KOZENY-CARMAN RELATIONS AND IMAGE PROCESSING METHODS FOR ESTIMATING DARCY'S CONSTANT

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ABSTRACT

A natural connection is demonstrated between Kozeny-Carman relations for porous media and the image processing techniques which have recently been applied to the problem of estimating the parameters in such relations. We show that the specific surface area term in the Kozeny-Carman relation is best estimated from a smoothed version of the actual material surface and then demonstrate that to measure this image specific surface the magnification of a cross-section of the porous material should be chosen so that a typical correlation length for the sample corresponds to a distance comparable to 100 discrete picture elements. Under these conditions, the assumptions typically made in the derivation of a Kozeny-Carman relation are entirely compatible with the resolution constraints imposed by digitizing the image. Thus, although the measured image specific surface may be considerably smaller in magnitude than the true specific surface area of the material (due to resolution constraints), this smaller value is nevertheless the required input to the Kozeny-Carman relation. The argument is based on a known comparison theorem relating the permeabilities of two porous materials which differ only by the addition (without rearrangement) of solid to the one more porous.

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1. INTRODUCTION

The rate at which a viscous fluid will flow through a porous medium is proportional to the pressure gradient, and the constant of proportionality is called Darcy's constant (Darcy, 1856), or the permeability k. The most successful methods of analyzing permeability have been the empirical methods associated with the names of Kozeny and Carman (Carman, 1979). These methods are generally based on very simple formulas derived using cylindrical tube models for the pores [for a recent review of various pore modeling techniques, see Quiblier (1984)]. The empiricism enters through constant scale factors that are chosen to fit data (Bear, 1972; Dullien, 1979). Walsh and Brace (1984) reexamined this approach and, in the same spirit, obtained a formula relating permeability to porosity, formation factor, and specific surface area. They found their formula provided a good means of correlating data on permeability and formation factor for low-porosity, low-permeability granites.

Recently image processing methods have been proposed as a means of estimating the parameters in the Kozeny-Carman relation (Berryman, 1985; Berryman and Blair, 1986). A cross-section of the porous material of interest is scanned by an electron microscope and the resulting image is digitized. Then, after segmenting the image into regions of material and void, the porosity and the two-point correlation function of the image are measured. The two-point correlation function of the imaged surface is then used to estimate the specific surface area using a theorem due to Debye et al. (1957). The permeabilities obtained in this way have been found to be in surprisingly good agreement with laboratory values. Yet, the Kozeny-Carman relation is itself only an approximation – depending as it does on smooth-walled cylindrical pores and an approximate dependence on the tortuosity. Similarly, the image processing method is not expected to measure the true specific surface area of naturally occurring materials, because the discretization of the image required in digital image processing places unavoidable constraints on the resolution. Despite the clear limitations of both of these methods we find that, when the two approximations are combined, the resulting estimates of permeability for the materials studied are quite reasonable (± 20%). Are these results merely fortuitous then, or is it possible to explain the observed accuracy of these results?

The purpose of this paper is to demonstrate that the approximations which are made in the derivation of a Kozeny-Carman relation are in fact entirely compatible with the resolution constraints imposed by digitizing the image of a cross-section of the material if the magnification is chosen according to a certain receipe: a typical correlation length for the sample should be approximately two orders of magnitude (100 times) larger than a discrete picture element (a pixel). Section 2 presents a derivation of the Kozeny-Carman relation with emphasis placed on the various generic approximations made in the standard derivations. We show that the surface area to be used in the Kozeny-Carman relation is that for a smoothed version of the true void/solid interface. Section 3 shows that the discretization of an image of a cross-section of the porous material automatically provides a method of smoothing the void/solid interface. The quantitative effect of this smoothing depends on the choice of magnification and on the size of the resulting picture elements relative to the scale of the fine structure determining the specific surface area. Section 4 demonstrates that an optimum choice of magnification can be found by analyzing the changes in image specific surface as a function of magnification.

2. KOZENY-CARMAN RELATIONS

Empirical relations between fluid permeability and various other measureable properties of a porous material have a long history dating back to work of Kozeny, Carman (1979), and many others [see Bear (1972), Dullien (1979), Quiblier (1984)]. These relations are generally derived from simple models of pore structure such as straight cylindrical tubes with smooth walls. Walsh and Brace (1984) recently used such a model to derive a formula relating permeability to porosity, formation factor, and specific surface area. These authors found that their formula provided a good means of correlating data on permeability and formation factor for low-porosity, low-permeability granites. Their formula has the general form

$$k = \frac{\phi^2}{cFs^2}, \tag{1}$$

where the factor c is a weak function of the shape of the tube cross-section in the model used to derive the formula. Typically, $c = 2$ for circular tubes and $c = 3$ for flat cracks. Approximate formulas such as Eq. (1) are well known in the study of permeability, and will be referred to as Kozeny-Carman relations. In version (1) of the Kozeny-Carman relation, the electrical formation factor F (*i.e.*, the electrical conductivity of a conducting pore fluid divided by the effective electrical conductivity of the saturated porous insulating material) supplies a measure of the tortuosity of the connected pore space, as we will show later in this section.

We emphasize that the formula in Eq. (1) is only an approximation, derived assuming that the pore space is well-characterized by a network of tubes with smooth surfaces and with simple connectedness properties. Consequently, care must be exercised in our choice of value for s to use as input to Eq. (1). If the particle surfaces are very rough, the pertinent "specific surface area" may not be the physical surface area that one would measure using (for example) the nitrogen adsorption technique (Knight and Nur, 1985). We propose that the pertinent value for s is one for a smoothed idealization of the real material. In the next section, we show that the digitization process automatically provides a natural means of doing the necessary smoothing.

Consider a single cylindrical tube of circular cross-section. Then Landau and Lifshitz (1959) show that the mass flux of fluid through such a tube with radius R is

$$Q = \frac{\pi}{8\nu} R^4 \frac{\Delta p}{\Delta z} \tag{2}$$

where ν is the kinematic viscosity and $\Delta p/\Delta z$ is the magnitude of the pressure gradient along the tube. It follows easily that the effective permeability introduced by a single tube embedded in a block of solid material with cross-sectional area A, length L along the tube axis, and volume $V = AL$, is therefore

$$k = (Q\nu/A)/(\Delta p/\Delta z) = \frac{\pi R^4}{8A}. \tag{3}$$

For this problem, the tube tortuosity is unity; the specific surface area is given by the tube surface area per unit volume $s = 2\pi R/A$; and the porosity is given by $\phi = \pi R^2/A$. To rewrite (3) in terms of the porosity and specific surface area, consider

$$k = const \times \phi^m/s^n. \tag{4}$$

Substituting the expressions for ϕ and s in terms of R and A into (4) and comparing with (3) while noting that R and A are independent, we find easily that the exponents m, n must satisfy $m - n = 1$ and $2m - n = 4$, so $m = 3$ and $n = 2$. Thus, we have the unique representation

$$k = \frac{1}{2}\frac{\phi^3}{s^2} \tag{5}$$

for the effective permeability of a single tube of circular cross-section oriented along the pressure gradient. The corresponding result for tubes of arbitrary ellipsoidal (major and minor axes a,b) cross-section is virtually identical to (5).

The result for the permeability is again supplied by Landau and Lifshitz (1959) and is given by $k = (\pi/4A)[a^3b^3/(a^2+b^2)]$. The porosity for an ellipsoidal tube is $\phi = \pi ab/A$ and the specific surface area is well approximated by $s \simeq 2\pi[(a^2+b^2)/2]^{\frac{1}{2}}/A$. The relationship between k, ϕ, and s is thus easily seen to be well approximated by (5). The fact that (5) is a good approximation for such a wide class of tube cross-sections provides an additional motivation for the choice of $c = 2$ in (1) used by Berryman and Blair (1986) as a conservative estimate of the permeability.

Now we need to know how the result (5) changes if the walls of the tube are rough instead of smooth. The comparison theorem in the Appendix gives us a simple method of estimating the effect of roughness. Consider Figure 1. If the outer circle has radius R_o and the inner circle (dashed line) has radius R_i, then the permeability of material containing such a rough-walled cylindrical pore must satisfy

$$\frac{1}{2}\frac{\phi_i^3}{s_i^2} = k_i \le k \le k_o = \frac{1}{2}\frac{\phi_o^3}{s_o^2}. \tag{6}$$

The key point to observe about (6) is this: If the magnitude of the asperities producing the roughness is small so that $R_o - R_i$ is small, then $\phi_o - \phi_i$ and $s_o - s_i$ are also small. Furthermore, it is clear that the porosity ϕ for the rough-walled sample satisfies

$$\phi_i \le \phi \le \phi_o. \tag{7}$$

[Wong et al.(1986) have recently shown experimentally that the pore volume is not fractal for some sandstones, shales, limestones, and dolomites, confirming our intuition as expressed in (7).] By contrast, it is definitely *not* true in general that the specific surface area s of the rough-walled sample satisfies $s_i \le s \le s_o$. In fact, if the surface is very rough [*e.g.*, fractal – see Wong et al.(1986)], then it is entirely possible that

$$s_i \le s_o \ll s. \tag{8}$$

Thus, Eq. (5) cannot be valid for rough-walled porous materials. Nevertheless, it follows from (6) and (7) that an approximation for k can be obtained by writing

$$k \simeq \frac{1}{2}\frac{\phi^3}{\bar{s}^2}, \tag{9}$$

where $\bar{s}$ is not the physical specific surface area, but an effective specific surface area which satisfies

$$s_i \le \bar{s} \le s_o. \tag{10}$$

The parameter $\bar{s}$ therefore has the significance of being the specific surface area of a smoothed – or poorly resolved – representation of the true void/solid interface. Furthermore, the error in (10) can be estimated easily from (3) and (6). If $R_o = R_i + \Delta R$, then for small ΔR we find that the relative error is bounded by

$$k/k_i - 1 \leq 4\frac{\Delta R}{R_i}. \tag{11}$$

Thus, for a single straight tube, if we make an error of one percent in estimating the tube radius, the error in the estimate of k should be no worse than four percent.

Pores of naturally occurring materials are virtually never straight tubes, and may not be very well approximated by bent tubes. However, to obtain an estimate of the importance of tortuosity on the fluid permeability, we again consider the model of Walsh and Brace (1984). Consider Figure 2. Over short sections of a tube, the pertinent pressure gradient is dp/dl where dl is directed along the tube axis. The applied pressure gradient is dp/dz with respect to the specimen axis. If the tube axis is at an angle θ with respect to the specimen axis, then

$$\frac{dp}{dl} = cos\theta\frac{dp}{dz} \tag{12}$$

and the actual path length relative to the apparent path length is given by the tortuosity

$$\tau = dl/dz = 1/cos\theta. \tag{13}$$

Now, the specific surface area is given by the tube surface area per unit volume $s = 2\pi R\tau/A$, and the porosity is given by $\phi = \pi R^2\tau/A$. Following the derivation of (5), we find that the permeability is therefore given by

$$k = (Q\nu/A)/(dp/dz) = \frac{\pi R^4}{8A}/(dl/dz) = \frac{1}{2}\frac{\phi^3}{s^2\tau^2}. \tag{14}$$

A similar calculation of the formation factor (Walsh and Brace, 1984) shows that

$$F = \tau^2/\phi. \tag{15}$$

To obtain a formula for Darcy's constant which will have some general applicability, we substitute (15) into (14) and recall our previous arguments concerning the necessity of replacing s by $\bar{s}$. The final result is

$$k = \frac{\phi^2}{2F\bar{s}^2}. \tag{16}$$

Eq. (16) is the form of the Kozeny-Carman relation we prefer.

It has been shown previously (Berryman and Blair, 1986) that formulas such as (16) should not be used without modification if $\phi \geq 0.5$. Furthermore, although we have shown that it is possible to account for the effects of roughness by choosing an appropriate averaged value of $\bar{s}$, we have not accounted for the fact that some of the surface area may lie in regions of secondary porosity, completely disconnected from the primary (connected) porosity contributing to the overall fluid permability. The presence of secondary porosity tends to inflate the values of both ϕ and $\bar{s}$ measured from images of cross sections. Although it is possible that such effects are not negligible for some porous materials of interest, we will nevertheless ignore the effects of secondary porosity in the present analysis. Another significant limitation of (16) arises from the implicit assumption that a single "representative" pore can be used to model the behavior of the porous medium as a whole. The implications of this assumption for the general applicability of (16) will be discussed in Section 5.

3. RESOLUTION OF IMAGE PROCESSING METHODS

For a porous material, we define a characteristic function $f(\vec{x}) = 0$ or 1. Then, we say that void regions have $f = 1$, while material regions have $f = 0$. The first two void-void correlation functions are then given by

$$\hat{S}_1 = \langle f(\vec{x}) \rangle = \phi, \tag{17}$$

and

$$\hat{S}_2(\vec{r}_1, \vec{r}_2) = \langle f(\vec{x} + \vec{r}_1) f(\vec{x} + \vec{r}_2) \rangle. \tag{18}$$

The brackets $\langle \cdot \rangle$ indicate a volume average over the spatial coordinate $\vec{x}$. The void volume fraction (or porosity) is given by ϕ. We will refer to the correlation functions defined above as the one- and two-point correlation functions, respectively. The one- and two-point correlations may be measured by processing images of material cross-sections (Berryman, 1985). In general, we assume that the porous medium of interest is statistically homogeneous so that on average only the differences in the coordinate values are significant (translational invariance). Furthermore, we often assume that the material is at least locally isotropic so that the averages depend only on the magnitude, not the orientation, of the arguments. With these assumptions, we find that the two-point correlation function simplifies to

$$\hat{S}_2(\vec{r}_1, \vec{r}_2) = \tilde{S}_2(\vec{r}_2 - \vec{r}_1) = S_2(\mid \vec{r}_2 - \vec{r}_1 \mid). \tag{19}$$

For isotropic materials, the most important properties of two-point correlation functions are (with $r = \mid \vec{r} \mid$)

$$S_2(0) = \phi, \tag{20}$$

$$\lim_{r \to \infty} S_2(r) = \phi^2, \tag{21}$$

and

$$S_2'(0) = -s/4, \tag{22}$$

where s is the specific surface area (internal surface area per unit volume). The important theorem stated in Eq. (22) concerning the relationship between the two-point correlation function and the specific surface area has been known since the work of Debye et al. (1957), and can be derived fairly simply from its definition (Berryman and Blair, 1986).

Although (17), (20), and (22) show that two of the quantities of interest for evaluating the Kozeny-Carman relation (16) may be obtained in principle from these correlation functions, the digital image processing methods used to calculate S_1 and S_2 necessarily introduce some error through the quantization process. The quantization error in the porosity depends on the magnification and on the magnitude of the porosity. In addition, for a fixed value of magnification, the measured values of the porosity have been observed to change by 2-3 % as the operator chooses different threshold values for segmenting the image. By contrast, the computed value of the image specific surface changes very little for the same choices of threshold. The image specific surface is however very strongly influenced by the choice of magnification as is illustrated in Figure 3. To see why this so, consider square pixels of width h. Then, any void pixel which is surrounded by solid pixels has void/solid interface of length $4h$. The maximum surface area of a digitized image clearly occurs when all void pixels are isolated from each other. Thus, if n is the number of void pixels and N is the number of pixels along each side of a square image array (*e.g.*, $N = 512$ for most of our applications), then the maximum specific surface in a binary image is given by $\bar{s}_{max} = 4hn/A = 4n/N^2h = 4\phi/h$. Since it is well-known that the plane figure with smallest perimeter per unit area is a circle, the minimum specific surface in a binary image is obtained when the void pixels coalesce into a shape as nearly circular as possible. The effective radius of such a figure is given approximately by $\bar{R} \simeq (\phi A/\pi)^{\frac{1}{2}}$ and the minimum surface area by $\bar{s}_{min} \simeq 2\pi\bar{R}/A = 2(\phi\pi/A)^{\frac{1}{2}} = (2\pi^{\frac{1}{2}}/N)(\phi^{\frac{1}{2}}/h)$. Thus, $\bar{s}$ satisfies

$$(2\pi^{\frac{1}{2}}/N)(\phi^{\frac{1}{2}}/h) \leq \bar{s} \leq 4\phi/h, \tag{23}$$

showing that the pixel size h scales the magnitude of possible values for $\bar{s}$. [The inequalities in (23) remain valid for all $\phi \geq \pi/4N^2$; however, we need not consider $\phi \leq 1/N^2$ since such low porosities correspond to having at most one void pixel in an $N \times N$ image array. For typical porosities (*i.e.*, $\phi \simeq 0.2$), the range of possible values covers about two orders of magnitude, so (23) is not terribly restrictive.] If $\bar{s}$ is close to the lower bound in (23), the digitized image must closely resemble the model used to derive (5). It is always possible to choose a value of the magnification so that the image to be analyzed contains only a few pores; when this is so, the value of $\bar{s}$ may in fact approach the lower bound in (23). Thus, it is possible that image processing can be used in this way to approach the circumstances assumed in the derivation of the Kozeny-Carman relation.

We will discuss a strategy for determining an optimum value for $\bar{s}$ in the next section.

4. OPTIMUM CHOICE OF IMAGE MAGNIFICATION

Berryman and Milton (1985) have shown that a rigorous bound on the permeability of a porous material is given by

$$k \leq \frac{2}{3}\frac{\int_0^\infty dr r\left[S_2(r) - \phi^2\right]}{(1-\phi)^2}. \tag{24}$$

To evaluate the integral in (24), Berryman and Blair (1986) made a crude approximation to the two-point correlation function. It follows from Eqs. (20)-(22) that, for small r, the two-point correlation function is given approximately by

$$\bar{S}_2(r) = \phi - \frac{1}{4}sr, \tag{25}$$

whereas for large r, it is given by

$$\bar{S}_2(r) = \phi^2. \tag{26}$$

The straight lines defined in Eqs. (25) and (26) intersect at the point

$$r_c = \frac{4\phi(1-\phi)}{s}. \tag{27}$$

Thus, if we extrapolate both Eqs. (25) and (26) beyond their strict range of validity using Eq. (25) for all $r \le r_c$ and Eq. (26) for all $r \ge r_c$, then the integral in (24) may be approximated as

$$\int_0^{r_c} drr\left[\bar{S}_2(r) - \phi^2\right] = \frac{8}{3}\frac{\phi^3(1-\phi)^3}{s^2}. \tag{28}$$

The resulting crude estimate of the bound on permeability is

$$k \le \frac{16}{9}\frac{\phi^3}{s^2}(1-\phi). \tag{29}$$

Although the numerical factor disagrees, the form of (29) is quite close to that of (5). This analysis shows that the approximation made to the correlation function in (25) and (26) is not grossly in error for our present purposes.

Now, the approximation to $S_2(r)$ embodied in (25)-(26) can be very poor if s is estimated from an image with high resolution, but it may be quite reasonable if estimated from one with fairly low resolution (see Figure 4). If we replace s in (25)-(27) by the low resolution estimate s_{LR}, we see that $r_c/(1-\phi) = 4\phi/s_{LR}$ becomes an estimate of the correlation length for the porous material. For the problem of cylindrical pores normal to the imaging plane, recall that $\phi = \pi R^2/A$ and $s = 2\pi R/A$, so $r_c/(1-\phi) = 4\phi/s = 2R$. Thus, r_c provides a direct estimate of the average pore diameter when the porosity is small, or an estimate of the pore radius for $\phi \simeq \frac{1}{2}$. Because of the low resolution image used to obtain r_c, its numerical value may be as much as a factor of 2 to 4 too low. This level of error makes the value of r_c unsatisfactory for making direct estimates of the permeability. However, this value is sufficiently accurate that it may be used to estimate the optimum value of magnification to choose for making the final estimate.

To tailor the image processing method for optimum application to permeability estimation, we again recall the result that small scale surface roughness has a negligible effect on the fluid permeability. Thus, our measurement of the image specific surface area will be adequate when the pixel size (h) is some small faction of the given material's natural correlation length (r_c). We propose two criteria for choosing the appropriate pixel size and its related magnification: (1) Choose the pixel size to be about one percent of the size of an average pore radius, or

$$h \simeq r_c/200. \tag{30}$$

This criterion assures that the measurement error in the average pore radius due to surface roughness and pixel quantization is on the order of one percent. (2) Increase the magnification until the pixels are comparable in size to the surface roughness of the material, but stop before an average pore occupies more than one quarter of the entire digitized image. This rule implies that the pixel size and natural correlation length of the material are related by $r_c^2/(1-\phi)^2 \leq \frac{1}{4}(Nh)^2$ or

$$h \geq 2r_c/N(1-\phi). \tag{31}$$

This criterion is suggested by the derivation of the Kozeny-Carman relation, based as it is on the analysis of flow through a single pore. The magnification clearly need not be increased to such an extent that an average pore diameter is *larger* than the width of the digitized image, for then the image would not contain even one entire pore. How much larger the image width should be than the pore diameter is unclear, but choosing a factor of two margin gives approximately equal statistical weight in the image to void and material pixels and, thus, the overall estimate of the magnitude of the interface region should be close to optimum.

The main point to understand about (30) and (31) is that they both show there is no advantage in increasing the magnification of our rock cross sections until h violates these limits. We expect to obtain the maximum amount of useful information about fluid flow properties when either (30) or (31) is satisfied.

To provide an explicit example, consider the results obtained for Berea sandstone in Table I. If we use the values of ϕ and $\bar{s}$ obtained at $100\times$ magnification, then the value of the correlation length is found to be $r_c = 20\mu m$. When $N = 512$, (31) predicts that the optimum magnification will produce a pixel width satisfying $h \geq 0.095\mu m$. This limiting size corresponds to an image magnification that lies between $1000\times$ and $2000\times$. We have typically found the best agreement between the predictions and the measurements of permeability for magnifications in the range $400\times$ – $1000\times$, which is consistent with these predictions.

5. DISCUSSION AND CONCLUSIONS

For Berea and the other sandstones studied (Berryman and Blair, 1986), we have demonstrated empirically the validity of the implicit assumption that an average or representative pore can be used to model the behavior of the whole porous medium. Thus, we have shown by example that (16) works well for some materials. However, counter-examples may also be found. If the porous material of interest possesses a wide distribution of pore sizes and shapes, then some modification of (16) may be necessary or a more sophisticated method altogether may be needed. If the model material were composed of M straight, parallel, ellipsoidal tubes and if the i–th tube contributed ϕ_i to porosity and s_i to the surface area per unit volume, the permeability along the tubes would be given by the sum of the individual contributions

$$k = \frac{1}{2}\sum_{i=1,M}\frac{\phi_i^3}{s_i^2}. \tag{32}$$

By contrast, Eq. (5) predicts that in this situation the permeability is given instead by

$$k = \frac{1}{2}(\sum_{i=1,M}\phi_i)^3/(\sum_{i=1,M}s_i)^2. \tag{33}$$

If all the tubes are identical, then $\phi = M\phi_0$, $s = Ms_0$, and

$$k = \frac{M}{2}\frac{\phi_0^3}{s_0^2} \tag{34}$$

follows from either (32) or (33). Therefore, Eqs. (5) and (16) are expected to be reasonable approximations when the distribution of pore sizes and shapes is narrow. If the distribution is wide, the methods used here would require substantial alteration. Formulas such as (32) could be evaluated using image processing methods by simply subdividing the image into M parts (one for each significant pore) and proceeding as before. Subdivision should work satisfactorily for estimating the local permeability of a thin section for then the tortuosity may be assumed to approach unity. However, subdivision may not work well for thick samples for then each of the M tubes might have very different tortuosities – in which case it would no longer be feasible to obtain the generalization of (32) corresponding to (16).

In conclusion, we have shown that the approximations made in the derivation of the Kozeny-Carman relation (5) are generally compatible with the resolution constraints imposed by digitizing the image of a cross-section of the material if the magnification is chosen so that the pixel size and the principal correlation length for the porous material are related by either (30) or (31). The key point is that in a well-defined range of magnifications the image specific surface measured from a digitized cross-section provides the estimate of the smoothed image specific surface area needed in the Kozeny-Carman relation. For the sandstones we have considered, the values of $\bar{s}$ for the lowest resolution images ($\simeq 200\times$ and below) provide crude estimates of the natural correlation length and pore size of the material [see Eq. (27)]. The values of $\bar{s}$ for higher resolution images (greater than $200\times$ but less than $2000\times$) are the ones we want for the Kozeny-Carman relation (16). Still higher magnifications will often be required to obtain the true specific surface area s of the material.

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APPENDIX

Suppose that a macroscopic pressure gradient $\Delta p/\Delta z$ *of unit magnitude* is applied along the z-axis of a porous medium with permeability k. Then, it is shown by Berryman (1986) that a general expression satisfied by k is

$$k \leq \frac{1}{2}\langle f\tau_{ij}\tau_{ij}\rangle, \tag{A1}$$

where $\langle\cdot\rangle$ is the spatial ($\vec{x}$) volume average, τ_{ij} is any trial stress distribution satisfying certain admissibility conditions, and $f(\vec{x})$ is the characteristic function defined in Section 3.

Now consider two porous media with stochastic functions f and g whose porosities are given by

$$\langle f\rangle = \phi_f \quad and \quad \langle g\rangle = \phi_g. \tag{A2}$$

Then, if we define the right hand side of (A1) to be $k_f(\tau)$, we have

$$k \leq k_f(\tau) = \frac{1}{2}\langle f\tau_{ij}\tau_{ij}\rangle. \tag{A3}$$

If the stochastic functions of these two materials are related by the inequality

$$g(\vec{x}) \geq f(\vec{x}) \quad \mathit{for\ all} \quad \vec{x}, \tag{A4}$$

then the material characterized by g is more porous than the one by f. We can construct the new material (g) satisfying (A4) by removing solid material from the original (f) without rearranging the original material. For any given trial stress distribution τ, it follows that

$$\langle f\tau_{ij}\tau_{ij}\rangle \leq \langle g\tau_{ij}\tau_{ij}\rangle. \tag{A5}$$

Substituting (A5) into (A3) and defining $k_g(\tau)$ by analogy to $k_f(\tau)$, we find easily that

$$k \leq k_f(\tau) \leq k_g(\tau), \tag{A6}$$

where k is the true permeability of the material charaterized by f.

The inequalities in (A6) are true for any admissible tensor τ. For example, if $\tau = \sigma_g$ where σ_g minimizes k_g, then

$$k \leq k_f(\sigma_g) \leq k_g(\sigma_g). \tag{A7}$$

Similarly, if σ_f minimizes k_f, then

$$k = k_f(\sigma_f) \leq k_g(\sigma_f). \tag{A8}$$

Thus, it follows from (A7) and (A8) that

$$k = k_f(\sigma_f) \leq k_f(\sigma_g) \leq k_g(\sigma_g). \tag{A9}$$

Thus, if a porous material differs from one less porous only by the removal of solid material (with no rearrangement of the remaining material), then the permeability of the less porous material is always less than or equal to that of the more porous one. The present derivation has been given previously by Berryman (1986). A similar theorem due to Hill and Power (1956) states that the viscous drag on a particle of arbitrary shape is always less than that on any larger particle whose bounding surface could completely enclose the smaller particle.

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Table I. Comparison of porosity and permeability values for Berea sandstone produced using image processing methods with those obtained from laboratory measurements. The formula used for the permeability value obtained from the image in all cases was $k_{KC} = \phi^2/2F\bar{s}^2$.

Magnification	Number of images	Pixel width $h(\mu m)$	Image specific surface $\bar{s}(\mu m^{-1})$	Porosity ϕ		Permeability k(D)		Formation factor F
				Image	Laboratory	Image	Laboratory	
20	1	8.547	0.0091	0.10	0.15-0.18[a]	0.935	0.023[a]	62[b]
50	1	4.049	0.0220	0.18	0.15-0.18[a]	0.516	0.023[a]	62[b]
100	2	1.447	0.0281	0.17	0.15-0.18[a]	0.312	0.023[a]	62[b]
200	1	0.720	0.0354	0.18	0.15-0.18[a]	0.197	0.023[a]	62[b]
490-500	4	0.294	0.1109	0.233	0.15-0.18[a]	0.021[c]	0.023[a]	62[b]
1000	1	0.144	0.1231	0.393	0.15-0.18[a]	0.016[c]	0.023[a]	62[b]

[a]Daily and Lin (1985)

[b]Dullien (1979)

[c]Computed using image porosity for magnification 100.

Figure Captions

Figure 1. Schematic drawing of a rough walled tube as modelled by the Kozeny-Carman relation. Surface roughness does not have a strong effect on the overall fluid permeability of a tube, because a slightly smaller tube is known to have a comparable permeability.

Figure 2. The effective tortuosity of a cylindrical tube at an angle θ to the specimen axis is $\tau = 1/cos\theta$. The effective porosity is $\phi = \pi R^2\tau/A$ and the effective specific surface area is $s = 2\pi R\tau/A$.

Figure 3. Image specific surface as a function of magnification for Berea sandstone. The values for the lowest resolution images ($200\times$ and below) provide estimates of the natural correlation length and pore size of the material [see Eq. (27)]. The values for higher resolution images (between $490\times$ and $2000\times$) provide estimates of the smoothed image specific surface area needed in the Kozeny-Carman relation. Still higher magnifications are required to obtain the true specific surface area of this material.

Figure 4. Two-point correlation functions for images of Berea sandstone at (a) $100\times$, (b) $200\times$, and (c) $500\times$. As the image magnification is increased from $100\times$ to $500\times$, the resolution of the particle surfaces increases to the point where the first few measured values of S_2 can be used to estimate the smoothed specific surface area needed in the Kozeny-Carman relation.