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On Elasticity of Porous Media

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Abstract

I. Subject of the paper: There are perfect and imperfect elastic media. Many of the media of imperfect elasticity are at least what we call of "differential elasticity," *i.e.*, they behave like media of perfect elasticity if they are subject to sufficiently small variations of stress. Elastic waves are transmitted through media of differential elasticity without perceptible loss of energy and without leaving changes of state in the medium. In the present paper, a special class of media of differential elasticity is dealt with, namely the porous media. Such a porous medium consists of a solid phase with a liquid or gaseous filling of the pores. The solid phase may be either a connected frame or an unsolidified aggregate of grains. II. On elasticity of isotropic media: The differential elasticity of an isotropic medium is described by two moduli of elasticity, which in general depend upon the average strain. Each other modulus of elasticity may be calculated from two of them. [See tabula (24).] III. Characteristic quantities of the porous system: We have to distinguish the following notions relating to the porous system: (a) the solid matter of which the frame is built (index $\hat{}$), (b) the frame, which is macroscopically isotropic (index $\bar{}$), (c) the filling of the pores (index $\tilde{}$), and (d) the porous system, consisting of the frame and the filling of the pores (no index). The proportions of the volumina and of the masses are described by the porosity and by different densities. IV. The stress of the porous system: It is divided into a hydrostatic pressure $\tilde{p}$, which reigns in the frame as well as in the pores, and an additional tensor stress $\bar{p}$, which reigns only in the frame. V. Differential elasticity of the open system: If the system is stressed while the pores are open on the surface, we obtain the elastic properties of the frame. VI. Differential elasticity of the closed system: If the pores are closed on the surface or if elastic waves run through the inner parts of a three-dimensional medium, the filling of the pores cannot circulate in case of small variations of stress. In that case, the system is called a closed system. The elastic moduli of the closed system stressed within the limits of differential elasticity may be calculated from the moduli of the solid material, of the frame, and of the filling of the pores, and from the porosity by the formulas (57) and (59). This enables us to calculate the velocities of the elastic longitudinal and transverse waves running through the porous medium [see (60)]. VII. Numerical example: By the developed theory, the difference between the velocities of longitudinal waves through dry sandstone and through sandstone saturated with water has been calculated. VIII. Porous system with anisotropic frame: Generalizing the method of section VI, the elastic moduli of the closed system can be calculated too in the case of an anisotropic frame. The results are shown in formulas (90) and (92).

I. Subject of the Paper

(*1*) There are materials which, within a more or less wide range of applied stress, may be considered as ideally elastic material, *i.e.*, materials for which a definite proportionality exists between the state of stress and the associated state of deformation. Examples of such materials at room temperature include the hard metals, most minerals, and frictionless liquids.

(*2*) There are other materials whose behavior departs considerably from that of ideally elastic materials, *e.g.*, such as plastics in which an applied stress causes irreversible deformation. These materials also include polyphase systems, such as partially lithified porous bodies or loose aggregates of grains in which the pores or interstices are filled with liquids or gases. With the application of stress to such systems, irreversible changes of state occur, perhaps because pore fluids migrate or the packing of grains in the aggregate is altered.

(*3*) Now it is a fact that even such non-ideally elastic systems behave as ideally elastic systems when the (changing state of) stress produces small variations about a given mean state of deformation; *i.e.*, the deformations that would be produced by such small stress variations are definitely reversible and proportional to them. Such stress variations occur, *e.g.*,, when elastic waves travel through the system. It is of course a well-known fact that when elastic waves, compressional as well as shear waves, propagate through a very weakly elastic system (*e.g.*, sand, loam, gravel) the energy losses or permanent changes in the system are negligible, just as in ideally elastic media.

(*4*) The present investigation will consider the elastic behavior of porous media under small stress variations. For a porous medium, understand a polyphase system consisting of a solid skeleton (framework) that is either a connected, porous solid body or a loose aggregate of solid grains wherein the pores filled with liquid or gas. Stress variations in such porous materials play a role, *e.g.*, in the dynamic stress-loading of certain building materials, as well as in geomechanics, acoustics, and seismics. An understanding of elastic behavior also forms a necessary foundation for the investigation of the mechanical processes by which the behavior of the observed media departs from ideally elastic behavior.

II. On Elasticity of Isotropic Media

(*5*) Let the skeleton (framework) of a porous medium consist of solid material which is elastic and isotropic. As a whole, let the skeleton (framework) be macroscopically isotropic as well. Finally, let the liquids and gases filling the pores also be isotropic. Next, the concepts from the theory of elasticity that we consider necessary concerning the elasticity of isotropic media will be listed [compare for example to Reference 5], along with an introduction to the symbols to be used later.

(*6*) Let x_1, x_2, and x_3 be the axes of a fixed rectangular coordinate system in space, and let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ be the unit vectors directed along these axes. The state of stress at a point of the medium is described by the stress vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$.

$$\mathbf{p}_1 = \mathbf{e}_1 p_1 + \mathbf{e}_2 p_6 + \mathbf{e}_3 p_5 = \text{the stress vector acting on a surface element normal to the } x_1 \text{ axis.}$$

$$\mathbf{p}_2 = \mathbf{e}_1 p_6 + \mathbf{e}_2 p_2 + \mathbf{e}_3 p_4 = \text{the stress vector acting on a surface element normal to the } x_2 \text{ axis.}$$

$$\mathbf{p}_3 = \mathbf{e}_1 p_5 + \mathbf{e}_2 p_4 + \mathbf{e}_3 p_3 = \text{the stress vector acting on a surface element normal to the } x_3 \text{ axis.}$$

(*7*) p_1, p_2, …, p_6 are the components of the stress tensor $\mathbf{P}$,

$$\mathbf{P} = \begin{pmatrix} p_1 & p_6 & p_5 \\ p_6 & p_2 & p_4 \\ p_5 & p_4 & p_3 \end{pmatrix}.$$

p_1, p_2, and p_3 are the normal stresses with compression positive and tension negative; p_4, p_5, and p_6 are the shearing stresses.

(8) $p = \frac{1}{3}(p_1 + p_2 + p_3)$ is the mean normal stress.

(9) If, at all points of a medium, there exists a state of stress $\mathbf{P}$ [which in general varies from point to point], then the medium is in a state of distortion Φ. It is however by no means assumsed that the relationship between Φ and $\mathbf{P}$ will be that of in an ideally elastic body. It need not even be assumed that Φ depends explicitly on $\mathbf{P}$; it may also be that Φ is determined not only from $\mathbf{P}$ itself, but also from the entire previous history of the medium. For example, in a plastic medium, a state of distortion Φ can indeed be caused by a stress $\mathbf{P}$, but the state of distortion remains more or less unaltered when the stress is again returned to zero.

(10) If the stress tensor undergoes a small change $\Delta\mathbf{P}$, *i.e.*, the stress components p_i $(i = 1, 2, \ldots, 6)$ change by small amounts Δp_i, then the state of distortion Φ becomes a state $\Phi + \Delta\Phi$. One can describe the change $\Delta\Phi$ in the neighborhood of a point A in the medium in state Φ, *e.g.*, by drawing three small straight lines the distances a_1, a_2, and a_3 out from A, and parallel to the coordinate axes. These lines, therefore, make right angles with one another. During the transition to the state of distortion $\Phi + \Delta\Phi$, A moves to A', the lines obtain the lengths a_1', a_2', and a_3', and the angles differ from 90° by small amounts. The six quantities:

$$\Delta e_1 = \frac{a_1' - a_1}{a_1}, \quad \Delta e_2 = \frac{a_2' - a_2}{a_2}, \quad \Delta e_3 = \frac{a_3' - a_3}{a_3},$$
$$\Delta e_4 = \frac{\pi}{2} - angle(a_2', a_3'), \quad \Delta e_5 = \frac{\pi}{2} - angle(a_3', a_1'), \quad \Delta e_6 = \frac{\pi}{2} - angle(a_1', a_2'),$$

are the components of the distortion increment $\Delta\Phi$.

(11) A medium behaves ideally elastic under small stress variations $\Delta\mathbf{P}$ when, for sufficiently small but otherwise arbitrary Δp_i, the following linear relations exist between the components of $\Delta\mathbf{P}$ and $\Delta\Phi$, whereby the c_{ij} represent quantities independent of Δe_j and Δp_i:

$$\Delta p_i = -\sum_{j=1}^{6} c_{ij}\Delta e_j \quad i = 1, 2, \ldots, 6.$$

The coefficient matrix c_{ij} is symmetric, *i.e.*, $c_{ij} = c_{ji}$. The c_{ij} are the elastic (stiffness) constants. The medium in this case should be termed "differentially elastic." According to the ideas discussed in paragraphs *(2)* and *(3)*, the c_{ij} generally depend on the state of deformation Φ, which is chosen to be the fixed initial state for the variations $\Delta\Phi$.

(12) If, in a medium, the relationship between Φ and $\mathbf{P}$ is clearly reversible, then the Δp_i are specified functions of the Δe_j after an arbitrary initial state Φ has been (definitely) chosen:

$$\Delta p_i = f_i(\Delta e_1, \Delta e_2, \ldots, \Delta e_6) \quad i = 1, 2, \ldots, 6.$$

A Taylor (series) expansion gives:

$$\Delta p_i = \left(\frac{\partial f_i}{\partial \Delta e_1}\right)_\Phi \Delta e_1 + \left(\frac{\partial f_i}{\partial \Delta e_2}\right)_\Phi \Delta e_2 + \ldots + \left(\frac{\partial f_i}{\partial \Delta e_6}\right)_\Phi \Delta e_6$$
$$+ \frac{1}{2}\left(\frac{\partial^2 f_i}{\partial \Delta e_1^2}\right)_\Phi (\Delta e_1)^2 + \ldots.$$

The index Φ on the parentheses denotes that the values of the (partial) derivatives inside the parentheses must be taken with respect to the initial state Φ, *i.e.*, for $\Delta e_1 = \Delta e_2 = \ldots = \Delta e_6 = 0$. For sufficiently small values of the Δe_j, one can limit the
GDp_i in the Taylor (series) expansion to linear terms. Furthermore, setting $-\left(\frac{\partial f_i}{\partial \Delta e_j}\right)_\Phi = c_{ij}$ gives us the equation from paragraph (*11*) exactly. It is therefore reasonable to apply the concept of a differentially elastic body to this case as well. In cases where the relation between Φ and $\mathbf{P}$ is linear, the c_{ij} are material constants independent of Φ. The medium is then not only a differentially, but also a fully ideally, elastic medium.

(*13*) When the differentially elastic medium of (*11*) is isotropic, then it is permissible to express the stiffness constants c_{ij} in the following manner in terms of two constants λ and μ:

$$
\begin{aligned}
c_{11} &= c_{22} = c_{33} = \lambda + \mu, \\
c_{44} &= c_{55} = c_{66} = \mu, \\
c_{12} &= c_{13} = c_{23} = c_{21} = c_{31} = c_{32} = \lambda,
\end{aligned}
$$

where all other c_{ij}'s vanish. λ and μ are the Lamé elastic constants which, according to paragraph (*12*), are therefore in general dependent on Φ. μ is the shear modulus of the medium.

(*14*) If ρ is the density of the medium, then $w = \sqrt{\mu/\rho}$ is the propagation velocity of elastic shear waves (also called transverse waves).

(*15*) $\Delta p = \frac{1}{3}(\Delta p_1 + \Delta p_2 + \Delta p_3)$ is the mean pressure increment.

(*16*) If a medium, in the state Φ, has a small volume V defined, then it experiences, upon a transition to the state $\Phi + \Delta\Phi$, a volume increment ΔV and the relationship

$$\Delta p = -k\frac{\Delta V}{V},$$

holds, where $k = \lambda + \frac{2}{3}\mu$. k is the bulk modulus (or incompressibility), and $1/k$ is the compressibility.

(*17*) If a specific, reversible relationship exists between V and p, then, according to (*12*), the Δ-quantities may be replaced by differentials, *i.e.*, $k = -V\,dp/dV$.

(*18*) Let a cylindrical core of the medium be enclosed in a container with rigid sidewalls (see Figure 1). The axis of the cylinder lies in the x_1 direction and the length of the cylinder is a_1. An additional pressure Δp_1 is applied to the ends, reducing the length of the cylinder to the value a_1'. From (*11*), the specific extension is $\Delta e_1 = (a_1' - a_1)/a_1 = \Delta a_1/a_1$. [If Δp_1 is positive, *i.e.*, compression, then Δe_1 is negative and so $-\Delta e_1$ is the dilatation]. Therefore, $M = -\Delta p_1/\Delta e_1 = -a_1\Delta p_1/\Delta a_1$ is an additional elastic constant of the medium.

(*19*) If there exists a unique, reversible relationship between the total pressure p_1 in the x_1-direction and the length a_1 of the cylinder, then according to paragraph (*12*), $M = -a_1 dp_1/da_1$.

(*20*) The relation of M to λ and μ is given from the elasticity equations of (*11*). Suppose the cross-section of the cylinder is rectangular with sides parallel to x_2 and x_3. Then, the strain due to Δp_1 gives the additional pressure Δp_2 and Δp_3 on the side faces while the rigidity of the walls is expressed in terms of $\Delta e_2 = \Delta e_3 = 0$. From (*11*) and (*13*), the only three elasticity equations for this case read:

$$
\begin{aligned}
\Delta p_1 &= -(\lambda + 2\mu)\Delta e_1 \\
\Delta p_2 &= \Delta p_3 = -\lambda\Delta e_1.
\end{aligned}
$$

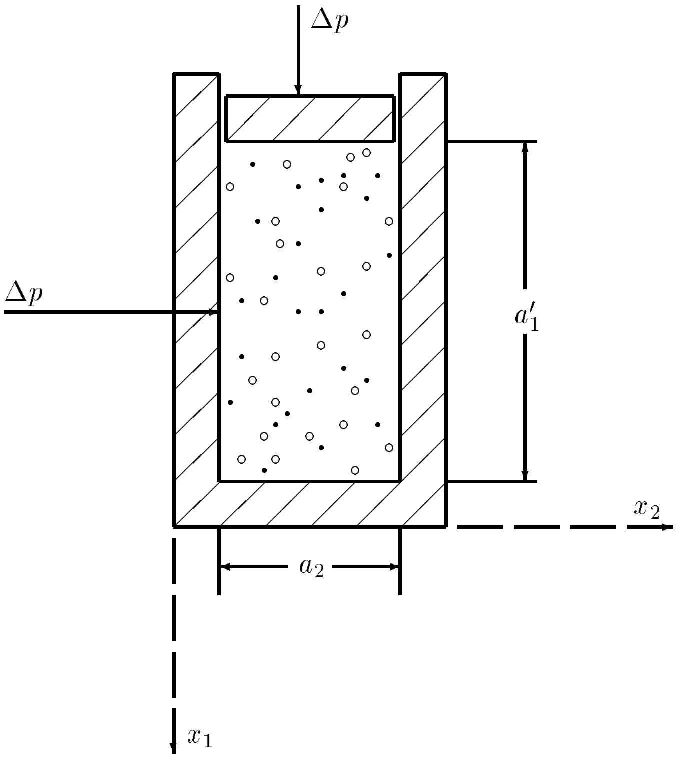

Figure 1:

It follows therefore that $M = \lambda + 2\mu$ and one sees that the pressure experiment presented in Figure 1 yields simultaneously the constants λ and M when one measures the quantities Δp_1, Δp_2, and Δe_1.

(*21*) If this medium is a solid body, then one can carry out the pressure experiment with free side faces, *i.e.*, with the conditions $\Delta p_2 = \Delta p_3 = 0$ (see Figure 2). Then, the original side lengths a_2 and a_3 of the rectangular cross section increase to magnitudes $a_2^{'}$ and $a_3^{'}$. This experiment gives rise to two additional elastic constants E and ν. Again, we have

$$\Delta e_1 = \frac{a_1^{'} - a_1}{a_1} = \frac{\Delta a_1}{a_1} \quad \text{and} \quad \Delta e_2 = \frac{a_2^{'} - a_2}{a_2} = \frac{\Delta a_2}{a_2},$$

so E and ν are defined by the equations:

$$E = -\frac{\Delta p_1}{\Delta e_1} = -a_1 \frac{\Delta p_1}{\Delta a_1}$$

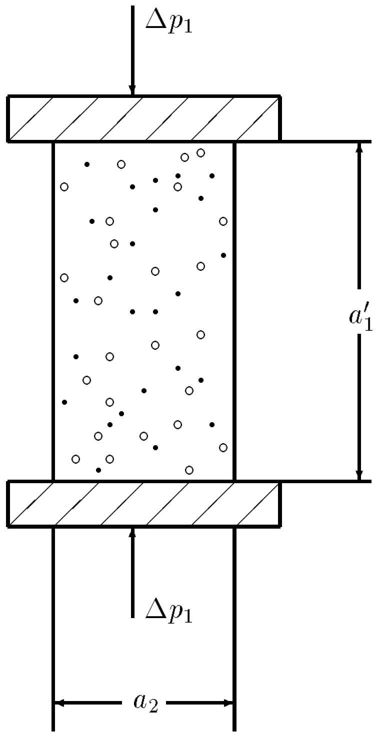

Figure 2:

and

$$\nu = -\frac{\Delta e_2}{\Delta e_1} = -\frac{a_1}{a_2}\frac{\Delta a_2}{\Delta a_1}.$$

E is the modulus of elasticity or Young's modulus, and ν is the Poisson ratio.

(*22*) If there exists a unique, reversible relation between the total pressure p_1 upon the end surfaces of the bar and the elongations a_1 and a_2 corresponding to paragraph (*12*), then

$$E = -a_1\frac{dp_1}{da_1} \quad \text{and} \quad \nu = -\frac{a_1}{a_2}\frac{da_2}{da_1}.$$

(*23*) The use of the elasticity equations (*11*) similar to the preceding in (*20*) gives

$$E = \mu\frac{3\lambda + 2\mu}{\lambda + \mu} \quad \text{and} \quad \nu = \frac{\lambda}{2(\lambda + \mu)}.$$

With the aid of these equations, E and ν can also be defined for cases in which an experiment of the type in Figure 2 is not directly feasible, as perhaps for an unconsolidated granular material, a liquid, or a gas.

(*24*) From any two of the six elastic constants, E, k, M, λ, μ, or ν of a differentially elastic, isotropic medium, the other four elastic constants may be calculated. All possible cases have been included in the Table (entries of which have been calculated by Mr. O. Wyler, Assistant at the Institute for Geophysics).

(*25*) In paragraph (*21*), pressure and tension waves having a velocity $\sqrt{E/\rho}$ propagate forward in the lengthwise direction of the bar with free side surfaces. Again, if the medium occupies infinite space, then the velocity of the pressure and tension waves (also called sound or longitudinal waves) is:

$$v = \sqrt{\frac{M}{\rho}} = \sqrt{\frac{1-\nu}{(1+\nu)(1-2\nu)}}\sqrt{\frac{E}{\rho}};$$

since, in this case, the medium will be strained according to (*18*). The notation and nomenclature in the literature do not always distinguish sufficiently between the two pressure moduli E and M. It is proposed here that E be called "the bar pressure modulus" and M "the volume pressure modulus."

(*26*) Liquids and gases do not transfer shear stresses, so that $\mu = 0$. The elastic behavior will be fixed by a single elastic constant, perhaps M is convenient. From (*24*), it follows that $E = 0$, $k = \lambda = M$, and $\nu = \frac{1}{2}$.

(*27*) For an ideal gas, the value of k may be found by using the equation of state

$$pV^{\kappa} = \text{constant.}$$

For an isothermal change of state, $\kappa = 1$; for an adiabatic change of state, $\kappa = c_p/c_v$, where c_p and c_v signify the specific heats for constant pressure and constant volume, respectively. For air, $c_p/c_v = 1.4$. From paragraph (*17*), considering an ideal gas, we obtain $k = -V\,dp/dV = \kappa p$.

III. Characteristic Quantities of the Porous System

(*28*) The porous system, whose elastic behavior is being investigated in this work, consists of a porous frame of solid material. The pores should communicate among themselves and be filled with either liquid or gas. The system is macroscopically homogeneous and isotropic, *i.e.*, the average mechanical property of a portion of the system, whose dimensions are large compared to the dimensions of the pore cross sections, are the same everywhere in the system and independent of direction. It will turn out that the system behaves like the one treated in Section II under conditions still more precisely to be prescribed, so that the use of the concepts and notation of that section is permitted.

(*29*) The skeleton is porous, but may be continuous solid or an unconsolidated aggregate of hard grains. Macroscopically, the concepts of Section II may be applied to the skeleton (*i.e.*, to the system without pore saturant). The pertinent notation will be provided as the bar index $^{-}$ over the symbol.

(*30*) The solid material from which the frame (skeleton) is built shall also be homogeneous and isotropic. The pertinent quantities will be designated by the caret index $\hat{}$ above the symbols.

(*31*) The pores should be filled with a frictionless liquid or a frictionless gas. The quantities related to the pore saturant will carry the tilde index $\tilde{}$.

(*32*) And so in the discussion of the mechanics of porous systems to follow, the distinct concepts become:

Concept	Notation for Index above a Symbol
a) solid material	$\hat{}$
b) skeleton = solid + pores	$\bar{}$
c) pore saturant	$\tilde{}$
d) system = skeleton + saturant	no index

(*33*) Let V be the volume of a piece cut out of the system, whose dimensions are large compared to the pore cross sections. Then:

$\hat{V} =$ volume of the solid matter contained in V.

$\tilde{V} =$ volume of the communicating pores contained in V.

$n = \dfrac{\tilde{V}}{V} =$ porosity and $V = \hat{V} + \tilde{V}$.

(*34*) For the mass of the system:

$m =$ mass of everything contained in V (bulk mass),

$\hat{m} =$ mass of the solid matter contained in V (solid mass),

$\tilde{m} =$ mass of the fluid saturant contained in V (fluid mass),

$m = \hat{m} + \tilde{m}$.

(*35*) Similarly, for the densities:

$\rho = \dfrac{m}{V} =$ bulk density of the system,

$\hat{\rho} = \dfrac{\hat{m}}{\hat{V}} =$ density of the solid matter,

$\tilde{\rho} = \dfrac{\tilde{m}}{\tilde{V}} =$ density of the fluid saturant,

$\bar{\rho} = \dfrac{\hat{m}}{V} =$ density of the skeleton.

(*36*) $\bar{\rho} = (1-n)\hat{\rho}$ and $\rho = \bar{\rho} + n\tilde{\rho}$ (In petrography and soil mechanics, $g\bar{\rho}$ is the "bulk weight," $g\hat{\rho}$ is the "specific weight of the skeleton," wherein g is the acceleration of gravity.)

(*37*) Cut out from the system a prism with basal area F and height h, therefore having a volume $V = F \cdot h$. F and h should be large compared to the cross section of the pores. A plane parallel to the base of the prism at a distance x from it intersects the prism with a cross-sectional area F. Then, $\hat{F}(x)$ is the portion of the cross section F that is comprised of solid matter and $\tilde{F}(x) = F - \hat{F}(x)$ is the fluid portion of the area. The mean value of $\tilde{F}(x)$ is also nF:

$$\frac{1}{h}\int_0^h \tilde{F}(x)\,dx = \frac{\tilde{V}}{h} = \frac{nV}{h} = nF.$$

We take the variation of $\tilde{F}(x)$ about the mean value to be negligibly small, so we have $\tilde{F}(x) = nF$ and therefore $\hat{F} = (1 - n)F$.

IV. The Stress in the Porous System

(*38*) Consider a point A in the interior of the given system and the environment of A, whose dimensions will be large relative to those of a typical pore cross section, but still small enough that we can assume a uniform hydrostatic pressure $\tilde{p}$ in the fluid occupying the connected pores.

(*39*) A plane passing through A cuts a surface F from the specified environment. N is the direction perpendicular to the plane and $\mathbf{e}_N$ a unit vector in this direction. As in (*37*), $\hat{F}$ will be the area occupied by the solid matter on F and $\tilde{F}$ that of the fluid. Forces will be transferred from one side of the plane to the other; their resultant will be $\mathbf{P}_N$. The resultant of the forces that act across $\hat{F}$, and thus across the solid matter, will be $\hat{P}_N$. The resultant of the forces that act across $\tilde{F}$, and thus across the pore saturant, is obviously $\tilde{F}\tilde{p}\mathbf{e}_N$, so the equation $\mathbf{P}_N = \hat{\mathbf{P}}_N + \tilde{F}\tilde{p}\mathbf{e}_N$ is valid.

(*40*) For the following study, it is advantageous to divide $\hat{\mathbf{P}}_N$ into two parts. One part is due to the hydrostatic pressure $\tilde{p}$. This is the force $\tilde{F}\tilde{p}\mathbf{e}_N$ that acts on the fluid surface $\tilde{F}$, or the total hydrostatic force $F\tilde{p}\mathbf{e}_N$ acting on the entire surface F. Therefore, the decomposition of the force on the solid gives the expression

$$(F - \tilde{F})\tilde{p}\mathbf{e}_N = \hat{F}\tilde{p}\mathbf{e}_N$$

or, breaking $\hat{\mathbf{P}}_N$ into its two component parts,

$$\hat{\mathbf{P}}_N = \hat{F}\tilde{p}\mathbf{e}_N + \bar{\mathbf{P}}_N.$$

(*41*) $\mathbf{P}_N = \bar{\mathbf{P}}_N + F\tilde{p}\mathbf{e}_N$ is the corresponding representation of $\mathbf{P}_N$.

(*42*) Now set $\mathbf{P}_N/F = p_N$ and $\bar{P}_N/F = \bar{p}_N$, so also $p_N = \bar{p}_N + \tilde{p}\mathbf{e}_N$. The general expression of the stresses p_N and $\tilde{p}\mathbf{e}_N$ for all possible directions N always takes the form of a tensor, so that the general expression for $\bar{p}_N$ also forms a tensor $\bar{\mathbf{P}}$ with components $\bar{p}_1$, $\bar{p}_2$, …, $\bar{p}_6$ [with notation corresponding to (*7*)]. Thus, the complete pressure tensor of the skeleton is now known.

V. Differential Elasticity of the Open System

(*43*) Let a portion of volume V be removed from the system. The dimensions of V will be large relative to the pore sizes. The stress state in V will be homogeneous. It is given in terms of the hydrostatic stress $\tilde{p}$ of the pore saturant and in terms of the residual stress tensor $\bar{\mathbf{P}}$ of the frame. It will be assumed that for this stress state, the frame will be differentially elastic. We can establish this while we change the stress state by $\Delta\tilde{p}$ and $\Delta\bar{\mathbf{P}}$, during which the pore saturant must have the opportunity to deform independently from the skeleton. The pores must therefore be open externally (hence, the terminology "open system") and the stress variations carried out so slowly that we will not be obliged to consider the friction of the saturating fluid circulating in the pores nor its sluggish resistance. In this manner, we keep the elastic property of the skeleton.

One may inquire whether this property is not easy to preserve with empty pores. For this reason, it is worthwhile noting that then $\tilde{p} = 0$ and in general the elasticity of the solid material and therefore also of the skeleton may depend on $\tilde{p}$. One must even account for the possibility that the elasticity of the skeleton may depend not only upon the pore pressure $\tilde{p}$, but also upon the character of the pore fluid saturant. For example, it can happen that the elastic properties of the skeleton are different according to whether the pores are filled with air or with water under the same pressure $\tilde{p}$. It is conceivable, for example, that the elasticity of the skeleton will change through accumulation of hygroscopic water on the pore walls. This hygroscopic water distinguishes itself from normal pore water through its larger density, higher pressure, and smaller mobility [for a comparison, see the paper by F. Zunker in Blanck (1930), pp. 66–89]. Additionally, it is here noted that the porosity in this work is determined by means of the volume $\tilde{V}$, which is occupied by pore water of normal pressure; on the other hand, the hygroscopic water is included in the frame material. As long as the volume of the hygroscopic water is negligibly small compared to the volume of the solid material, one is permitted to assume that the material composing the skeleton, is nevertheless isotropic and homogeneous. On the other hand, one is not permitted to make use of this theory without further study for cases in which the pore size is so small, or rather the solid phase is so fine-grained, that the previously neglected effects must be reconsidered.

Consideration must also be given to capillary phenomena. In capillaries, $\tilde{p}$ can be negative (in tension). Furthermore, the liquid and gas will be separated by a meniscus when the pores are partly filled with liquid and partly with gas, or in the case of liquid-filled pores on the surface of the system bordering on the gas-filled external space. The strain of the frame induced by these meniscii is also to be considered in $\bar{\mathbf{P}}$ [one thinks about the pressure dilatation, see Haefeli (1938), section III, Figures 7 and 8]. Besides this, the hydrostatic pressure $\tilde{p}$ in the pores at both sides may have different meniscus values. The volume V considered in the preceding section is therefore not permitted to contain meniscii in the interior which divide the pore volume into non-negligible different partial volumes in which the stress $\tilde{p}$ possesses different values.

(44) When the solid matter, the skeleton, and the pore-fluid saturant are assumed to be differentially elastic, the general stress variations $\Delta\tilde{p}$ and $\Delta\bar{\mathbf{P}}$ can be separated, and the effect of each partial stress treated independently. The effect of a general stress variation will then be found through superposition; that is, simple addition.

(45) $\hat{k}$ is the bulk modulus of the solid matter [see *(16)*, *(17)*, and *(32)*]. Let a stress variation $\Delta\tilde{p} \neq 0$, $\Delta\bar{\mathbf{P}} = 0$ be considered. From *(40)* to *(42)*, this signifies that the skeleton will be exposed on all sides to the additional hydrostatic pressure $\Delta\tilde{p}$. Consequently, the frame shrinks the same fraction at all points, retaining a form similar to its original geometric form. Because of this, we have the self-similarity condition $\Delta\hat{V}/\hat{V} = \Delta V/V$. On the other hand, $\Delta\hat{V}/\hat{V} = -\Delta\tilde{p}/\hat{k}$. Therefore,

$$\frac{\Delta V}{V} = -\frac{\Delta\tilde{p}}{\hat{k}}.$$

(46) A further result of the self-similarity is the constancy of the porosity, that is, for $\Delta\tilde{p} \neq 0$ and $\Delta\bar{\mathbf{P}} = 0$, then $\Delta n = 0$.

(47) The variation of the mean normal stress, $\Delta\bar{p} = \frac{1}{3}(\Delta\bar{p}_1 + \Delta\bar{p}_2 + \Delta\bar{p}_3)$, is due to $\Delta\bar{\mathbf{P}}$.

(48) The variation in stress $\Delta\tilde{p} = 0$, $\Delta\bar{\mathbf{P}} \neq 0$, and $\Delta\bar{p} = 0$ gives a deformation of the frame without a volume change, that is, $\Delta V = 0$. Such a deformation may be described by a constant $\bar{\mu}$, the shear modulus of the skeleton. Further, it is easy to make the obvious hypothesis for this stress variation that $\Delta n = 0$.

(49) Finally, consider a stress variation $\Delta\tilde{p} = 0$, $\Delta\bar{p} \neq 0$. The bulk modulus $\bar{k}$ of the skeleton can be determined by measuring ΔV, namely

$$\bar{k} = -V\frac{\Delta\bar{p}}{\Delta V}.$$

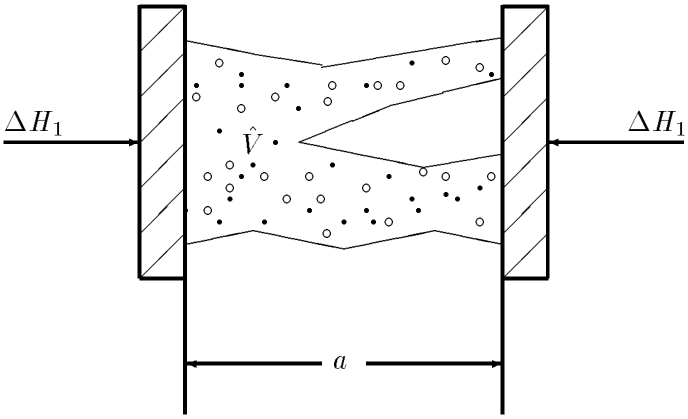

Figure 3:

(*50*) In order to determine the porosity variation Δn, which corresponds to the stress variation of (*49*), an integration theorem from elasticity theory is used [see Love (1907), p. 207, theorem 3]. This theorem, with notation adjusted to that of our present case, has the following content: a homogeneous, isotropic, differentially elastic body of arbitrary shape, of volume $\hat{V}$ and possessing the bulk modulus $\hat{k}$, will be compressed between two parallel planes separated by distance a by the impressed force ΔH_i (see Figure 3). The volume decrease (when decrease is reckoned as negative) is then

$$\Delta \hat{V}_i = -\frac{a \Delta H_i}{3\hat{k}}.$$

(*51*) In the case under consideration, the volume $\hat{V}$ is the solid part of the skeleton of our porous system. Remove from this system a cube of volume V [see the beginning of (*43*)] with edges of length a and with those edges parallel to the coordinate axes. A variation in stress (*49*) causes the three opposing pairs of surfaces of the cube to be compressed by the resulting pressure forces $\Delta H_i = a^2 \Delta \bar{p}_i$, $i = 1, 2, 3$. Applying the theorem (*50*) three times and summing the small volume changes gives:

$$\Delta \hat{V} = \sum_{i=1}^{3} \Delta \hat{V}_i = -\frac{a^3}{3\hat{k}} \sum_{i=1}^{3} \Delta \bar{p}_i.$$

Recalling (*47*) and also that $a^3 = V$, it follows that

$$\Delta\hat{V} = -V\frac{\Delta\bar{p}}{\hat{k}}.$$

(*52*) From (*33*), $\hat{V} = (1-n)V$. When the Δ-quantities are small, one can handle them to a satisfactory approximation as differentials, so that $\Delta\hat{V} = (1-n)\Delta V - V\Delta n$, thus $\Delta\hat{V}/V = (1-n)\Delta V/V - \Delta n$. By using (*51*) and (*49*), it therefore follows:

$$\Delta n = \left(\frac{1}{\hat{k}} - \frac{1-n}{\bar{k}}\right)\Delta\bar{p}.$$

VI. Differential Elasticity of the Closed System

(*53*) As in section **V**, the differentially elastic behavior of a volume V of the system will be studied, while it is under the influence of initial stresses $\tilde{p}$ and $\mathbf{P}$, and experiencing the additional stresses $\Delta\tilde{p}$, $\Delta\bar{\mathbf{P}}$. In contrast to the discussion of section **V**, the external channels of the pore saturant from V will be sealed and flow from outside to the interior of V prevented.

(*54*) When V is an isolated part of the system in the load test to be undertaken, the pores on the surface must be closed (hence the name "closed system").

(*55*) One can imagine that V is a constituent of a very much larger portion V_0 of the system, with V lying in the interior so far from the surface of V_0 that the load variations $\Delta\tilde{p}$ and $\Delta\bar{\mathbf{P}}$ produce no perceptible flow through the surface of V. The minimum distance that V must be removed from the surface of V_0 in order to fulfill these conditions depends on the rate of the load variation. For sudden changes in the load, the distance does not need to be as large as for slow variations. An example of a situation corresponding to (*55*) is that of elastic waves propagating through the interior of V_0.

(*56*) The closed system V is differentially elastic for sufficiently small load variations and we may assume that there are no flows of pore fluid that could influence the elastic processes. Since the system is assumed to be microscopically isotropic, the theory of section **II** can be applied. Following (*32*) d), the notation of that section will be adapted. The elastic behavior of the system will be described in terms of two elastic constants. Here the two constants k and μ will be chosen. As the circumstances demand, others may be calculated from the table of (*24*). Let us once more be reminded, using section **II** as a starting point, that the deformation change $\Delta\Phi$ will be brought about by a load variation $\Delta\tilde{p}$, $\Delta\bar{\mathbf{P}}$, where a deformation state Φ will be assumed, caused by the action of a stress state $\tilde{p}$, $\bar{\mathbf{P}}$ (= initial stress), and that the elastic constants depend on Φ and therefore on $\tilde{p}$ and $\bar{\mathbf{P}}$.

(*57*) Corresponding to (*48*), the stress variation $\Delta\tilde{p} = 0$, $\Delta\bar{\mathbf{P}} \neq 0$, $\Delta\bar{p} = 0$ is considered. This gives a deformation change without a change in the volume V. It is therefore obvious to take an additional hypothesis that, for such a stress variation, whether the system is open or closed, the deformation change is independent of the choice of μ or $\bar{\mu}$; that is, we can take $\mu = \bar{\mu}$.

(*58*) In order to determine k, expose this volume on all surfaces to an additional hydrostatic pressure Δp. According to (*40*) through (*42*), Δp will be composed of two parts. $\Delta\tilde{p}$ is the portion of Δp acting on *all* the surface elements of V, that also determines the additional hydrostatic pressure in the pores. $\Delta\bar{p}$ is the remainder of Δp, that acts only upon the elements of the surface defined as the solid. Then:

a) $\Delta p = \Delta\tilde{p} + \Delta\bar{p}$. The desired bulk modulus of the closed system is defined by the next equation:

b) $\Delta V/V = -\Delta p/k$.

c) $\Delta\tilde{V}/\tilde{V} = -\Delta\tilde{p}/\tilde{k}$ describes the compressibility of the pore fluid saturant.

d) $\Delta V/V = -\Delta\tilde{p}/\hat{k} - \Delta\bar{p}/\bar{k}$ determines the compressibility of the frame caused by both constituents of Δp in a), according to results of (*45*) and (*49*).

e) $\Delta\hat{V}/V = -(1-n)\Delta\tilde{p}/\hat{k} - \Delta\bar{p}/\hat{k}$ describes the compressibility of the solid materials from (*45*) and (*51*) using also $\hat{V} = (1-n)V$.

f) $\tilde{V} = nV$ (definition of porosity).

g) $\Delta V = \Delta\hat{V} + \Delta\tilde{V}$ since the system is closed and pore fluid saturant can neither enter nor exit.

(*59*) From the seven equations (*58*) a)-g), six quantities, $\tilde{V}$, ΔV, $\Delta\tilde{V}$, $\Delta\hat{V}$, $\Delta\tilde{p}$, and $\Delta\bar{p}$ will be eliminated. The quantity V then also drops out and we obtain for the compressional modulus of the closed system

$$k = \hat{k}\left(\frac{\bar{k}+Q}{\hat{k}+Q}\right), \quad \text{where} \quad Q = \frac{\hat{k}(\hat{k}-\bar{k})}{n(\hat{k}-\tilde{k})}.$$

(*60*) Using (*57*) and (*59*), the elastic constants μ and k of the closed system are expressed in terms of the porosity and in terms of the elastic constants of the solid materials, the skeleton, and the pore fluid saturant. From (*36*), the density of the system is $\rho = \bar{\rho} + n\tilde{\rho}$ and from (*24*), $M = k + 4\mu/3$ is its volume compressibility modulus. Finally, from (*14*) and (*25*), the velocities v and w of longitudinal and transverse waves propagating through the system are given by

$$v = \sqrt{\frac{M}{\rho}} \quad \text{and} \quad w = \sqrt{\frac{\mu}{\rho}}.$$

(*61*) In the special case where the bulk modulus $\hat{k}$ of the saturating fluid is negligibly small (*59*) becomes $k = \bar{k}$.

(*62*) If $n = 0$, then also $k = \hat{k}$.

(*63*) If the pore fluid saturant has the same compressiblity as the solid matter, then $k = \tilde{k} = \hat{k}$.

(*64*) If the compressibility of the solid matter is negligible with respect to that of the pore fluid saturant, then we may set $\hat{k} = \infty$. (*59*) then yields $k = \bar{k} + \tilde{k}/n$.

VII. Numerical Example

(*65*) As experience has shown, the propagation velocity of elastic waves through a given rock is dependent on its water content. It is possible, as noted in (*43*), that this dependence can come partly from the fact that the nature of the pore-fluid saturant influences the elastic behavior of the frame. To establish such an influence, it is necessary to know the dependence existing between the propagation velocity and the density and elastic behavior of the pore-fluid saturant. This dependence is calculable from our analysis. A numerical example of such a calculation will be given.

(*66*) Concerning a sandstone under normal pressure and temperature, the following quantities are given by Grubenmann *et al.* (1915), p. 118, rock sample #1200 (numerical data without specified units signifies units of the cgs system):

Apparent porosity	13.3 %
True porosity	17.1%
Bulk density of dry rocks	2.23 g/cm^3.

Further, from Röthlisberger (1949):

Longitudinal wave velocity in dry rocks	2.3 km/s
Shear wave velocity in dry rocks	1.3 km/s

(*67*) As in (*32*) d), the quantities related to the entire system will be denoted without an index over the symbols. In the example under consideration, there are two cases to be distinguished, namely the dry rock (skeleton plus air as pore saturant) and the water saturated rock (skeleton plus water as the pore fluid). The quantities referring to the dry rock will be denoted with the subscript t ["t," since "trocken" is the German word for "dry" – *eds.*] and the quantities for the water saturated rock will remain without subscripts.

(*68*) The capacity of the rocks to take on water up to full saturation is measured by the apparent porosity, which is $n = 0.133$. Included in the true porosity, $n_w = 0.171$, are the pores inaccessible to water, that are accounted for within the solid material in this theory. If V is the total volume of a rock sample, $\tilde{V}$ the volume of the pores accessible to water, and $\tilde{V}'$ the volume of the pores inaccessible to water, then

$$n = \frac{\tilde{V}}{V},$$

$$n_w = \frac{\tilde{V} + \tilde{V}'}{V}, \quad [\text{"}w\text{," since "wahr" is German for "true" – } \textit{eds.}],$$

$$n_f = \frac{\bar{V}'}{V - \tilde{V}} = \frac{n_w - n}{1 - n} = 0.044 \quad [\text{"}f\text{," since "fest" is German for "solid" – } \textit{eds.}]$$

with the latter expression being the porosity of the solid material.

(*69*) $\tilde{\rho}_f = 0.0013$ is the density of air. From (*36*), the density of the rock with air-filled pores is $\rho_t = 2.23 = \bar{\rho} + n\tilde{\rho}_t$. We see that $\rho_t \simeq \bar{\rho} = 2.23$.

(*70*) From (*36*), $\hat{\rho} = \bar{\rho}/(1-n) = 2.57$ is the density of the solid matter.

(*71*) The density $\hat{\rho}$ and the porosity n_f of the solid matter, as well as its mineral composition, justifies taking the elastic constants of the solid to be the values for granite. Thus, we take $k = 25 \times 10^{10}$ [for elastic constants of rocks, see Birch *et al.* (1942) or Niggli (1948)].

(*72*) From static experiments on sandstones with open pores, it is well known that the bulk modulus $\bar{k}$ of the frame lies between 10^{10} and 10^{11}. On the other hand, the bulk modulus of dry air at atmospheric pressure, from (*27*), has the order of magnitude 10^6. The dry sandstone corresponds to (*61*) above; that is the elasticity of a dry sandstone (frame plus air) may be assumed to be the same as the elasticity of the frame. The elastic constants are to be determined from the known velocities:

$$v_t = 2.3 \times 10^5 = \sqrt{\frac{\bar{M}}{\bar{\rho}}}$$

and

$$w_t = 1.3 \times 10^5 = \sqrt{\frac{\bar{\mu}}{\bar{\rho}}}.$$

Therefore, it follows that $\bar{M} = 11.8 \times 10^{10}$, $\bar{\mu} = 3.77 \times 10^{10}$, and, from (*24*),

$$\bar{k} = \bar{M} - 4\bar{\mu}/3 = 6.8 \times 10^{10}.$$

(*73*) The propagation velocity of longitudinal waves in water is 1.435 km/s and the density of water is $\tilde{\rho} = 1$. From (*25*) and (*26*), it follows that $\tilde{k} = \tilde{M} = \tilde{\rho}\tilde{v}^2 = 2.06 \times 10^{10}$.

(*74*) All quantities that are needed to calculate k for water-saturated sandstone from (*59*) are now known. We obtain:

$$Q = 1.23 \times 10^{10} \qquad \text{and} \qquad k = 1.28 \times 10^{10}.$$

From (*57*) and (*72*), $\mu = \bar{\mu} = 3.77 \times 10^{10}$; from (*36*) and (*69*), $\rho = \bar{\rho} + n\tilde{\rho} = 2.36$; and finally the propagation velocity of longitudinal waves in water-saturated sandstones is $v = \sqrt{(k + 4\mu/3)/\rho}$. We obtain $v = 2.75\,km/s$, compared to $v_t = 2.3\,km/s$ for dry sandstone, assuming the elasticity of the skeleton is the same in both cases.

VIII. Porous System with Anisotropic Frame

(*75*) In sections **III-VII**, the frame has been assumed microscopically isotropic in the manner of (*28*). In that case, the differential frame elasticity is described in terms of the two constants $\bar{k}$ and $\bar{\mu}$, introduced in (*48*) and (*49*).

In contrast to the preceding sections, the frame will now be considered anisotropic, while the solid material and the pore saturant will be assumed isotropic. For the anisotropic skeleton, the variation $\Delta\bar{\mathbf{P}}$ of the residual stress tensor and the components of the distortion Δe_i introduced in (*10*) are related by a system of equations corresponding to (*11*):

$$\Delta\bar{p}_i = -\sum_{j=1}^{6} \bar{c}_{ij}\Delta e_j, \qquad i = 1, 2, \ldots, 6,$$

where the matrix of the coefficients $\bar{c}_{ij}$ again satisfies the symmetry conditions $\bar{c}_{ij} = \bar{c}_{ji}$.

(*76*) When the determinant of the coefficient matrix $\bar{c}_{ij}$ is assumed different from zero, the following solution of the system gives for the Δe_i:

$$\Delta e_i = -\sum_{j=1}^{6} \bar{\gamma}_{ij}\Delta\bar{p}_j,$$

with $\bar{\gamma}_{ij} = \bar{\gamma}_{ji}$.

(*77*) It remains to introduce the following auxillary quantities:

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$$
$$\varepsilon_4 = \varepsilon_5 = \varepsilon_6 = 0.$$

(*78*) As in section **VI**, the differential elasticity of a closed system will be considered. Corresponding to (*40*), (*41*), and (*42*), the components Δp_i of the change in the stress tensor are given as follows:

$$\Delta p_i = \Delta \bar{p}_i + \varepsilon_i \Delta \tilde{p}, \qquad i = 1, 2, \ldots, 6.$$

(*79*) The closed system behaves microscopically as a homogeneous, anisotropic medium, corresponding to (*76*), which can be expressed using the system of equations:

$$\Delta e_i = -\sum_{j=1}^{6} \gamma_{ij} \Delta p_j, \qquad i = 1, 2, \ldots, 6.$$

In the following, the elastic constants γ_{ij} of the closed system will be calculated from the elastic constants $\bar{\gamma}_{ij}$ of the skeleton, *i.e.*, of the open system, from the bulk moduli $\hat{k}$ and $\tilde{k}$ of the solid material and of the pore fluid, and from the porosity n.

(*80*) The elimination of $\tilde{V}$ and $\Delta\tilde{V}$ from equations (*58*) c), f), and g) gives

$$\frac{1}{n}\left(\frac{\Delta V}{V} - \frac{\Delta \hat{V}}{V}\right) = -\frac{\Delta \tilde{p}}{\tilde{k}},$$

as the expression determining the compressibility of the pore saturant.

(*81*) The compression of the solid matter is described, in agreement with (*58*) e), by

$$\frac{\Delta \hat{V}}{V} = -(1-n)\frac{\Delta \tilde{p}}{\hat{k}} - \frac{1}{3\hat{k}}\left(\Delta \bar{p}_1 + \Delta \bar{p}_2 + \Delta \bar{p}_3\right).$$

(*82*) From (*78*), the variation of Δp_i is composed of two terms added together. The term $\varepsilon_i \Delta \tilde{p}$ corresponds to a deformation of the skeleton calculated from (*45*), and the other term corresponds to (*76*). The superposition of these two deformations gives:

$$\Delta e_i = -\varepsilon_i \frac{\Delta \tilde{p}}{3\hat{k}} - \sum_{j=1}^{6} \bar{\gamma}_{ij} \Delta \bar{p}_i, \qquad i = 1, 2, \ldots, 6.$$

(*83*) Now the following notation is introduced:

$$\begin{aligned} \bar{S}_i &= \bar{\gamma}_{i1} + \bar{\gamma}_{i2} + \bar{\gamma}_{i3} = \bar{\gamma}_{1i} + \bar{\gamma}_{2i} + \bar{\gamma}_{3i}, \\ a_i &= \bar{S}_i - \frac{\varepsilon_i}{3\hat{k}}, \qquad i = 1, 2, \ldots, 6, \\ \frac{1}{\alpha} &= n\left(\frac{1}{\tilde{k}} - \frac{1}{\hat{k}}\right). \end{aligned}$$

(*84*) From (*82*) follows:

$$\Delta e_1 + \Delta e_2 + \Delta e_3 = \frac{\Delta V}{V} = -\frac{\Delta \tilde{p}}{\hat{k}} - \sum_{j=1}^{6} \bar{S}_j \Delta \bar{p}_j.$$

(*85*) The elimination of both quantities $\Delta V/V$ and $\Delta\hat{V}/V$ from (*80*), (*81*), and (*84*) gives, using the notation of (*83*):

$$\Delta\tilde{p} = \alpha\sum_{j=1}^{6} a_j \Delta\bar{p}_j.$$

(*86*) Substitute the value of Δp_j given in (*78*) into (*79*), then set the expressions (*79*) and (*82*) for Δe_j equal to one another and arrange the result in terms of stresses, obtaining:

$$\Delta\tilde{p}\left(\sum_{h=1}^{6}\varepsilon_h\gamma_{ih} - \frac{\varepsilon_i}{3\hat{k}}\right) = \sum_{j=1}^{6}\left(\bar{\gamma}_{ij} - \gamma_{ij}\right)\Delta\bar{p}_j, \qquad i = 1,2,\ldots,6.$$

(*87*) The elimination of $\Delta\tilde{p}$ from (*85*) and (*86*) gives the following equation:

$$\alpha\left(\sum_{h=1}^{6}\varepsilon_h\gamma_{ih} - \frac{\varepsilon_i}{3\hat{k}}\right)\sum_{j=1}^{6} a_j\Delta\bar{p}_j - \sum_{j=1}^{6}\left(\bar{\gamma}_{ij} - \gamma_{ij}\right)\Delta\bar{p}_j = 0, \quad i = 1,2,\ldots,6.$$

(*88*) The $\Delta\bar{p}_j$ are linearly independent quantities, and so the coefficients of each Δp_j must vanish, giving:

$$\gamma_{ij} + \alpha a_j\left(\gamma_{i1} + \gamma_{i2} + \gamma_{i3}\right) = \bar{\gamma}_{ij} + \alpha a_j\frac{\varepsilon_i}{3\hat{k}}, \qquad i,j = 1,2,\ldots,6.$$

(*89*) For the index i in (*88*) with j varying from 1 through 6, we obtain six linear equations for the six unknowns, γ_{i1}, γ_{i2}, …, γ_{i6}. The three equations for $j = 1,2,3$ contain only the three unknowns γ_{i1}, γ_{i2}, and γ_{i3} and may therefore be separately determined. The determinant of the coefficients for this system of three equations is:

$$D \equiv \det\begin{pmatrix} 1+\alpha a_1 & \alpha a_1 & \alpha a_1 \\ \alpha a_2 & 1+\alpha a_2 & \alpha a_2 \\ \alpha a_3 & \alpha a_3 & 1+\alpha a_3 \end{pmatrix} = 1 + \alpha(a_1 + a_2 + a_3).$$

(*90*) We have to calculate the three unknowns γ_{i1}, γ_{i2} and γ_{i3}, so we can insert the value of their sum into (*88*). Then from (*88*), we always obtain, for $j = 4,5,6$, one equation that contains only one of the unknowns, γ_{i4}, γ_{i5}, or γ_{i6}. The final expression for calculation of the γ_{ij} has the same form for each, namely

$$\gamma_{ij} = \bar{\gamma}_{ij} - \frac{\alpha}{D}a_i a_j, \qquad i,j = 1,2,\ldots,6.$$

The goal, declared in (*79*), to calculate the elastic constants γ_{ij} for a closed system is thereby attained. As expected, the matrix of these elastic coefficients is symmetric, *i.e.*, $\gamma_{ij} = \gamma_{ji}$.

(*91*) Naturally we must be able to rederive the results for the isotropic skeleton treated in section **VI** as given in (*57*) and (*59*) from the more general results (*90*). The desired derivation is short: $\bar{\lambda} = \bar{k} - \frac{2}{3}\bar{\mu}$ and $\bar{\mu}$ is the elastic constant of the frame. The matrix of the coefficients c_{ij} then follows from (*13*). Finally, the $\bar{\gamma}_{ij}$ are to be calculated from (*76*). One obtains:

$$\bar{\gamma}_{11} = \bar{\gamma}_{22} = \bar{\gamma}_{33} = \frac{\bar{\lambda} + \bar{\mu}}{\bar{\mu}(3\bar{\lambda} + 2\bar{\mu})},$$

$$\bar{\gamma}_{44} = \bar{\gamma}_{55} = \bar{\gamma}_{66} = \frac{1}{\bar{\mu}},$$

$$\bar{\gamma}_{12} = \bar{\gamma}_{13} = \bar{\gamma}_{23} = \bar{\gamma}_{21} = \bar{\gamma}_{31} = \bar{\gamma}_{32} = -\frac{\bar{\lambda}}{2\bar{\mu}(3\bar{\lambda} + 2\bar{\mu})}.$$

All other $\bar{\gamma}_{ij} = 0$.

$$\bar{S}_{ij} = \frac{\varepsilon_i}{3\bar{k}} \quad \text{and} \quad a_i = \frac{\varepsilon_i}{3}\left\{\frac{1}{\bar{k}} - \frac{1}{\hat{k}}\right\}, \qquad i = 1, 2, \ldots, 6,$$

$$D = 1 + \alpha\left(\frac{1}{\bar{k}} - \frac{1}{\hat{k}}\right).$$

With the quantity Q introduced in (*59*), we get

$$\gamma_{ij} = \bar{\gamma}_{ij} - \frac{\left(\frac{1}{\bar{k}} - \frac{1}{\hat{k}}\right)\varepsilon_i\varepsilon_j}{9\left(\frac{\bar{k}}{Q} + 1\right)}.$$

In accordance with (*57*), it follows that $\mu = \bar{\mu}$ is the shear modulus of the closed system. The bulk modulus k of the closed system can be obtained from the equation $1/3k = \gamma_{11} + \gamma_{12} + \gamma_{13}$, resulting in (*59*).

(*92*) Assuming the determinant of the coefficient matrix γ_{ij} in (*79*) to be different from zero, we can solve the system for the Δp_{ij}, which gives a system of equations of the form (*11*). Using well-known operations on determinants, the coefficients c_{ij} may be expressed in terms of the γ_{ij}. Here according to (*90*) these are functions of $\bar{\gamma}_{ij}$, so the c_{ij} are also. Corresponding to (*75*) and (*76*), the $\bar{\gamma}_{ij}$ themselves may be expressed in terms of the $\bar{c}_{ij}$, so that finally the c_{ij} will be obtained as functions of the $\bar{c}_{ij}$ (and naturally also of quantities, $\hat{k}$, $\tilde{k}$, and n). Without showing the steps of the calculation, we are led to the general expression for the c_{ij} (I owe to Mr. Oswald Wyler, a graduate mathematician at the Electrotechnische Hochschule, a hint that leads to the simple form of the following results):

$$c_{ij} = \bar{c}_{ij} + \frac{\alpha}{D^*}b_i b_j, \qquad i, j = 1, 2, \ldots, 6.$$

(*93*) The meanings of the new symbols introduced in (*92*) are as follows:

$$b_i = \varepsilon_i - \frac{\bar{c}_{1i} + \bar{c}_{2i} + \bar{c}_{3i}}{3\hat{k}}, \qquad i = 1, 2, \ldots, 6;$$

$$D^* = 1 + \frac{\alpha}{3\hat{k}}(b_1 + b_2 + b_3).$$

Sought / Given	E	k	M	λ	μ	ν
E,k	—	—	$\frac{3k(3k+E)}{9k-E}$	$\frac{3k(3k-E)}{9k-E}$	$\frac{3kE}{9k-E}$	$\frac{1}{2}-\frac{E}{6k}$
E,M	—	$\frac{3M-E+w_1}{6}$	—	$\frac{M-E+w_1}{4}$	$\frac{3M+E-w_1}{8}$	$\frac{E-M+w_1}{4M}$
E,λ	—	$\frac{E+3\lambda+w_2}{6}$	$\frac{E-\lambda+w_2}{2}$	—	$\frac{E-3\lambda+w_2}{4}$	$\frac{w_2-E-\lambda}{4\lambda}$
E,μ	—	$\frac{\mu E}{3(3\mu-E)}$	$\frac{\mu(4\mu-E)}{3\mu-E}$	$\frac{\mu(E-2\mu)}{3\mu-E}$	—	$\frac{E}{2\mu}-1$
E,ν	—	$\frac{E}{3(1-2\nu)}$	$\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}$	$\frac{\nu E}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	—
k,M	$\frac{9k(M-k)}{M+3k}$	—	—	$\frac{3k-M}{2}$	$\frac{3}{4}(M-k)$	$\frac{3k-M}{3k+M}$
k,λ	$\frac{9k(k-\lambda)}{3k-\lambda}$	—	$3k-2\lambda$	—	$\frac{3}{2}(k-\lambda)$	$\frac{\lambda}{3k-\lambda}$
k,μ	$\frac{9k\mu}{3k+\mu}$	—	$k+\frac{4}{3}\mu$	$k-\frac{2}{3}\mu$	—	$\frac{3k-2\mu}{2(3k+\mu)}$
k,ν	$3k(1-2\nu)$	—	$\frac{3k(1-\nu)}{1+\nu}$	$\frac{3k\nu}{1+\nu}$	$\frac{3k(1-2\nu)}{2(1+\nu)}$	—
M,λ	$\frac{(M+2\lambda)(M-\lambda)}{M+\lambda}$	$\frac{M+2\lambda}{3}$	—	—	$\frac{M-\lambda}{2}$	$\frac{\lambda}{M+\lambda}$
M,μ	$\frac{\mu(3M-4\mu)}{M-\mu}$	$M-\frac{4}{3}\mu$	—	$M-2\mu$	—	$\frac{M-2\mu}{2(M-\mu)}$
M,ν	$\frac{(1-2\nu)(1+\nu)M}{1-\nu}$	$\frac{(1+\nu)M}{3(1-\nu)}$	—	$\frac{\nu M}{1-\nu}$	$\frac{(1-2\nu)M}{2(1-\nu)}$	—
λ,μ	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\lambda+\frac{2}{3}\mu$	$\lambda+2\mu$	—	—	$\frac{\lambda}{2(\lambda+\mu)}$
λ,ν	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	$\frac{\lambda(1+\nu)}{3\nu}$	$\frac{\lambda(1-\nu)}{\nu}$	—	$\frac{\lambda(1-2\nu)}{2\nu}$	—
μ,ν	$2\mu(1+\nu)$	$\frac{2\mu(1+\nu)}{3(1-2\nu)}$	$\frac{2\mu(1-\nu)}{1-2\nu}$	$\frac{2\mu\nu}{1-2\nu}$	—	—

$$w_1 = +\sqrt{(M-E)(9M-E)},$$
$$w_2 = +\sqrt{(E+\lambda)^2+8\lambda^2}.$$

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