G}^{hi} e_h \tag{39}$$

and

$$\mathbf{M}^*\sigma = \sum v_i \mathbf{M}_i \mathbf{H}^{hi}\sigma_h, \tag{40}$$

where we have taken the reference material to be the host material $r \to h$.

Now the strains and stresses of the host material are related to those of the composite by

$$e_h = \mathbf{G}^{*h}e \quad \text{and} \quad \sigma_h = \mathbf{H}^{*h}\sigma. \tag{41}$$

Then, substituting (41) into (39) and (40) and recalling the identities

$$\mathbf{G}^{*i} = \mathbf{G}^{hi}\mathbf{G}^{*h} \quad \text{and} \quad \mathbf{H}^{*i} = \mathbf{H}^{hi}\mathbf{H}^{*h}, \tag{42}$$

together with

$$\sum v_i \mathbf{G}^{*i} = \mathbf{I} = \sum v_i \mathbf{H}^{*i}, \tag{43}$$

we find

$$(\mathbf{L}^* - \mathbf{L}_h)e = \sum v_i(\mathbf{L}_i - \mathbf{L}_h)\mathbf{G}^{*i}e \tag{44}$$

and

$$(\mathbf{M}^* - \mathbf{M}_h)\sigma = \sum v_i(\mathbf{M}_i - \mathbf{M}_h)\mathbf{H}^{*i}\sigma. \tag{45}$$

So far no approximations have been made in arriving at (44) and (45). They are completely equivalent to (19) and (20).

Now, to obtain Wu's scheme, we make the usual substitution $\mathbf{G} \to \mathbf{T}$. Then, for arbitrary e and σ, we have

$$\mathbf{L}^*_{Wu} - \mathbf{L}_h = \sum v_i(\mathbf{L}_i - \mathbf{L}_h)\mathbf{T}^{*i} \tag{46}$$

and

$$\mathbf{M}^*_{Wu} - \mathbf{M}_h = \sum v_i(\mathbf{M}_i - \mathbf{M}_h)\mathbf{W}^{*i}. \tag{47}$$

These two formulas are exactly the multiphase generalizations of Wu's formulas (Berryman, 1980), and they are consistent in the sense that either one can be derived from the other using the uniqueness of the matrix inverses.

Appendix B. Differential effective medium

The differential effective medium (DEM) approach is based on the concept that a composite can be built up by infinitesimal additions of inclusions into a host material. Then, as the included material accumulates to a finite amount, the new composite becomes the host material, and so on until the desired concentration of included material is achieved. This method leads easily to the differential equation for the stiffness tensor

$$(1-y)\frac{d}{dy}\mathbf{L}^*_{DEM} = (\mathbf{L}_i - \mathbf{L}^*_{DEM})\,\mathbf{T}^{*i}, \tag{48}$$

where $y = v_i$ and the initial value of the stiffness tensor is given by $\mathbf{L}^*_{DEM}(y=0) = \mathbf{L}_h$. This formula may be separated into coupled equations for the bulk and shear moduli:

$$(1-y)\frac{d}{dy}K^*_{DEM} = \left(K_i - K^*_{DEM}\right) P^{*i} \tag{49}$$

and

$$(1-y)\frac{d}{dy}\mu^*_{DEM} = \left(\mu_i - \mu^*_{DEM}\right) Q^{*i}. \tag{50}$$

The coupling between these equations occurs in the factors P and Q (see Table 1 for examples). The coupled system is easily integrated using a Runge-Kutta scheme.

To check whether the DEM gives consistent results when applied to the compliance tensor, consider the analogue of (48) for the compliance

$$(1-y)\frac{d}{dy}\mathbf{M}^*_{DEM} = \left(\mathbf{M}_i - \mathbf{M}^*_{DEM}\right) \mathbf{W}^{*i}, \tag{51}$$

where $\mathbf{W}^{*i} = \mathbf{L}_i \mathbf{T}^{*i} \mathbf{M}^*$ and the initial value of the compliance tensor is given by $\mathbf{M}^*_{DEM}(y = 0) = \mathbf{M}_h$. Equation (51) implies that

$$(1-y)\frac{d}{dy}\left(\frac{1}{K^*_{DEM}}\right) = \left(\frac{1}{K_i} - \frac{1}{K^*_{DEM}}\right) K_i P^{*i} \frac{1}{K^*_{DEM}} \tag{52}$$

and

$$(1-y)\frac{d}{dy}\left(\frac{1}{\mu^*_{DEM}}\right) = \left(\frac{1}{\mu_i} - \frac{1}{\mu^*_{DEM}}\right) \mu_i Q^{*i} \frac{1}{\mu^*_{DEM}}. \tag{53}$$

Both equations (52) and (53) are easily shown to reduce respectively to (49) and (50), so DEM does satisfy this internal consistency requirement. This result is not surprising, since any theory that is realizable must possess this property.

It is not difficult to show that the generalized DEM of Norris (1985) also satisfies this consistency requirement.

Assume that there are two types of inclusions labelled type-1 and type-2 imbedded in a host material. The changes in the volume fractions of the inclusions with a timelike parameter t are prescribed by definite functions $v_1(t)$ and $v_2(t)$. Then, the differential equation for the stiffness tensor is

$$\begin{aligned}\dot{\mathbf{L}}^*(t) &= \left(\mathbf{L}_1 - \mathbf{L}^*\right) \mathbf{T}^{*1}\dot{v}_1 + \left(\mathbf{L}_2 - \mathbf{L}^*\right) \mathbf{T}^{*2}\dot{v}_2 \\ &\quad + \left[\left(\mathbf{L}_1 - \mathbf{L}^*\right) \mathbf{T}^{*1} v_1 + \left(\mathbf{L}_2 - \mathbf{L}^*\right) \mathbf{T}^{*2} v_2\right] \frac{\dot{v}}{1-v},\end{aligned} \tag{54}$$

where $v(t) = v_1(t) + v_2(t)$ and $\mathbf{L}^*(0) = \mathbf{L}_h$. The analogous equation for the compliance tensor is

$$\begin{aligned}\dot{\mathbf{M}}^*(t) &= \left(\mathbf{M}_1 - \mathbf{M}^*\right) \mathbf{W}^{*1}\dot{v}_1 + \left(\mathbf{M}_2 - \mathbf{M}^*\right) \mathbf{W}^{*2}\dot{v}_2 \\ &\quad + \left[\left(\mathbf{M}_1 - \mathbf{M}^*\right) \mathbf{W}^{*1} v_1 + \left(\mathbf{M}_2 - \mathbf{M}^*\right) \mathbf{W}^{*2} v_2\right] \frac{\dot{v}}{1-v},\end{aligned} \tag{55}$$

where $\mathbf{W}^{*i} = \mathbf{L}_i \mathbf{T}^{*i} \mathbf{M}^*$.

References

Avellaneda, M., (1987), Iterated homogenization, differential effective medium theory and applications, *Commun. Pure Appl. Math. 40*, 527–554.

Benveniste, Y. (1987), A new approach to the application of Mori-Tanaka's theory in composite materials, *Mech. Mat. 6*, 147–157.

Berge, P. A., J. G. Berryman, and B. P. Bonner (1993), Influence of microstructure on rock elastic properties, *Geophys. Res. Lett. 20*, 2619–2622.

Berge, P. A., B. P. Bonner, and J. G. Berryman (1995), Ultrasonic velocity-porosity relationships for sandstone analogs made from fused glass beads, *Geophysics 60*, 108–119.

Berge, P. A., G. J. Fryer, and R. H. Wilkens (1992), Velocity-porosity relationships in the upper oceanic crust: Theoretical considerations, *J. Geophys. Res. 97*, 15239–15254.

Berryman, J. G. (1980), Long-wavelength propagation in composite elastic media II. Ellipsoidal inclusions, *J. Acoust. Soc. Am. 68*, 1820–1831.

Berryman, J. G. (1992), Single-scattering approximations for coefficients in Biot's equations of poroelasticity, *J. Acoust. Soc. Am. 91*, 551–571.

Berryman, J. G. (1994), Role of porosity in estimates of composite elastic constants, *Trans. ASME J. Energy Resources Tech. 116*, 87–96.

Berryman, J. G., and P. A. Berge (1993), Rock elastic properties: Dependence on microstructure, in: C. S. Cheng and J. W. Ju, eds., *Homogenization and Constitutive Modeling for Heterogeneous Materials*, AMSE, New York, pp. 1–13.

Budiansky, B. (1965), On the elastic moduli of some heterogeneous materials, *J. Mech. Phys. Solids 13*, 223–227.

Cheng, C. H. A. (1978), *Seismic Velocities in Porous Rocks: Direct and Inverse Problems*, Ph.D. thesis, MIT, Cambridge, MA.

Cheng, C. H., and M. N. Toksöz (1979), Inversion of seismic velocities for the pore aspect ratio spectrum of a rock, *J. Geophys. Res. 84*, 7533–7543.

Christensen, R. M. (1990), A critical evaluation for a class of micromechanics models, *J. Mech. Phys. Solids 38*, 379–404.

Cleary, M. P., L.-W. Chen, and S.-M. Lee (1980), Self-consistent techniques for heterogeneous media, *ASCE J. Engng. Mech. 106*, 861–887.

Dvorak, G. J., and Y. Benveniste (1992), On transformation strains and uniform fields in multiphase elastic media, *Proc. R. Soc. London* **A 437**, 291–310.

Eshelby, J. D. (1957), The determination of the elastic field of an ellipsoidal inclusion and related problems, *Proc. Roy. Soc. London Ser. A 241*, 376–396.

Fate, W. A. (1975), High temperature elastic moduli of polycrystalline silicon nitride, *J. Appl. Phys. 46*, 2375–2377.

Ferrari, M. (1991), Asymmetry and the high concentration limit of the Mori-Tanaka effective medium theory, *Mech. Mat. 11*, 251–256.

Ferrari, M., and M. Filipponi (1991), Appraisal of current homogenizing techniques for the elastic response of porous and reinforced glass, *J. Am. Ceramic Soc. 74*, 229–231.

Fisher, E. S., M. H. Manghnani, and J. L. Routbort (1991), A study of the elastic properties of Al_2O_3 and Si_3N_4 matrix composites with SiC whisker reinforcement, in: S. K. Das, C. P. Ballard, and F. Marikar, eds., *High Performance Composites for the 1990's*, The Minerals, Metals and Materials Society, Warrendale, PA, pp. 365–370.

Fisher, E. S., M. H. Manghnani, and J.-F. Wang (1992), Elastic properties of Al_2O_3 and Si_3N_4 matrix composites with SiC whisker reinforcement, *J. Am. Ceramics Soc. 75*, 908–914.

Gubernatis, J. E., and J. A. Krumhansl (1975), Macroscopic engineering properties of polycrystalline materials: Elastic properties, *J. Appl. Phys. 46*, 1875–1883.

Hashin, Z., and S. Shtrikman (1963), A variational approach to the elastic behavior of multiphase materials, *J. Mech. Phys. Solids 11*, 127–140.

Hill, R. (1963), Elastic properties of reinforced solids: Some theoretical principles, *J. Mech. Phys. Solids 11*, 357–372.

Hill, R. (1965), A self-consistent mechanics of composite materials, *J. Mech. Phys. Solids 13*, 213–222.

Hori, M., and S. Nemat-Nasser (1993), Double-inclusion model and overall moduli of multiphase composites, *Mech. Mat.* **14**, 189–206.

Ju, J. W., and T. M. Chen (1994a), Micromechanics and effective moduli of elastic composites containing randomly dispersed ellipsoidal inhomogeneities, *Acta Mechanica* **103**, 103–121.

Ju, J. W., and T. M. Chen (1994b), Effective elastic moduli of two-phase composites containing randomly dispersed spherical inhomogeneities, *Acta Mechanica* **103**, 123–144.

Korringa, J., R. J. S. Brown, D. D. Thompson, and R. J. Runge (1979), Self-consistent imbedding and the ellipsoidal model for porous rocks, *J. Geophys. Res. 84*, 5591–5598.

Kuster, G. T., and M. N. Toksöz (1974), Velocity and attenuation of seismic waves in two-phase media: I. Theoretical formulation, *Geophysics 39*, 587–606.

Milton, G. W. (1985), The coherent potential approximation is a realizable effective medium theory, *Commun. Math. Phys. 99*, 463–500.

Mori, T., and K. Tanaka (1973), Average stress in matrix and average elastic energy of materials with misfitting inclusions, *Acta Metal. 21*, 571–574.

Nemat-Nasser, S., and M. Hori (1993), *Micromechanics: Overall Properties of Heterogeneous Materials*, North-Holland, Amsterdam, pp. 345–357.

Nemat-Nasser, S., N. Yu, and M. Hori (1993), Bounds and estimates of overall moduli of composites with periodic microstructure, *Mech. Mat.* **15**, 163–181.

Norris, A. N. (1985), A differential scheme for the effective moduli of composites, *Mech. Mat. 4*, 1–16.

Norris, A. N. (1989), An examination of the Mori-Tanaka effective medium approximation for multiphase composites, *ASME J. App. Mech. 56*, 83–88.

Toksöz, M. N., C. H. Cheng, and A. Timur (1976), Velocities of seismic waves in porous rocks, *Geophysics 41*, 621–645.

Walpole, L. J. (1969), On the overall elastic moduli of composite materials, *J. Mech. Phys. Solids 17*, 235–251.

Walsh, J. B. (1969), New analysis of attenuation in partially melted rock, *J. Geophys. Res. 74*, 4333–4337.

Walsh, J. B., W. F. Brace, and A. W. England (1965), Effect of porosity on compressibility of glass, *J. Am. Ceramic Soc.* **48**, 605–608.

Weng, G. J. (1984), Some elastic properties of reinforced solids, with special reference to isotropic ones containing spherical inclusions, *Internat. J. Engng. Sci. 22*, 845–856.

Wneg, G. J. (1990), The theoretical connection between Mori-Tanaka's theory and the Hashin-Shtrikman bounds, *Int. J. Eng. Sci.* **28**, 1111–1120.

Wilkens, R. H., G. J. Fryer, and J. Karsten (1991), Evolution of porosity and seismic structure of upper oceanic crust: Importance of aspect ratios, *J. Geophys. Res. 96*, 17981–17995.

Willis, J. R. (1977), Bounds and self-consistent estimates for the overall properties of anisotropic composites, *J. Mech. Phys. Solids* **25**, 185–202.

Wu, T. T. (1966), The effect of inclusion shape on elastic moduli of a two-phase material, *Internat. J. Solids Structures 2*, 1–8.

Zimmerman, R. W. (1991a), Elastic moduli of a solid containing spherical inclusions, *Mech. Mat. 21*, 17–24.

Zimmerman, R. W. (1991b), *Compressibility of Sandstones*, Elsevier, Amsterdam, Chapter 11.

TABLE 1. Three examples of coefficients P and Q for spherical and nonspherical scatterers. The superscripts h and i refer to host and inclusion phases, respectively. Special characters are defined by $\beta = \mu[(3K+\mu)/(3K+4\mu)]$, $\gamma = \mu[(3K+\mu)/(3K+7\mu)]$, and $\zeta = (\mu/6)[(9K+8\mu)/(K+2\mu)]$. The expression for spheres and needles were derived by Wu (1966) and Walpole (1969). The expressions for penny-shaped cracks were derived by Walsh (1969) and assume $K_i/K_h << 1$ and $\mu_i/\mu_h << 1$. The aspect ratio of the cracks is α.

Inclusion shape	P^{hi}	Q^{hi}
Spheres	$\frac{K_h+\frac{4}{3}\mu_h}{K_i+\frac{4}{3}\mu_h}$	$\frac{\mu_h+\zeta_h}{\mu_i+\zeta_h}$
Needles	$\frac{K_h+\mu_h+\frac{1}{3}\mu_i}{K_i+\mu_h+\frac{1}{3}\mu_i}$	$\frac{1}{5}\left(\frac{4\mu_h}{\mu_h+\mu_i}+2\frac{\mu_h+\gamma_h}{\mu_i+\gamma_h}+\frac{K_i+\frac{4}{3}\mu_h}{K_i+\mu_h+\frac{1}{3}\mu_i}\right)$
Penny cracks	$\frac{K_h+\frac{4}{3}\mu_i}{K_i+\frac{4}{3}\mu_i+\pi\alpha\beta_h}$	$\frac{1}{5}\left(1+\frac{8\mu_h}{4\mu_i+\pi\alpha(\mu_h+2\beta_h)}+2\frac{K_i+\frac{2}{3}(\mu_i+\mu_h)}{K_i+\frac{4}{3}\mu_i+\pi\alpha\beta_h}\right)$

Table 2. General characteristics of various approximation schemes.

Approximation Scheme	Reference Material	Embedding Material	Embedded Material	Scattering Analogy	Realizable Model
KT	host	host	inclusions/composite	yes	no
MT	host	host	inclusions & host	no	no
Wu	host	composite	inclusions & host	no	no
DEM	host	composite	inclusions only	yes	yes
SC	composite	composite	all constituents	yes	yes

FIGURE CAPTIONS

Figure 1: Bulk moduli of porous glass foam computed from compressibilities measured by Walsh *et al.* (1965). Measured values (triangles) are compared to theoretical predictions of the Mori-Tanaka theory and the Kuster-Toksöz (1974) theory, which happen to coincide for composites with spherical inclusions. The same solid curve is also obtained when either theory assumes the voids are spherical while the glass particles take any other ellipsoidal shape. Dashed lines were computed assuming spherical glass and needle-shaped voids, while dotted lines assumed spherical glass with penny-shaped voids.

Figure 2: (a) Bulk moduli (dots) of dry sintered glass bead samples measured by Berge *et al.* (1993, 1995). Measured values are compared to predictions of the KT and MT theories for spherical inclusions (solid line), needle-shaped inclusions (dashed lines), and penny-shaped inclusions (dotted lines). (b) Shear moduli (dots) of water-saturated sintered glass bead samples measured by Berge *et al.* (1993, 1995). Displayed curves of the KT and MT theories use the same line patterns as described in (a).

Figure 3: (a) Bulk moduli (dots) of water-saturated sintered glass bead samples of Berge *et al.* (1993, 1995), and (b) shear moduli (dots) of these samples. Theoretical curves presented as in Figure 2.

Figure 4: (a) Bulk moduli (squares) and (b) shear moduli (squares) of porous silicon nitride samples of Fate (1975) compared to predictions of KT and MT theories. Line patterns for theoretical results are as in Figures 2 and 3. The moduli for a pure silicon nitride sample of Fisher *et al.* (1991, 1992) are also plotted as squares on the zero porosity axes in (a) and (b).