Chapter 1
Resolution of Iterative Inverses in Seismic Tomography*

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Abstract

With applications to seismic traveltime tomography in mind, methods have been developed for computing both the model and data resolution matrices for iterative inverses such as those produced by the Lanczos scheme for finding a matrix inverse.

1 Introduction

The central role of resolution in inversion has been emphasized by Backus and Gilbert [1], who base their general inversion methods on techniques designed to optimize the resolution of the resulting geophysical model obtained from processed data. Tomographic reconstruction schemes based on seismic data — one important example of the more general inversion problem — have well-defined resolution properties [4]. Yet, these properties are not computed as often as they might be because of the common misconception [3] that resolution matrices can only be computed using singular value decomposition (SVD). Iterative inversion methods such as conjugate gradients, conjugate directions, Lanczos, and LSQR are used more commonly than SVD in tomography codes, because of their smaller storage and computing requirements [9]. The state of our knowledge of tomographic resolution would therefore improve if methods for computing resolution matrices were available for the most common iterative matrix inversion schemes.

This paper shows how resolution matrices can be computed for the Lanczos matrix inversion scheme [5]. Generalization for other iterative inversion schemes is straightforward.

2 Tridiagonalization Method of Lanczos

Consider the linear inversion problem of the form $\mathbf{Ms} = \mathbf{t}$, where $\mathbf{s}$ is the unknown. For a crosshole seismic tomography problem [2], $\mathbf{M}$ is an $m \times n$ ray-path matrix, $\mathbf{t}$ is an m-vector of first arrival traveltimes, and $\mathbf{s}$ is the model slowness (inverse velocity) vector. The method of Lanczos [5] solves this problem by introducing a sequence of orthonormal vectors $\mathbf{z}^{(k)}$ through a process of tridiagonalization. To obtain the minimum norm least-squares solution of $\mathbf{Ms} = \mathbf{t}$, the method may be applied to the normal equations

$$\mathbf{M}^T\mathbf{Ms} = \mathbf{M}^T\mathbf{t}, \tag{1}$$

since they have the form $\mathbf{Ax} = \mathbf{b}$ with square symmetric matrix $\mathbf{A} = \mathbf{M}^T\mathbf{M}$, unknown vector $\mathbf{x} = \mathbf{s}$, and data vector $\mathbf{b} = \mathbf{M}^T\mathbf{t}$.

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2.1 Algorithm

With $\mathbf{A} = \mathbf{M}^T\mathbf{M}$ and $\mathbf{b} = \mathbf{M}^T\mathbf{t}$, the Lanczos algorithm is a projection procedure equivalent to the following:

$$\mathbf{z}^{(1)}\left(\mathbf{z}^{(1)}\right)^T\mathbf{b} = \mathbf{b}, \tag{2}$$

$$\left[\mathbf{z}^{(1)}\left(\mathbf{z}^{(1)}\right)^T + \mathbf{z}^{(2)}\left(\mathbf{z}^{(2)}\right)^T\right]\mathbf{A}\mathbf{z}^{(1)} = \mathbf{A}\mathbf{z}^{(1)}, \tag{3}$$

and, for $k \geq 2$,

$$\left[\mathbf{z}^{(k-1)}\left(\mathbf{z}^{(k-1)}\right)^T + \mathbf{z}^{(k)}\left(\mathbf{z}^{(k)}\right)^T + \mathbf{z}^{(k+1)}\left(\mathbf{z}^{(k+1)}\right)^T\right]\mathbf{A}\mathbf{z}^{(k)} = \mathbf{A}\mathbf{z}^{(k)}. \tag{4}$$

By simple induction on the recursion formulas, all the basis vectors satisfy $\left(\mathbf{z}^{(i)}\right)^T\mathbf{z}^{(j)} = \delta_{ij}$, so ideally the $\mathbf{z}$ vectors are orthonormal. Of course, these conditions are not satisfied exactly for finite precision calculations — leading to some practical difficulties that will not be discussed here for lack of space [7].

For the application to traveltime tomography, the pertinent constants are defined by $N_1 = \left|\mathbf{M}^T\mathbf{t}\right| = \left(\mathbf{z}^{(1)}\right)^T\mathbf{M}^T\mathbf{t}$, $D_k = \left(\mathbf{z}^{(k)}\right)^T\mathbf{M}^T\mathbf{M}\mathbf{z}^{(k)}$ for $k = 1, 2, \ldots$, and $N_{k+1} = \left(\mathbf{z}^{(k+1)}\right)^T\mathbf{M}^T\mathbf{M}\mathbf{z}^{(k)}$ for $k = 1, 2, \ldots$. We then see that the equations (2)–(4) determine a tridiagonal system of the form

$$\mathbf{z}^{(k+1)}\mathbf{e}_k^T N_{k+1} + \mathbf{Z}_k\mathbf{T}_k = \mathbf{M}^T\mathbf{M}\mathbf{Z}_k \quad \text{for} \quad 2 \leq k \leq r, \tag{5}$$

where the tridiagonal matrix of coefficients is defined by

$$\mathbf{T}_k = \begin{pmatrix} D_1 & N_2 & & & \\ N_2 & D_2 & N_3 & & \\ & N_3 & D_3 & N_4 & \\ & & \ddots & \ddots & \ddots \\ & & & N_k & D_k \end{pmatrix} \quad \text{for} \quad 2 \leq k \leq r, \tag{6}$$

and where the matrix $\mathbf{Z}_k = \left(\mathbf{z}^{(1)} \quad \mathbf{z}^{(2)} \quad \mathbf{z}^{(3)} \quad \ldots \quad \mathbf{z}^{(k)}\right)$ is composed of the resulting orthonormal vectors. In practical implementations of the algorithm, the constants N_{k+1} are generally found through the normalization condition implicit in (4).

Assuming infinite precision, the process stops when $k = r$ (the rank of the matrix) because then $N_{r+1} \equiv 0$ (or is numerically negligible). It follows from (5) that this tridiagonalization process results in the identity

$$\mathbf{M}^T\mathbf{M} = \mathbf{Z}_r\mathbf{T}_r\mathbf{Z}_r^T. \tag{7}$$

Since $\mathbf{T}_r$ is invertible, the Moore-Penrose inverse [8] of the normal matrix is given by

$$\left(\mathbf{M}^T\mathbf{M}\right)^\dagger = \mathbf{Z}_r\left(\mathbf{T}_r\right)^{-1}\mathbf{Z}_r^T. \tag{8}$$

The solution to the least-squares inversion problem may therefore be written as $\mathbf{s} = \mathbf{Z}_r\left(\mathbf{T}_r\right)^{-1}\mathbf{Z}_r^T\mathbf{M}^T\mathbf{t} = N_1\mathbf{Z}_r\left(\mathbf{T}_r\right)^{-1}\mathbf{e}_1$. The first column of the inverse of $\mathbf{T}_r$ (the only one needed) may be found using an elementary recursion relation.

2.2 Resolution

Since the Lanczos algorithm directly produces a sequence of orthonormal vectors in the model space, it is straightforward to see that the model resolution matrix for this method is

$$\mathcal{R}_{model} = \mathbf{M}^T\mathbf{M}\left(\mathbf{M}^T\mathbf{M}\right)^{\dagger} = \mathbf{Z}_r\mathbf{Z}_r^T, \tag{9}$$

which is also clearly symmetric. We compute the data resolution matrix using the fact that $\mathcal{R}_{data} = \mathbf{M}\mathbf{M}^{\dagger} = \mathbf{M}\left(\mathbf{M}^T\mathbf{M}\right)^{\dagger}\mathbf{M}^T$ together with (8), so

$$\mathcal{R}_{data} = \mathbf{M}\mathbf{Z}_r\left(\mathbf{T}_r\right)^{-1}\mathbf{Z}_r^T\mathbf{M}^T. \tag{10}$$

Both resolution matrices are symmetric if we compute the full Lanczos inverse.

Assuming only that orthogonality has been maintained among the vectors $\mathbf{z}^{(k)}$ (a non-trivial assumption for realistic applications [7]), the matrix $\mathbf{Z}_r$ is just the span of the model space associated with the ray-path matrix $\mathbf{M}$. Thus, (9) is understood intuitively as the result one obtains by finding a new orthonormal set of basis vectors for the span of the resolution matrix. In other methods such as the Paige-Saunders LSQR algorithm [6], it is also possible to obtain a corresponding formula for the data resolution. However, for the Lanczos algorithm, the data resolution must be computed indirectly from the available information using (10).

3 Conclusions

The method of computing resolution matrices discussed here may be easily generalized to a variety of other iterative and approximate inversion methods including Paige-Saunders inverses [6], which are expected to be more useful than the Lanczos scheme in practice for solving problems in seismic traveltime tomography [3].

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