WEIGHTED LEAST-SQUARES CRITERIA FOR ELECTRICAL IMPEDANCE TOMOGRAPHY

Jeffrey S. Kallman

James G. Berryman
Lawrence Livermore National Laboratory
Electronics Engineering
P. O. Box 808 L-156
Livermore, CA 94550

Abstract – Methods are developed for design of electrical impedance tomographic reconstruction algorithms with specified properties. Assuming a starting model with constant conductivity, an algorithm with the following properties is found: (1) The optimum constant for the starting model is determined automatically. (2) The weighted least-squares error between the predicted and measured power dissipation data is as small as possible. (3) The variance of the reconstructed conductivity from the starting model is minimized. (4) Potential distributions with the largest volume integral of gradient squared have the least influence on the reconstructed conductivity, and therefore distributions most likely to be corrupted by contact impedance effects are deemphasized. (5) Cells with most coverage tend to deviate least from the background value. The resulting algorithm maps the reconstruction problem into a vector space where the contribution to the inversion from the background conductivity remains invariant, while the optimum contributions in orthogonal directions are found. For a starting model with nonconstant conductivity, the reconstruction algorithm has analogous properties.

Introduction

In [1], Berryman describes a weighted least-squares method for linear traveltime tomography that has a number of desirable properties. In this paper we develop a weighted least-squares method for electrical impedance tomography that has similar properties. In the past electrical impedance tomography has been used to find conductivity models consistent with measured potentials due to injected currents. We work the problem from a different perspective, one which is a closer analog to the traveltime tomography problem. Our goal is to find a conductivity model consistent with measured power dissipation data (within some specified measurement tolerance). This dissipation data is acquired by injecting a current between two electrodes and measuring the voltage between the electrodes.

First, let us clarify the analogy between electrical impedance tomography and traveltime tomography. In traveltime tomography we try to determine the distribution of slowness $s = \frac{1}{v}$ (where v is the wave velocity) in a region between a set of wave sources and receivers of known position given first arrival timetime data t_i measured between the m pairs of sources and receivers (indexed by $i = 1, \ldots, m$). The data are the times t it takes for waves to get from sources to receivers. Assuming that the waves are of a high enough frequency to be treated as if they were traveling along rays, the waves that move most quickly from sources to receivers have particular ray paths. For the i-th source-receiver pair, Fermat's principle says that the first arrival traveltime for the i-th ray path is given by

$$t_i(s) = \int s dl_i^{(path)} \tag{1}$$

where $l_i^{(path)}$ is the arc length along the path between the source and receiver that minimizes the traveltime. Since the slowness can never be negative or zero, this integral is always positive unless the source and receiver are in the same place. This case is excluded, so traveltime is strictly positive.

Similarly, in electrical impedance tomography we try to determine the distribution of conductivity σ in the region between and around a set of current sources and sinks of known position given power dissipation data p_i measured between the m pairs of sources and sinks (indexed by $i = 1, \ldots, m$). The data are the powers p that are dissipated when the currents are applied. For the i-th source-sink pair, Dirichlet's principle states that the power dissipated is

$$p_i(\sigma) = \int_{Volume} \sigma |\nabla \phi_i(\sigma)|^2 d^3x \tag{2}$$

where $\phi_i(\sigma)$ is the potential field which minimizes the power dissipation for the conductivity distribution $\sigma(\vec{x})$. Since the conductivity is never negative, the power dissipation is also strictly positive.

From the previous two paragraphs it should be clear that the following quantities are analogous to each other: slowness s is the analog of conductivity σ, differential arc length $dl_i^{(path)}$ is the analog of the differential squared magnitude of the electric field $|\nabla\phi_i(\sigma)|^2 d^3x$, and the first arrival traveltime $t_i(s)$ is the analog of power dissipation $p_i(\sigma)$. With these preliminaries out of the way, we can continue.

Let p be the measured power dissipation, an m-vector such that $p^T = (p_1, \ldots, p_m)$ where p_i is the power dissipation of the i-th current probe (a T superscript implies the transpose). Although various methods of parameterizing conductivity models have been studied, we restrict our attention here to a model based on cells of constant conductivity. Thus, if we are working on a two-dimensional cross section and have a rectangular geometry, we may form our model conceptually by dividing the region between and around the current injection points into rectangular cells of constant conductivity. On the other hand, if our problem is three-dimensional, the cells are blocks of constant conductivity. Then σ is the model conductivity n-vector $\sigma^T = (\sigma_1, \ldots, \sigma_n)$ with σ_j being the conductivity of the j-th cell, satisfying

$$K\sigma = p \tag{3}$$

where K is an $m \times n$ matrix whose matrix elements $k_{i,j}$ are determined by the square magnitude of the electric field of the i-th current probe through the j-th cell of the model (*i.e.* $k_{i,j} = \int_{cell_j} |\nabla\phi_i|^2 d^3x$). This matrix will be called the E-square matrix. In general, we do not have good *a priori* approximations to the square magnitude of the electric field if the medium is very inhomogeneous (*i.e.*, having contrasts of 15% or more). However, for the present application, we will assume that the E-square matrix has been fixed with a known set of (what are possibly trial) square magnitudes of the electric field.

We do not restrict our analysis to small changes $\Delta\sigma$ in σ relative to some known background [2,3]. We take explicit account of the fact that the linear weighted least-squares problem we study is actually derived from a nonlinear reconstruction problem. The (possibly) substantial uncertainties in the E-square matrix K due to nonuniform conductivity effects are acknowledged and treated from the outset. The importance of this point of view derives from the fact that the errors introduced into the reconstruction by a poor choice of K may be far more significant than the measurement errors introduced through p. The method developed here will ultimately be used as one step in a comprehensive nonlinear inversion algorithm such as that of Dines and Lytle[4]. However, we should emphasize that the analysis of the present work is still restricted to the linear inverse problem associated with (3). Extensions to the nonlinear inversion problem will be presented elsewhere [5].

Once a set of trial E-squares and corresponding values of $k_{i,j} = \int_{cell_j} |\nabla\phi_i|^2 d^3x$ are known, the model vector σ may be underdetermined if $m < n$ or overdetermined if $m > n$. Thus, finding a "solution" to (3) requires the use of a generalized inverse [6] which in turn often implies a least-squares estimate [8] of the model conductivity vector. A general objective function to be minimized might have the form suggested by Herman[9]

$$\alpha_\mu(\sigma) = (p - K\sigma)^T W_1 (p - K\sigma) + \mu(\sigma - \sigma_b)^T W_2 (\sigma - \sigma_b), \tag{4}$$

where W_1 and W_2 are (respectively) $m \times m$ and $n \times n$ real, symmetric weight matrices, μ is some scalar determined by the relative importance of the second term compared to the first, and p_b is some special background value of the power dissipation vector to which the final result should be close. Many criteria have been proposed in order to ensure a unique solution to the reconstruction problem. For example, positivity of both the power dissipation vector p and the conductivity σ is clearly required for any physical model, but for electrical impedance tomography maintaining the positivity of p and σ is seldom a problem. Therefore, we will make no special effort to control this feature through the objective function itself. Some of the most important features of the reconstructed conductivity needing control are: (*i*) The (weighted) least-squares error between the measured and predicted power dissipation data should be as small as possible. (*ii*) For linear tomographic inversion with a good starting model, the variance of the reconstructed conductivity from the specified background should be as small as possible for the given data [10]. (*iii*) The (possibly weighted) mean conductivity should be the best possible in some sense. Additional criteria that have been proposed for the analogous case of traveltime tomography include: (*iv*) Rays of greatest length should be weighted least in the reconstruction [11]. This is analogous to putting the least weight on the most diffuse current distributions. (*v*) Cells with the most ray coverage in them should be the ones with the most accurate reconstructed values. This is analogous to saying that cells with the most power dissipated in them should be the ones with the most accurate reconstructed values. The motivation for each of these criteria and the impact they have on determining the values of the weights and other constants in (4) will be developed more fully in the discussion that follows.

As shown by the application to both traveltime tomography and electrical impedance tomography, the techniques developed here may be used in many currently existing least-squares tomographic inversion methods.

Homogeneous Background

We treat the minimization problem for homogeneous background σ_b in this section, and then generalize to more complex background conductivities in the next section. In a comprehensive nonlinear inversion algorithm, we often expect constant conductivity to be the initial guess.

A. Optimum homogeneous conductivity

The crudest sort of reconstruction one can imagine is to take the available data and form the optimum constant conductivity vector consistent with that data. By optimum constant, we mean that it uses all the data, that it produces a minimum error in some appropriate least-squares functional, and that it agrees with the exact result if the medium is actually constant. If the medium is homogeneous (constant conductivity), the exact value for the conductivity is easily seen to be given by

$$s_0 = \frac{p_i}{\int_{Volume} |\nabla \phi_i|^2 d^3x} \tag{5}$$

for any particular measurement i, or by

$$s_0 = \frac{\sum_i p_i}{\sum_i \int_{Volume} |\nabla \phi_i|^2 d^3x} \tag{6}$$

where p_i is the power dissipated due to the ith current probe and $\int_{Volume} |\nabla \phi_i|^2 d^3x$ is the integral of the squared magnitude of the electric field vector due to current probe i over the entire volume. The conductivity scalar s_0 is the total power dissipated by the current probes in all measurements divided by the sum of the integrals of square magnitude of the electric field over all space.

When the medium is not actually homogeneous, a least-squares error criterion can be used based on minimizing the functional

$$\psi(s) = (p - Ksv)^T W_3 (p - Ksv), \tag{7}$$

where W_3 is another $m \times m$ real, symmetric weight matrix and $v^T = (1, \ldots, 1)$ is an n-vector of ones. The minimum of (7) occurs when

$$v^T K^T W_3 (p - Ks_0 v) = 0, \tag{8}$$

or equivalently when

$$s_0 = \frac{v^T K^T W_3 p}{v^T K^T W_3 K v}. \tag{9}$$

Now Eq. (9) can be simplified by introducing some new notation. Define the row sums R_i and column sums C_j

$$R_i = \sum_{j=1}^{n} k_{i,j} = \int_{Volume} |\nabla\phi_i|^2 d^3x, \qquad C_j = \sum_{i=1}^{m} k_{i,j} = \sum_{i=1}^{m} \int_{Volume_j} |\nabla\phi_i|^2 d^3x. \tag{10}$$

The quantity R_i is seen to be the integrated square magnitude of the electric field for current probe i over the entire volume. The quantity C_j is the sum over all current probes i of the integrated square magnitude of the electric field in cell j, so we will call this the "coverage" of cell j. Any cell with $C_j = 0$ is uncovered and therefore lies outside the span of our data for the present choice of current paths. We retain only the covered cells in the reduced conductivity vector $\tilde{\sigma}$ of length $\tilde{n} \leq n$. The matrix K may similarly be reduced to $\widetilde{K}$ by deleting the corresponding columns of zeros. Finally, define diagonal $m \times m$ and $\tilde{n} \times \tilde{n}$ matrices R and C respectively whose diagonal elements are given by the nonzero sums in (10). For simplicity, we assume that $\tilde{n} = n$ in the following discussion. Then, the diagonal matrices R and C are given by

$$R = \begin{pmatrix} R_1 & & \\ & \ddots & \\ & & R_m \end{pmatrix} \tag{11}$$

and

$$C = \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_n \end{pmatrix}. \tag{12}$$

Now in addition to the n-vector v we have already introduced, we define $u^T = (1, \ldots, 1)$, an m-vector of ones. Then,

$$Kv = Ru \tag{13}$$

and

$$K^T u = Cv \tag{14}$$

follow easily from the definitions. Substituting (13) into (9) we obtain

$$s_0 = \frac{u^T R W_3 p}{u^T R W_3 R u}. \tag{15}$$

For comparison, note that (6) may be rewritten as

$$s_0 = \frac{u^T p}{u^T R u}. \tag{16}$$

Now, neglecting a possible (but clearly irrelevant) arbitrary scale factor in the weight matrix W_3, we see that (15) will agree with (16) if

$$W_3 R u = u, \tag{17}$$

i.e., if u is a right eigenvector of $W_3 R$ with unit eigenvalue. Of course, (17) does not uniquely determine $W_3 R$ since we see easily that two possible choices are given by

$$W_3 R = I, \tag{18}$$

where I is the $n \times n$ identity matrix, and by

$$W_3 R = R^{-1} K C^{-1} K^T. \tag{19}$$

In either case, the resulting W_3 is real and symmetric as required. Although other choices clearly exist, the simplest choice for W_3 appears to be the one determined by (18), *i.e.*,

$$W_3 = R^{-1}. \tag{20}$$

With this choice, (15) and (16) are identical. On the other hand, (19) leads to a weight matrix that may not be positive definite – an undesirable property for a weight matrix.

The special background value of σ for our analysis of (4) in the remainder of this section will be taken to be

$$\sigma_b = s_0 v, \tag{21}$$

where s_0 is determined by (16).

B. Weighted least-squares error

Next we consider (4) with $\mu = 0$. Then, the error functional

$$\alpha_0(\sigma) = (p - K\sigma)^T W_1 (p - K\sigma) \tag{22}$$

should be minimized with respect to σ. The standard result is that the minimum occurs for σ satisfying

$$K^T W_1 K \sigma = K^T W_1 p. \tag{23}$$

If the square matrix $K^T W_1 K$ is nonsingular, *i.e.*, has no zero eigenvalues, then (23) is easily solved by inverting the matrix; this result is just the well-known solution associated with the Gauss-Markov theorem [7]. However, in many cases of practical interest, this matrix is singular and some method from the theory of generalized inverses will be required [11]. One particularly simple method to analyze involves the iterative solution of (23) [9,12-14]. However, the success of this method requires that the matrix whose pseudoinverse is to be approximated in this way must have its eigenvalues λ restricted to the unit interval $0 \leq \lambda \leq 1$. We can precondition the matrix by multiplying (23) by some positive definite, diagonal matrix D^{-1} and reassociating matrix factors so that

$$(D^{-1}K^TW_1KD^{-1})D\sigma = D^{-1}K^TW_1p. \tag{24}$$

The diagonal elements of D are chosen so that the eigenvalues of the matrix in parentheses lie in the unit interval. We define an n-vector z

$$z = D\sigma \tag{25}$$

and, since the matrix in parentheses in (24) is clearly symmetric, a matrix A can always be found such that

$$A^TA = D^{-1}K^TW_1KD^{-1}. \tag{26}$$

Then, (23) may be rewritten as

$$A^TAz = D^{-1}K^TW_1p \tag{27}$$

which can be solved by iteration according to

$$z^{(0)} = D^{-1}K^TW_1p, \tag{28}$$

$$z^{(k+1)} = z^{(0)} + (I - A^TA)z^{(k)}. \tag{29}$$

It is well-known [15,16] that the $z^{(k)}$ sequence converges to the solution of (27) obtained from the Moore-Penrose generalized inverse as $k \to \infty$. We will develop this result in more detail by examining the eigenvalue (singular value) decomposition of A^TA in a manner similar to that of Aki and Richards [17].

Now using a method of Lanczos [18] that we call "completing the square", we will introduce the $(m+n) \times (m+n)$ real, symmetric square matrix H determined by K

$$H = \begin{pmatrix} 0 & K \\ K^T & 0 \end{pmatrix} \tag{30}$$

and the corresponding eigenvalue problem

$$\begin{pmatrix} 0 & K \\ K^T & 0 \end{pmatrix} \begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix} = \lambda \begin{pmatrix} R & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix}. \tag{31}$$

In (31), the vectors u_λ and v_λ are of length m and n respectively. We see in particular that (13) and (14) are of the form (31) so that the vector $(u^T, v^T) = (1, \ldots, 1)$ is an eigenvector of H with eigenvalue unity. This fact is significant, and provides the motivation for studying the eigenvalue problem (31), as we shall show.

We will now proceed to manipulate (31) into a more illuminating form. First, we invert the diagonal matrix on the right side of (31) and multiply through, obtaining

$$\begin{pmatrix} 0 & R^{-1}K \\ C^{-1}K^T & 0 \end{pmatrix} \begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix} = \lambda \begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix}. \tag{32}$$

[Note that, if $(u_\lambda^T, v_\lambda^T)^T$ is an eigenvector of (32) with eigenvalue λ, then $(u_{-\lambda}^T, -v_{-\lambda}^T)^T$ is also an eigenvector with eigenvalue $-\lambda$.] Next we square the matrix on the left side of (32) to find

$$\begin{pmatrix} R^{-1}KC^{-1}K^T & 0 \\ 0 & C^{-1}K^TR^{-1}K \end{pmatrix} \begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix} = \lambda^2 \begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix}. \tag{33}$$

Finally, we multiply (33) by the matrix

$$\begin{pmatrix} R^{\frac{1}{2}} & 0 \\ 0 & C^{\frac{1}{2}} \end{pmatrix}$$

(these square root matrices are well-defined since R and C both have only real, positive, diagonal elements) and then reassociate some terms to reach the desired result

$$\begin{pmatrix} R^{-\frac{1}{2}}KC^{-1}K^TR^{-\frac{1}{2}} & 0 \\ 0 & C^{-\frac{1}{2}}K^TR^{-1}KC^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} R^{\frac{1}{2}}u_\lambda \\ C^{\frac{1}{2}}v_\lambda \end{pmatrix} = \lambda^2 \begin{pmatrix} R^{\frac{1}{2}}u_\lambda \\ C^{\frac{1}{2}}v_\lambda \end{pmatrix}. \tag{34}$$

Comparing (34) and (26), we see that, if the following choices are made for the weight matrix

$$W_1 = R^{-1} \tag{35}$$

(note that this is the same choice that was made for W_3) and the preconditioning matrix

$$D = C^{\frac{1}{2}}, \tag{36}$$

then the matrix A becomes

$$A = R^{-\frac{1}{2}}KC^{-\frac{1}{2}} \tag{37}$$

and (34) may be rewritten as

$$\begin{pmatrix} AA^T & 0 \\ 0 & A^TA \end{pmatrix} \begin{pmatrix} w_\lambda \\ x_\lambda \end{pmatrix} = \lambda^2 \begin{pmatrix} w_\lambda \\ x_\lambda \end{pmatrix}, \tag{38}$$

where $w_\lambda = R^{\frac{1}{2}}u_\lambda$ and $x_\lambda = C^{\frac{1}{2}}v_\lambda$. That the positive eigenvalues of A all lie in the range [0,1] as desired follows from the fact that, by construction, the matrices KC^{-1} and K^TR^{-1} have all their column sums equal to unity. Furthermore, since $(u^T, v^T)^T$ is a strictly positive eigenvector with eigenvalue $\lambda = 1$ from (28) and (29), we know that this eigenvalue is both maximal and simple from the Perron-Frobenius theory of non-negative matrices [19,20].

[A technical point arising here is this: The matrix A^TA must be "primitive," *i.e.,* some positive power of it must have all positive matrix elements. This condition will often be satisfied for problems in tomography. If it is not satisfied, we separate A^TA into submatrices for which it is satisfied and proceed with the analysis as presented for each of the submatrices.] In addition, we know that the eigenvectors $x_{\lambda_j}, x_{\lambda_{j'}}$ for distinct eigenvalues $\lambda_j^2, \lambda_{j'}^2$ are orthogonal, and we may assume that those eigenvectors which share a common eigenvalue have been orthogonalized.

Returning to the iterative scheme (28)-(29) now that we have an explicit representation for A, we find that

$$z^{(0)} = A^T R^{-\frac{1}{2}} p = \sum_{j=1}^{n} \zeta_j x_{\lambda_j}, \tag{39}$$

where

$$\zeta_j = x_{\lambda_j}^T A^T R^{-\frac{1}{2}} p \tag{40}$$

assuming that the eigenvectors have been normalized so that $x_\lambda^T x_\lambda = 1$. Furthermore, the eigenvalue decomposition of A^TA is given by

$$A^T A = \sum_{j=1}^{n} x_{\lambda_j} \lambda_j^2 x_{\lambda_j}^T. \tag{41}$$

Thus, we have in general that

$$(I - A^T A)^p z^{(0)} = \sum_{j=1}^{n} (1 - \lambda_j^2)^p \zeta_j x_{\lambda_j}. \tag{42}$$

It follows easily from (29) that the k-th iterate may be expressed as

$$z^{(k)} = \sum_{p=1}^{k} (I - A^T A)^p z^{(0)} \tag{43}$$

so from (42) we find that

$$z^{(k)} = \sum_{j=1}^{n} \sum_{p=0}^{k} (1 - \lambda_j^2)^p \zeta_j x_{\lambda_j}. \tag{44}$$

Let the eigenvalues be ordered in decreasing magnitude so that

$$1 = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_r > \lambda_{r+1} = \ldots = \lambda_n = 0, \tag{45}$$

where r is the rank of A^TA. Then, (44) may be rewritten as

$$z^{(k)} = \zeta_1 x_{\lambda_1} + \sum_{j=2}^{r} \sum_{p=0}^{k} (1 - \lambda_j^2)^p \zeta_j x_{\lambda_j}. \tag{46}$$

Using the fact that (for $\lambda_j \neq 0$)

$$\sum_{p=0}^{N}(1-\lambda_j^2)^p = \frac{1-(1-\lambda_j^2)^{N+1}}{1-(1-\lambda_j^2)} = \frac{1-(1-\lambda_j^2)^{N+1}}{\lambda_j^2}, \tag{47}$$

we see that

$$z^{(N)} = \zeta_1 x_{\lambda_1} + \sum_{j=2}^{r} \frac{\zeta_j}{\lambda_j^2}[1-(1-\lambda_j^2)^{N+1}]x_{\lambda_j}. \tag{48}$$

Clearly, the rate of convergence of $z^{(k)}$ to $z^{(\infty)}$ depends only on the value of the lowest nonzero eigenvalue λ_r since the term $(1-\lambda_r^2)^k$ is the most slowly decreasing factor appearing in (46) as $k \to \infty$. Eq. (48) gives the eigenvector decomposition of the solution of (27) that would be obtained using the Moore-Penrose generalized inverse of A^TA.

Two important facts should now be noted about (46). First, since the right null spaces of A and A^TA are the same, (46) is orthogonal to the null space of A^TA for all k. This fact follows immediately from (40), since $\zeta_j = 0$ for any j such that $Ax_{\lambda_j} = 0$. This is also a very useful fact because it means that the solution $z^{(\infty)}$ is the unique solution of minimum norm z^Tz consistent with the data; thus, any other z for which (22) has a minimum will be equal to $z^{(\infty)}$ plus some terms from the null space of A^TA. Second, the coefficient of the eigenvector x_{λ_1} is the same for all k.

Eq. (46) provides a solution to the problem of finding the conductivity with minimum error for (22) when $W_1 = R^{-1}$. Interestingly, it does more that. Consider the term $\zeta_1 x_{\lambda_1}$ which has constant coefficient (as $k \to \infty$) starting from the initial iterate $z^{(0)}$ of (28). We see that

$$\zeta_1 = x_{\lambda_1}^T A^T R^{-\frac{1}{2}} p = v^T K^T R^{-1} p/(v^T C v)^{\frac{1}{2}} \tag{49}$$

and, using (13) together with the fact that $v^TCv = u^TRu$, we therefore find that

$$\zeta_1 x_{\lambda_1} = s_0 C^{\frac{1}{2}} v \tag{50}$$

where s_0 is given by (16). Recalling that $s_0 v$ is the optimum homogeneous background, we see that this result is quite remarkable. Eq. (50) shows that, by transforming into this particular vector space using the conditioning matrices R and C, we have found a solution of a weighted least-squares problem that automatically has minimum (weighted) norm consistent with the data – but that also has exactly the right value of the coefficient for the optimum constant background. This latter property is guaranteed to be satisfied at any stage of the iterative process for obtaining the solution of (27). Furthermore, the weighted least-squares criterion (22) with $W_1 = R^{-1}$ is the only one with all these properties for the constant background problem – this fact follows by running through the steps of the derivation of (31)-(34) backwards assuming that R and C are arbitrary diagonal matrices and then noting that (31) can only be consistent with (13), (14), and (16) for this *unique* choice of R and C.

A similar weighting technique for traveltime tomography appears in an unpublished manuscript of Burkhard [16]. The goal of obtaining a convergent iterative method motivated Burkhard to make choices corresponding to (35) and (36). The choice for weight matrix $W_1 = R^{-1}$ is also similar to one in traveltime tomography, discussed by Frank and Balanis [10]. Their arguments for such weights are: (i) Signal to noise ratio is expected to be better on shorter ray paths than on longer ones, since the overall attenuation will typically be smaller. (ii) Shorter trial ray paths are more likely to correspond to real paths that remain completely in the image plane for two-dimensional reconstruction problems. These arguments carry over, with some modification due to the substantial differences in the physics of the two problems, to electrical impedance tomography: (i) Large gradients imply large currents and the largest of these are anticipated to occur close to the injection electrodes; thus, nonlinear effects such as contact impedance are most likely to contaminate the power measurements when the gradients are large. Weighting against a large E-square integral emphasizes measurements taken using smaller currents, and so towards those presumably less corrupted by nonlinear effects. In addition it emphasizes electrode placements that are far apart over placements that are close together. These points can be seen by examining the closed form solution to the E-square integral in three dimensions for a point current source and sink pair of current I a distance L apart in a uniform conductor of infinite extent

$$\tilde{R} = \int_{Volume} |\nabla\phi|^2 d^3x = \frac{I^2}{4\pi\sigma^2 L}. \tag{51}$$

So weighting by $\tilde{R}^{-1}$ emphasizes smaller currents and larger separations. (ii) Smaller gradients imply smoother potential field distributions. Furthermore, the minimum E-square integral occurs when the potential field satisfies the Laplacian $\nabla^2\phi = 0$. Thus, weighting against large gradients is most desirable when the conductivity distribution is either constant or fairly smooth with low contrasts, consistent with the linear least-squares approach under discussion here.

C. Minimum variation from background

Now consider the second term in (4)

$$\beta(\sigma) = (\sigma - \sigma_b)^T W_2 (\sigma - \sigma_b). \tag{52}$$

The motivation for using (52) as a criterion is that, although there may be many "solutions" to (3), the ones with least deviation from some prescribed (smooth) background are most likely to have no spurious structure. It is not desirable to have a reconstruction method that produces “interesting” features that are not real. High frequencies in the reconstruction should therefore be damped by some means. A stationary point of (52) occurs when σ satisfies

$$W_2(\sigma - \sigma_b) = 0, \tag{53}$$

i.e., when $\sigma - \sigma_b$ is in the null space of W_2. If the weight matrix is positive definite, then (53) implies that

$$\sigma = \sigma_b. \tag{54}$$

We could choose $W_2 = I$, but this choice ignores the fact that some of the cells have significantly more coverage than others. The cells with most coverage ought to be given the most weight in (52) and clearly any cell with no coverage should have no influence on the final result. The simplest choice of weight matrix satisfying these requirements is

$$W_2 = C. \tag{55}$$

Then, we see that (52) becomes

$$(\sigma - \sigma_b)^T C(\sigma - \sigma_b) = (z - z_b)^T (z - z_b) \tag{56}$$

where $z = C^{\frac{1}{2}}\sigma$ and

$$z_b = \zeta_1 x_{\lambda_1}. \tag{57}$$

Criteria based on minimum variance from some background such as (52) are appropriate only as secondary criteria [9]. Forcing σ to the value $\sigma = \sigma_b$ at the absolute minimum of (52) is too strong a constraint. A more reasonable constraint to impose is that the solution should have its component in the direction of σ_b equal in magnitude to σ_b but other components orthogonal to this direction should also be allowed, *i.e.,* $z_b^T(z - z_b) = 0$. In other words, the solution should be restricted to a hyperplane whose projection along the direction of z_b has the correct magnitude. We see that (46) has precisely this property. So it is possible to find a conductivity vector achieving the absolute minimum of (22) consistent with the data, while also satisfying a sensible minimum variance criterion.

D. Minimizing the general objective function

Now consider minimizing the full objective function (4) with the particular choices of weight matrices motivated in the preceding analysis

$$\alpha_\mu(\sigma) = (p - K\sigma)^T R^{-1}(p - K\sigma) + \mu(\sigma - \sigma_b)^T C(\sigma - \sigma_b). \tag{58}$$

Using (37) and $z = C^{\frac{1}{2}}\sigma$, (58) may be transformed to

$$\alpha_\mu(z) = (y - Az)^T(y - Az) + \mu(z - z_b)^T(z - z_b) \tag{59}$$

where A is again given by (37) and $y = R^{-\frac{1}{2}}p$. The minimum of (59) occurs when z satisfies

$$A^T(Az - y) + \mu(z - z_b) = 0, \tag{60}$$

or equivalently (since $A^T A z_b = z_b$),

$$(A^T A + \mu I)(z - z_b) = A^T y - z_b. \tag{61}$$

Using (39) and (57), we notice immediately that

$$z_b^T(A^T y - z_b) = 0, \tag{62}$$

which means that the right-hand side of (61) is orthogonal to z_b. Notice further that the eigenvectors of $A^T A$ are also eigenvectors of $A^T A + \mu I$, since

$$(A^T A + \mu I)x_{\lambda_j} = (\lambda_j^2 + \mu)x_{\lambda_j}. \tag{63}$$

Therefore, the solution of (61) with

$$\Delta z = z - z_b, \tag{64}$$

and

$$\Delta z^{(0)} = A^T y - z_b, \tag{65}$$

has the form

$$\Delta z = \sum_{j=2}^{r} \frac{\zeta_j}{\lambda_j^2 + \mu} x_{\lambda_j}, \tag{66}$$

where for $2 \le j \le r$

$$\zeta_j = x_{\lambda_j}^T \Delta z^{(0)} \tag{67}$$

are precisely the same numbers given by (40). For finite $\mu > 0$, (66) is the unique solution of (61) since the matrix on the left-hand side of (61) is nonsingular.

The complete "solution" of (3) is now $\sigma = C^{-\frac{1}{2}}z$, with z being the solution of (61), plus some conductivity vector from the null space of the E-square matrix K. In the absence of additional information to constrain the final result, the vector from the null space may be omitted.

Eq. (61) has some clear computational advantages over (27). In particular, if the smallest nonzero eigenvalue λ_r is very small [on the order of the round off error for computations of O(1)], then the computation of (46) and (48) will be numerically unstable, while the presence of a small but finite μ in (61) will stabilize the computation of (66).

Note that the result (61) has much in common with Marquardt's [21,22] algorithm for solving such problems using damped least-squares. In the presence of severe nonlinearity – which may often be the case for tomographic reconstruction of conductivity data – it may be advisable to use the damped algorithm with an optimum constant $\mu < 1$ found through trial and error as suggested by Aki and Richards [17]. Various alternative methods for choosing μ have been considered and many of these are summarized in the review by Titterington [23].

Finally, note that μ may be viewed as an interpolation parameter. As $\mu \to 0$, (66) $\to$ (48); as $\mu \to \infty$, (66) $\to$ (54). For finite but small μ, (66) interpolates between (48) and (54) while providing a stable approximation to (48).

Inhomogeneous Background

We will now generalize the discussion of the preceding section to the problem of finding appropriate minimization functionals of the form (4) when σ_b is not a constant conductivity vector. This situation will generally arise in the later stages of a comprehensive nonlinear inversion algorithm [4,5].

A. Functional design

If σ_b is not constant, then (13) and (14) no longer play an important role in the design of the least-squares functional. The question arises as to whether some vectors other than u and v can play a similar role. The main issue is whether a vector $(\bar{u}^T, \bar{v}^T)^T$ can be constructed that has unit eigenvalue for some problem of the form

$$\begin{pmatrix} 0 & K \\ K^T & 0 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \bar{\lambda} \begin{pmatrix} \bar{R} & 0 \\ 0 & \bar{C} \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}. \tag{68}$$

It turns out that such a vector can always be found. Furthermore, there is sufficient freedom in the choice of the vector that an additional condition may be placed on the minimization functional if desired.

Define

$$\bar{v} = \sigma_b/s \tag{69}$$

where s is some positive constant conductivity value chosen for convenience to make $\bar{v}$ dimensionless [one reasonable choice would be $s = s_0$ where s_0 is given by (16)], and let $\bar{u} = u$ for now. Then

$$K\bar{v} = \bar{R}\bar{u} \tag{70}$$

implies that the diagonal components of $\bar{R}$ satisfy

$$\bar{R}_i = \sum_{j=1}^{n} k_{i,j}(\sigma_b)_j/s = (p_b)_i/s, \tag{71}$$

where $(p_b)_i$ is the predicted power dissipation for the i-th current probe. Similarly,

$$K^T\bar{u} = Cv = \bar{C}\bar{v} \tag{72}$$

where the diagonal components of $\bar{C}$ satisfy

$$\bar{C}_j = sC_j/(\sigma_b)_j. \tag{73}$$

A derivation analogous to the one leading to (38) then gives

$$\begin{pmatrix} \bar{A}\bar{A}^T & 0 \\ 0 & \bar{A}^T\bar{A} \end{pmatrix} \begin{pmatrix} \bar{w}_{\bar{\lambda}} \\ \bar{x}_{\bar{\lambda}} \end{pmatrix} = \bar{\lambda}^2 \begin{pmatrix} \bar{w}_{\bar{\lambda}} \\ \bar{x}_{\bar{\lambda}} \end{pmatrix} \tag{74}$$

where

$$\bar{A} = \bar{R}^{-\frac{1}{2}}K\bar{C}^{-\frac{1}{2}} \tag{75}$$

with $\bar{w}_{\bar{\lambda}} = \bar{R}^{\frac{1}{2}}\bar{u}_{\bar{\lambda}}$ and $\bar{x}_{\bar{\lambda}} = \bar{C}^{\frac{1}{2}}\bar{v}_{\bar{\lambda}}$. Because our design of the eigenvalue problem begins by first constructing a strictly positive vector $(\bar{u}^T, \bar{v}^T)^T$ with eigenvalue unity, it again follows from the Perron-Frobenius theory of nonnegative matrices [19,20] that this eigenvalue is simple and that it is the maximum eigenvalue of $\bar{A}^T\bar{A}$ and $\bar{A}\bar{A}^T$, and also $\bar{A}$. The only caveat is that $\bar{A}^T\bar{A}$ must be assumed "primitive" as defined in Section 2. Thus, by construction, the eigenvalues $\bar{\lambda}_j^2$ lie in the range [0,1].

An additional degree of freedom arises in the problem with inhomogeneous background, since no constraint analogous to (16) applies. The choice $\bar{u} = u$ is the simplest, but it is not required. In fact, we are free to specify either $\bar{u}$ or $\bar{R}$ in (70) – once one is specified the other is determined. The choice is constrained only by the requirements that all the components of $\bar{u}$ must be positive, and that the diagonal components of $\bar{R}$ must be positive (the off-diagonal components vanish). A different choice for $\bar{u}$ also leads to a different choice of $\bar{C}$ in (72), *i.e.*, simply replace C_j in (73) by the j-th component of the vector

$K^T\bar{u}$. This additional degree of freedom implies that some other constraint on the final results can be forced by an appropriate choice of these weights. Although not required, the particular choices given by (71) and (73) have several features in common with the choices made in Section 2 which may be enough to recommend them generally: For example, (73) still has the desirable property that the elements of the weight matrix $\bar{C}$ are directly proportional to the elements of the cell coverage matrix C, and therefore tend to weight the cells with the most coverage most heavily. Similarly, (71) leads to a weighting favorable to current probes with the least power dissipation – which has similar logic to recommend it as that used by Frank and Balanis [10] for the shortest path-length weighting in traveltime tomography.

The analysis of (74) exactly parallels that of (38). Furthermore, if we make the choices

$$W_1 = \bar{R}^{-1} \tag{76}$$

and

$$W_2 = \bar{C} \tag{77}$$

in (4), then all of the analysis of Section 2 applies to the inhomogeneous background problem with barred quantities replacing unbarred.

We conclude that the analysis of the nonconstant background problem is virtually identical to that for constant background once the weight matrices have been determined.

B. Nonlinear inversion

A general nonlinear iterative inversion algorithm based on the preceding analysis may be summarized as follows: Given power dissipation data but no *a priori* knowledge of a sensible background conductivity model, first assume a constant background and compute the best estimate of the model conductivity using (28)-(29) with the weights (35) and (36). If the resulting least-squares error is too large, use the new model conductivity as the background and repeat the calculation. Continue until the resulting least-squares error is comparable to the error in the measured power dissipation data. Note that, except for the particular choices of weight matrices and an as yet unspecified algorithm for computing the squared magnitude of the electric field for the background model, this algorithm is analogous to that used by Lytle and Dines [24] in the case of traveltime tomography.

Discussion

Tomographic reconstruction of a conductivity model from power dissipation data based on Eq. (3) implicitly assumes that a background conductivity σ_b is known: the squared magnitude of the electric field used to determine the E-square matrix K should be based on the application of Laplace's equation to the construction of a sensible potential field through such a model. The background conductivity model should be expected to play a special (and explicit) role in the reconstruction process. The methods developed in this paper show that weighted least-squares criteria can always be found that map the reconstruction problem into a vector space where the contribution to the inversion from the background conductivity remains invariant; meanwhile the reconstruction proceeds to find optimum contributions in orthogonal directions. The resulting algorithm produces a hybrid conductivity model both consistent with the data in the least-squares sense and possessing the least possible variance from the background.

A problem that has not been addressed in this paper is that of obtaining the necessary power dissipation measurements. Obtaining power dissipation measurements is a problem because it is difficult to measure the voltages across the current injection electrodes reliably – partially because a substantial contact impedance develops at the interface between the region being probed and the electrodes. As the contact impedance is a function of the current magnitude, the data collection process can be nonlinear. If the application makes it possible, the effects of contact impedance can be eliminated or reduced to some extent by using electrodes with large surface areas [25]. In any case, we may be limited to using power dissipation measurements with substantial inaccuracy.

Applications and extensions of these methods to nonlinear conductivity inversion will be presented elsewhere.

Acknowledgments

We thank D. M. Goodman and T. J. Yorkey for helpful conversations. This work was performed under the auspices of the U. S. Department of Energy by the Lawrence Livermore National Laboratory under contract No. W-7405-ENG-48 and supported specifically by the Engineering Research Program, by the DOE Office of Basic Energy Sciences, Division of Engineering and Geosciences, and by the DOE Office of Technology Development.

References

[1] J. G. Berryman, "Weighted least-squares criteria for seismic traveltime tomography," *IEEE Trans. Geosci. Remote Sensing*, vol. 27, pp. 302-309, May, 1989.

[2] J. A. Scales, A. Gersztenkorn, and S. Treitel, "Fast l_p solution of large, sparse, linear systems: Application to seismic travel time tomography," *J. Comput. Phys.*, vol. 75, pp. 314-333, April, 1988.

[3] R. P. Bording, A. Gersztenkorn, L. R. Lines, J. A. Scales, and S. Treitel, "Applications of seismic travel-time tomography," *Geophys. J. R. astr. Soc.*, vol. 90, pp. 285-303, August, 1987.

[4] K. A. Dines and R. J. Lytle,"Analysis of electrical conductivity imaging," *Geophysics*, vol. 46, pp. 1025-1036, July, 1981.

[5] J. G. Berryman and R. V. Kohn, "Variational constraints for electrical-impedance tomography," *Phys. Rev. Lett.*, vol. 65, pp. 325-328, July, 1990.

[6] C. R. Rao and S. K. Mitra, *Generalized Inverse of Matrices and its Applications*, New York: Wiley, 1971.

[7] A. Albert, *Regression and the Moore-Penrose Pseudoinverse*, New York: Academic, 1972.

[8] R. Penrose, "On best approximate solutions of linear matrix equations," *Proc. Cambridge Phil. Soc.*, vol. 52, pp. 17-19, January, 1956.

[9] G. T. Herman, *Image Reconstruction from Projections – The Fundamentals of Computerized Tomography*, New York: Academic, pp. 103-106 and 210-221, 1980.

[10] M. S. Frank and C. A. Balanis, " Methods for improving the stability of electromagnetic geophysical inversions," *IEEE Trans. Geosci. Remote Sensing*, vol. 27, pp. 339-343, May, 1989.

[11] F. Natterer, *The Mathematics of Computerized Tomography*, New York: Wiley, 1986.

[12] P. Gilbert, "Iterative methods for the three-dimensional reconstruction of an object from projections," *J. Theor. Biol.*, vol. 36, pp. 105-117, July, 1972.

[13] K. A. Dines and R. J. Lytle, "Computerized geophysical tomography," *Proc. IEEE*, vol. 67, pp. 1065-1073, July, 1979.

[14] S. Ivansson, "Remark on an earlier proposed iterative tomographic algorithm," *Geophys. J. R. astr. Soc.*, vol. 75, pp. 855-860, December, 1983.

[15] A. Ben-Israel and D. Cohen, "On iterative computation of generalized inverses and associated projections," *SIAM J. Num. Anal.*, vol. 3, pp. 410-419, September, 1966.

[16] N. R. Burkhard, "Resolution and error of the back projection technique algorithm for geophysical tomography," Lawrence Livermore National Laboratory Rep., UCRL-52984, September, 1980.

[17] K. Aki and P. G. Richards, *Quantitative Seismology – Theory and Methods*, Vol. II, San Francisco: Freeman, Sect. 12.3, 1980.

[18] C. Lanczos, *Linear Differential Operators*, New York: Van Nostrand, Chapt. 3, pp. 100-162, 1961.

[19] E. Seneta, *Non-negative Matrices – An Introduction to Theory and Applications*, New York: Wiley, pp. 1-6, 1973.

[20] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, New York: Academic, pp. 27-32, 1979.

[21] D. W. Marquardt, "An algorithm for least squares estimation of non-linear parameters," *J. Soc. Indust. Appl. Math.*, vol. 11, pp. 431- 441, June, 1963.

[22] D. W. Marquardt, "Generalized inverses, ridge regression, biased linear estimation, and nonlinear estimation," *Technometrics*, vol. 12, pp. 591-612, August, 1970.

[23] D. M. Titterington, "Common structure of smoothing techniques in statistics," *Intern. Stat. Rev.*, vol. 53, pp. 141-170, 1985.

[24] R. J. Lytle and K. A. Dines, "Iterative ray tracing between boreholes for underground image reconstruction," *IEEE Trans. Geosci. Remote Sensing*, vol. GE-18, pp. 234-240, July, 1980.

[25] K.-S. Cheng, S. J. Simske, D. Isaacson, J. C. Newell, and D. G. Gisser, "Errors due to measureing voltage on current-carrying electrodes in electric current computed tomography," *IEEE Trans. Biomed. Eng.*, vol. BME-37, pp. 60-65, January, 1990.