Connecting Theory to Experiment in Poroelasticity

Steven R. Pride

Institut de Physique du Globe de Paris, Département de Géomagnétisme, 4 place Jussieu, 75252 Paris cedex 05, France

James G. Berryman

Lawrence Livermore National Laboratory, P.O. Box 808 L-202, Livermore, California 94550

Abstract

The variables controlled and measured in elastostatic laboratory experiments (the volume changes, shape changes, confining stresses, and pore pressure) are exactly related to the appropriate variables of poroelastic field theory (the gradients of the volume-averaged displacement fields and the volume-averaged stresses). The relations between the laboratory and volume-averaged strain measures require the introduction of a new porous-material geometrical term. In the anisotropic case, this term is a tensor that is related both to the presence of porosity gradients and to a type of weighted surface porosity. In the isotropic case, the term reduces to a scalar and depends only on the surface-porosity parameter. When this surface-porosity parameter is identical to the usual volume porosity, the relations initially proposed by Biot and Willis are recovered. The exact statement of the poroelastic strain-energy density is derived and is used to define both the laboratory strain measures and the laboratory elastic moduli. Only two restrictions are placed on the materials being treated: 1) the fluid is homogeneous in each sample and 2) the material possesses a rigidity. However, the entire work is limited to linear deformations and long (relative to sample size) wavelengths of applied stress.

Key words: B. constitutive behaviour; B. granular material; B. porous material; B. geological material; C. volume averaging.

1 Introduction

The problem of determining the laws that control the linear elastostatics of a fluid-saturated porous material has been studied for many years. A short list of papers pertinent to the present study includes Biot (1941), Gassmann (1951), Biot and Willis (1957), Biot (1962), Mandl (1964), Nur and Byerlee (1971), Brown and Korringa (1975), Rice and Cleary (1976), Burridge and Keller (1981), Zimmerman *et al.* (1986), Berryman and Milton (1991), Thompson and Willis (1991), Pride *et al.* (1992), Berryman and Wang (1995), and Tuncay and Corapcioglu (1995). However, despite more than fifty years of research, there still exist fundamental questions not yet thoroughly resolved.

In this paper, we relate the variables measured and controlled in elastostatic laboratory experiments (the confining stresses, the volume changes, and the shape changes) to the field quantities of poroelastic theory (the volume-averaged stress and displacement fields in the fluid and solid phases). No rigorous connection has yet been established between the laboratory measures and the averaged field quantities. Biot and Willis (1957) simply postulated the connections, while the volume-averaging based arguments of Pride *et al.* (1992) made the limiting assumption that porosity gradients were absent and, in general, did not address the issue with sufficient care. Only two assumptions about the nature of the porous material are made here: 1) the fluid is homogeneous throughout a given sample; and 2) the material possesses a framework of grains that continuously spans each sample. This second assumption is equivalent to assuming that the material possesses a rigidity. The material can otherwise possess arbitrary heterogeneities including porosity gradients and multiple solid constituents. The grains as well as the porous material itself may be anisotropic. By using an averaging volume with the same size and shape as the laboratory sample, we will show that the relations assumed by Biot and Willis are not exact in general.

We limit ourselves to linear deformation, *i.e.,* the dependence of the poroelastic moduli on the average fluid and solid stress levels is not explicitly used in this paper. We also limit ourselves to cases where the applied stress has wavelengths much larger than both the dimensions of the individual grains and the dimensions of the sample (or averaging volume) used to define the macroscopic properties. An interesting and still poorly understood aspect of poroelasticity occurs when the applied wavelengths begin to approach the dimensions of the granular sample. Consider a small surface element on the solid part of a sample's exterior surface. As the applied wavelengths begin to approach the sample's dimensions, the traction vector over such a small element should no longer be taken as constant but should, more properly, be resolved into a mean traction and a mean stress moment (this effective stress moment quantifies the variation of the traction over the element). Thus grain rotation enters the problem and it is precisely such small-wavelength induced rotations and effective stress moments that are being ignored here. Eringen was the first to propose a theory for such situations (see Eringen, 1968, for a review of his "micro-polar continuum" approach) while Walton (1988) gives an account of some of the interesting wave modes that are possible in a granular material when such small-wavelength effects are explicitly permitted. Finally, consistent with the assumption of long wavelengths (and, therefore, low frequencies), the response is assumed to be thermodynamically reversible.

The paper is divided into nine sections. In Sec. 2, the volume averaging is defined along with the gradient of the averaged fields. The approach taken and several of the results obtained are new. In Sec. 3, the deformation measures of the "jacketed" laboratory experiment are introduced. In Sec. 4, the implications of mechanical equilibrium are explored with the two-fold objective of 1) relating the confining stress of experiments to the volume-averaged stress of poroelastic theory and 2) defining the porous-continuum statement of the conservation of linear momentum. In Sec. 5, the exact statement of the long-wavelength poroelastic strain-energy density is derived. It is used to define the laboratory strain measures as well as the constitutive laws. In Sec. 6, rigorous connection is made between the laboratory strain measures and the gradients of the averaged displacement fields. It is here that a new porous-material geometry term (related to the presence of porosity gradients) must be introduced. This is also where the deviation from the assumptions of Biot and Willis (1957) and Biot (1962) takes place. In Sec. 7, we demonstrate that all of the key results of the paper are valid when the sample is not sealed in a jacket but is, instead, embedded within a larger porous body like the earth.

The key relations are independent of the particular boundary conditions acting on a sample's exterior surface. In Sec. 8, we put our key results into a form where they may be used to solve poroelasticity boundary-value problems. Finally, in Sec. 9, we give a brief summary and discussion.

2 Averages, Gradients of Averages, and Averages of Gradients

The averaging relations to be defined in this section are free of any restrictions on material properties including the two mentioned in the introduction. They can also be applied to any physics problem in porous media and are not limited to the poroelasticity problems of interest in this paper. We begin by defining the volume-averaging procedure.

Let Ω_∞ denote the entire set of points occupied by the porous material being considered. This region can be partitioned into fluid and solid subregions as $\Omega_\infty = \Omega_{f\infty} + \Omega_{s\infty}$. A weight function $W(\mathbf{r}', \mathbf{r}) = W(\mathbf{r}' - \mathbf{r})$ is introduced where $\mathbf{r}'$ and $\mathbf{r}$ are position vectors in Ω_∞. This function is defined to have a value of one in a region surrounding $\mathbf{r}' = \mathbf{r}$ (this region may be many grain diameters in width) and it is assumed to fall off to zero for points $\mathbf{r}'$ far from $\mathbf{r}$.

The volume measures of $W(\mathbf{r}' - \mathbf{r})$ are defined

$$\begin{aligned} V_o &= \int_{\Omega_\infty} W(\mathbf{r}' - \mathbf{r})\, d^3\mathbf{r}' && (1) \\ V_{fo}(\mathbf{r}) &= \int_{\Omega_{f\infty}} W(\mathbf{r}' - \mathbf{r})\, d^3\mathbf{r}' && (2) \\ V_{so}(\mathbf{r}) &= \int_{\Omega_{s\infty}} W(\mathbf{r}' - \mathbf{r})\, d^3\mathbf{r}'. && (3) \end{aligned}$$

The fact that V_o is independent of $\mathbf{r}$ is a consequence of the functional dependence $W(\mathbf{r}', \mathbf{r}) = W(\mathbf{r}' - \mathbf{r})$. The subscript "$o$" is used here and throughout to indicate that a quantity has no dependence on the applied deformation. When connection is made to laboratory measurements, we will take $W(\mathbf{r}' - \mathbf{r})$ to be the step function defined as

$$W(\mathbf{r}' - \mathbf{r}) = \begin{cases} 1 & \text{for } \mathbf{r}' \text{ interior to } \partial E_o(\mathbf{r}) \\ 0 & \text{for } \mathbf{r}' \text{ exterior to } \partial E_o(\mathbf{r}), \end{cases} \qquad (4)$$

where $\partial E_o(\mathbf{r})$ is the surface enclosing the laboratory sample prior to deformation when the center of the sample is located at $\mathbf{r}$. If this surface $\partial E_o(\mathbf{r})$ is defined by the equation $S(\mathbf{r}' - \mathbf{r}) =$

0 [*e.g.*, $S(\mathbf{r}' - \mathbf{r}) = r' - r - \rho(\theta' - \theta, \varphi' - \varphi) = 0$, where ρ is the function defining the step position in spherical coordinates], then the weighting function in Eq. (4) can be expressed as $W = H(S)$ where H is the step function. In this same special case, V_o, V_{fo} and V_{so} are the sample's total volume, pore volume and grain volume, respectively, prior to deformation. In general, the porosity $\phi_o(\mathbf{r})$ is defined from either

$$V_{fo}(\mathbf{r}) = \phi_o(\mathbf{r})\, V_o \qquad \text{or} \qquad V_{so}(\mathbf{r}) = [1 - \phi_o(\mathbf{r})]\, V_o, \tag{5}$$

where V_o, $V_{fo}(\mathbf{r})$ and $V_{so}(\mathbf{r})$ come from Eqs. (1)–(3). In what follows, the possible presence of porosity gradients is always permitted.

Consider an arbitrary field ψ (scalar, vector, or tensor) that is labeled ψ_f for points in the fluid and ψ_s for points in the solid grains. The averages of these fields are defined

$$\overline{\psi}_f(\mathbf{r}) \equiv \frac{1}{V_{fo}(\mathbf{r})} \int_{\Omega_{f\infty}} W(\mathbf{r}' - \mathbf{r}) \psi_f(\mathbf{r}')\, d^3\mathbf{r}' \tag{6}$$

$$\overline{\psi}_s(\mathbf{r}) \equiv \frac{1}{V_{so}(\mathbf{r})} \int_{\Omega_{s\infty}} W(\mathbf{r}' - \mathbf{r}) \psi_s(\mathbf{r}')\, d^3\mathbf{r}'. \tag{7}$$

An overbar will always be used to define the average value of a quantity throughout the phase in which it is naturally defined.

Taking the gradients of these averaged fields gives

$$\nabla \left[\phi_o(\mathbf{r}) \overline{\psi}_f(\mathbf{r})\right] = \frac{1}{V_o} \int_{\Omega_{f\infty}} \nabla W(\mathbf{r}' - \mathbf{r}) \psi_f(\mathbf{r}')\, d^3\mathbf{r}' \tag{8}$$

$$= -\frac{1}{V_o} \int_{\Omega_{f\infty}} \left[\nabla' W(\mathbf{r}' - \mathbf{r})\right] \psi_f(\mathbf{r}')\, d^3\mathbf{r}' \tag{9}$$

and, similarly,

$$\nabla \left[(1 - \phi_o(\mathbf{r}))\, \overline{\psi}_s(\mathbf{r})\right] = -\frac{1}{V_o} \int_{\Omega_{s\infty}} \left[\nabla' W(\mathbf{r}' - \mathbf{r})\right] \psi_s(\mathbf{r}')\, d^3\mathbf{r}', \tag{10}$$

where ∇' is denoting the derivative with respect to primed coordinates. For the special case of the step-function weighting $W = H(S)$, we have that $\nabla' H(S) = \delta(S)\, \nabla' S$. Since $\delta(S) = \delta(r' - r - \rho)/|\nabla' S|$ and since the outward normal $\mathbf{n}$ to the surface is defined by $\mathbf{n} \equiv \nabla' S/|\nabla' S|$, Eqs. (9) and (10) reduce to

$$\nabla \left[\phi_o(\mathbf{r}) \overline{\psi}_f(\mathbf{r})\right] = \frac{1}{V_o} \int_{\partial E_{fo}(\mathbf{r})} \mathbf{n}_f(\mathbf{r}') \psi_f(\mathbf{r}')\; d^2\mathbf{r}' \tag{11}$$

$$\nabla \left[(1 - \phi_o(\mathbf{r}))\, \overline{\psi}_s(\mathbf{r})\right] = \frac{1}{V_o} \int_{\partial E_{so}(\mathbf{r})} \mathbf{n}_s(\mathbf{r}') \psi_s(\mathbf{r}')\; d^2\mathbf{r}'. \tag{12}$$

We have partitioned the closed surface ∂E_o into fluid and solid portions as $\partial E_o = \partial E_{fo} + \partial E_{so}$ ($\mathbf{n}_f$ and $\mathbf{n}_s$ are thus the outward normals to ∂E_{fo} and ∂E_{so}). Equations (11) and (12) have been derived in different manners by both Slattery (1967, 1981) and Pride *et al.* (1992).

Equations (9) and (10) are perfectly well defined for sufficiently smooth $W(\mathbf{r}'-\mathbf{r})$. However, a problem may arise in Eqs. (11) and (12) [*i.e.*, when $W(\mathbf{r}'-\mathbf{r})$ is a step function] for the special case where a portion of the grain-fluid interface exactly coincides with the surface ∂E_o. The problem is the ambiguity in defining the partitioning $\partial E_o = \partial E_{fo} + \partial E_{so}$ when the grain-fluid interface occupies more than just a line on ∂E_o, *i.e.*, are such areas of the grain-fluid interface to be called fluid or solid? One can correctly argue that there are no naturally occuring grain surfaces that can identically overlay the mathematically perfect surface ∂E_o (which might be a sphere for example). This argument is adopted here and Eqs. (11) and (12) are taken as well defined results throughout the remainder of the paper. However, one could certainly construct a mathematical model of a porous material in which the grain-fluid interface exactly overlays ∂E_o. For such classes of model materials, a well-defined expression for the gradient of the average can only come from Eqs. (9) and (10) using a continuous weight function (the weight function may fall off quite rapidly but must do so in a smooth fashion).

Equations (11) and (12) are appropriate definitions of the gradient when the phases are well mixed or when the phases continuously span the averaging region as is the case in this study. However, when the phases are poorly mixed or when a suspension of particles becomes sufficiently dilute that a phase is not represented on the surface $\partial E_o(\mathbf{r})$ for certain points $\mathbf{r}$, then the gradient of the average defined by Eqs. (11) and (12) will have a spatial variation that is too rapid to be of use in a volume-averaged theory. For such cases, a smoothly-weighted volume average is more appropriate. The weight function should fall off from one to zero over distances larger than the average distance between the suspended particles. A weight function can always be chosen so that the gradients defined by Eqs. (9) and (10) have spatial variations that are appropriate for the problem being treated. It will be seen, however, that in poroelastic problems, it is only possible to relate the moduli measured in laboratory experiments exactly to the moduli required in a volume-averaged theory when the weight function is a step function having the same shape as the laboratory sample.

We have been considering the gradient of averaged fields. Let us next consider the average of a gradient. Starting with the definition of the volume average, we have

$$\begin{aligned}
\phi_o \overline{\nabla' \psi_f} &\equiv \frac{1}{V_o} \int_{\Omega_{f\infty}} W \nabla' \psi_f \, d^3\mathbf{r}' &(13)\\
&= -\frac{1}{V_o} \int_{\Omega_{f\infty}} (\nabla' W)\, \psi_f \, d^3\mathbf{r}' + \frac{1}{V_o} \int_{\Omega_{f\infty}} \nabla' \left(W \psi_f\right) \, d^3\mathbf{r}' &(14)\\
&= \nabla \left(\phi_o \overline{\psi}_f\right) - \frac{1}{V_o} \int_{\partial G_\infty} \mathbf{n}_s W \psi_f \, d^2\mathbf{r}', &(15)
\end{aligned}$$

where ∂G_∞ denotes the grain-fluid interface throughout the interior of Ω_∞ and we have used the fact that $\mathbf{n}_f = -\mathbf{n}_s$ along this surface. Equation (9) was used for the first term of Eq. (14), while Green's theorem was used on the second [note that the weight function is zero on the external surface $\partial E_{f\infty}$ surrounding $\Omega_{f\infty}$]. Similarly, we obtain for the solid phase

$$(1-\phi_o)\overline{\nabla' \psi_s} = \nabla \left[(1-\phi_o)\, \overline{\psi}_s\right] + \frac{1}{V_o} \int_{\partial G_\infty} \mathbf{n}_s W \psi_s \, d^2\mathbf{r}'. \qquad (16)$$

Equations (15) and (16) exactly relate the average of the gradient to the gradient of the average and are key results for averaging the differential equations that control the physics of porous materials. They are the generalization to the case of weighted volume averaging of a result due to Slattery (1967). When the step function of Eq. (4) is used, Eqs. (15) and (16) reduce to the Slattery result. Interestingly, Saffman (1971) obtained similar equations for ensemble averaging in porous media. In their appendix, Joseph *et al.* (1990) demonstrated that the volume-averaged and ensemble-averaged equations have identical form.

As stated, we are concerned here with porous materials that have a continuous packing of grains across any given sample. Therefore, throughout the remainder of this paper, only the step-function weighting of Eq. (4) will be employed in any volume averaging. In this case, the porosity gradient is defined by setting $\psi_f = \psi_s = 1$ in Eqs. (11) and (12) to obtain

$$\begin{aligned}
\nabla \phi_o &= \frac{1}{V_o} \int_{\partial E_{fo}} \mathbf{n}_f \, d^2\mathbf{r}' &(17)\\
&= -\frac{1}{V_o} \int_{\partial E_{so}} \mathbf{n}_s \, d^2\mathbf{r}'. &(18)
\end{aligned}$$

These relations show that a porosity gradient exists if there is more grain area on one side of a sample than on the other. Let us define Ω_o $(= \Omega_{fo} + \Omega_{so})$ as the region contained within the bounds of ∂E_o. Green's theorem states that

$$\frac{1}{V_o}\int_{\Omega_{so}} \nabla'\psi_s \, d^3\mathbf{r}' = \frac{1}{V_o}\int_{\partial E_{so}} \mathbf{n}_s\psi_s \, d^2\mathbf{r}' + \frac{1}{V_o}\int_{\partial G_o} \mathbf{n}_s\psi_s \, d^2\mathbf{r}' \tag{19}$$

so that, upon taking $\psi_s = 1$, we obtain as well

$$\nabla\phi_o = \frac{1}{V_o}\int_{\partial G_o} \mathbf{n}_s \, d^2\mathbf{r}', \tag{20}$$

where ∂G_o is that part of the grain-fluid interface contained within Ω_o. Thus, the porosity gradient is proportional to the integral of the outward normal over either ∂E_{fo} or ∂G_o, or the inward normal over ∂E_{so}. All three of these relations will prove useful.

3 The Jacketed Experiment

We now define the canonical experiment of poroelastostatics known as the "jacketed experiment." A porous sample is sealed in a thin jacket and then immersed in a reservoir characterized by a spatially-uniform stress tensor $\boldsymbol{\tau}_c$ (the subscript "c" denotes "confining"). This tensor is separable into isotropic and deviatoric parts as $\boldsymbol{\tau}_c = -p_c\mathbf{I} + \boldsymbol{\tau}_c^D$ where p_c is called the confining pressure. A small-radius tube filled with pore fluid is passed through the jacket allowing the fluid in the pores of the sample to be connected to an independent external fluid reservoir characterized by a pressure $\overline{p}_f$ (the overbar is indicating that in the static experiment, the fluid pressure of the reservoir is the same as the fluid pressure throughout the pores of the sample). The control variables in the experiment are thus $\boldsymbol{\tau}_c$ and $\overline{p}_f$. After small increments $\delta\boldsymbol{\tau}_c$ or $\delta\overline{p}_f$ (or both) are slowly applied in the two reservoirs, the deformation of the sample is recorded and macroscopic moduli defined. Throughout the entire paper, we assume that the jacket is sufficiently thin that its deformation contributes negligibly to the total deformation. The results that we obtain are "exact" only to the extent that this proviso holds true.

The deformation is characterized by two (primary) volumetric changes as well as shape changes (deviatoric shear). The volumetric changes are: 1) the volume change of the sample as a whole; and 2) the fluid volume that passed either into or out of the sample via the tube. This latter quantity is often difficult to measure in the laboratory because the tubing compliance usually needs to be considered. Nonetheless, we assume that such calibrations have been made, and that both volume changes have been recorded. The precise definition of the deviatoric shear strain to be recorded in the laboratory is given at the end of this section. We begin by considering the detailed definitions of the various volumetric strains.

Let $\Omega = \Omega_f + \Omega_s$ denote the region within the jacket at any instant. The measure of Ω is V. Before any pressure increments have been applied, this total sample volume V is taken as V_o. We also define $V_\phi = \phi V$ as the total volume of porespace within Ω at any instant. Fluid mass may enter or leave Ω (via the tube) without affecting these definitions. We can also define $V_s = (1-\phi)V$ as the volume occupied by the solid grains at any instant.

If we additionally keep track of the volume V_f occupied by the total mass of fluid initially inside of the sample, then we have that $V_f = V_\phi + V_\zeta$, where V_ζ represents the volume of fluid that either left or entered Ω via the tube due to applied pressure increments. Following Biot and Willis (1957), we can introduce a dimensionless parameter ζ called the "increment in fluid content," where $V_\zeta = -\zeta V$. Using the definitions

$$V = V_o + \delta V \tag{21}$$

$$V_\phi = \phi V = V_{fo} + \delta V_\phi \tag{22}$$

$$\phi = \phi_o + \delta\phi \tag{23}$$

$$V_s = (1-\phi)V = V_{so} + \delta V_s \tag{24}$$

$$V_\zeta = -\zeta V = \delta V_f - \delta V_\phi \tag{25}$$

$$\zeta = \delta\zeta \qquad (i.e., \ \zeta_o = 0), \tag{26}$$

a variety of volumetric-strain measures can be defined including $\delta V/V$, $\delta\phi$, and

$$\frac{\delta V_\phi}{V_\phi} = \frac{\phi_o}{\phi}\frac{\delta V}{V} + \frac{\delta\phi}{\phi} \tag{27}$$

$$\zeta = \phi\frac{\delta V_\phi}{V_\phi} - \frac{\delta V_f}{V}. \tag{28}$$

These strain measures, when divided by δp_c and $\delta\overline{p}_f$ (or linear combinations of δp_c and $\delta\overline{p}_f$) at equilibrium and in the absence of applied shear stress ($i.e.$, $\boldsymbol{\tau}_c^D = 0$), define various possible poroelastic compressibility moduli.

It is perhaps useful to connect present results to the laboratory compressibility laws proposed by Brown and Korringa (1975). In the absence of applied shear stress, Brown and Korringa proposed the laws

$$-\begin{bmatrix} \delta V_\phi/V_\phi \\ \delta V/V \end{bmatrix} = \begin{bmatrix} 1/K_\phi & 1/K_p \\ 1/K_s & 1/K^* \end{bmatrix} \begin{bmatrix} \delta\overline{p}_f \\ \delta p_c - \delta\overline{p}_f \end{bmatrix}, \tag{29}$$

where K^*, K_p, K_s, and K_ϕ are the four experimentally defined poroelastic incompressibilities. As alluded to, the volumetric strains most easily measured in the jacketed experiment are $\delta V/V$ and ζ. To replace $\delta V_\phi/V_\phi$ by ζ in Eq. (29), we need to know $\delta V_f/V$ [*c.f.*, Eq. (28)]. The change in fluid volume is defined

$$\delta V_f \equiv \int_{\Omega_f} d^3\mathbf{R} - \int_{\Omega_{fo}} d^3\mathbf{r}' \tag{30}$$

$$= \int_{\Omega_{fo}} J_f(\mathbf{r}')\, d^3\mathbf{r}' - \int_{\Omega_{fo}} d^3\mathbf{r}', \tag{31}$$

where $\mathbf{R} = \mathbf{r}' + \mathbf{u}_f(\mathbf{r}')$. In the small-deformation limit being considered, the Jacobian goes as $J_f(\mathbf{r}') = 1 + \nabla' \cdot \mathbf{u}_f(\mathbf{r}') = 1 - \delta\overline{p}_f/K_f$ with K_f the fluid's bulk modulus, so that

$$\frac{\delta V_f}{V} = \frac{\delta V_f}{V_o} = -\frac{\phi_o}{K_f}\delta\overline{p}_f \tag{32}$$

and

$$\zeta = \phi_o\left(\frac{\delta V_\phi}{V_\phi} + \frac{\delta\overline{p}_f}{K_f}\right). \tag{33}$$

This then leads to the variant of the experimental compressibility law of interest in this work

$$\begin{bmatrix} \zeta \\ -\delta V/V_o \end{bmatrix} = \begin{bmatrix} \phi_o\,(1/K_p + 1/K_f - 1/K_\phi) & -\phi_o/K_p \\ 1/K_s - 1/K^* & 1/K^* \end{bmatrix} \begin{bmatrix} \delta\overline{p}_f \\ \delta p_c \end{bmatrix}. \tag{34}$$

In Sec. 5, the form of this law will be seen to be consistent with the form of the poroelastic strain-energy density, thus giving the Maxwell relation

$$-\frac{\phi_o}{K_p} = \frac{1}{K_s} - \frac{1}{K^*}, \tag{35}$$

so that there are only three independent compressibilities in poroelasticity [note that a compressibility is being defined as the change in a volume due to a change in an isotropic stress when the deviatoric stresses are held constant—such a definition is independent of the possible presence of anisotropy]. Brown and Korringa (1975) derived Eq. (35), but only under the unnecessary restriction that $\nabla\phi_o = 0$ as will be discussed in Sec. 5.

The pressure dependence of the porosity increment then follows as

$$\delta\phi = \left(\frac{1}{K_s} - \frac{(1-\phi_o)}{K^*}\right)\left[\delta p_c - \left(1 + \phi_o\,\frac{1/K_\phi - 1/K_s}{1/K_s - (1-\phi_o)/K^*}\right)\delta\overline{p}_f\right]. \tag{36}$$

As Brown and Korringa (1975) first pointed out, the only way to have $K_\phi = K_s$ (and, therefore, $\delta\phi$ only a function of the difference $\delta p_c - \delta\overline{p}_f$) is if the grain material in the sample is isotropic, having everywhere the same bulk modulus K_m. In this case, and only in this case, when $\delta p_c = \delta\overline{p}_f$, the framework of grains scales in a self-similar fashion so that $\delta\phi = 0$ and $K_\phi = K_s = K_m$.

To this point, we have only been considering the volumetric changes. We now give the precise definitions of the entire set of strain measurements to be obtained experimentally. In elasticity problems, the only meaningful definition of strain is that it is the conjugate variable to stress in the expression for the work done in an elastic deformation. In the jacketed laboratory experiment, the stress variables are the increments in confining stress $\delta\boldsymbol{\tau}_c$ and fluid pressure $\delta\bar{p}_f$. We will prove in Sec. 5 that the following definitions of the overall strain tensor $\mathbf{e}$ and increment of fluid content ζ are conjugate to $\delta\boldsymbol{\tau}_c$ and $\delta\bar{p}_f$, respectively:

$$\zeta \equiv -\frac{1}{V_o}\int_{\partial E_{To}} \mathbf{n}_T \cdot \mathbf{u}_f \, d^2\mathbf{r}' \tag{37}$$

$$\mathbf{e} \equiv \frac{1}{V_o}\int_{\partial E_{Jo}} \mathbf{n}_J \mathbf{u}_J \, d^2\mathbf{r}', \tag{38}$$

where ∂E_{Jo} denotes the exterior surface of the jacket and ∂E_{To} denotes the cross-section of the small tube where it pierces the jacket and enters the pore space of the sample. The laboratory measurables $\mathbf{n}_J$ and $\mathbf{u}_J$ are, respectively, the outward normal (prior to deformation) and displacement of the jacket surface while $\mathbf{n}_T$ is the outward normal to the tube cross section ∂E_{To}. The integral over ∂E_{To} is exactly the volume V_ζ of fluid that entered (or left) the sample through the tube. This definition of ζ is thus consistent with that given above. The definition (38) says that we must also measure the average value of the tensor $\mathbf{n}_J\mathbf{u}_J$ over the surface of the jacketed sample. This is the ideal laboratory strain measure that experimentalists should always strive to record.

Some insight into the definition (38) can be obtained by noting that since $|\partial E_{To}| \ll |\partial E_{Jo}|$, the jacket surface ∂E_{Jo} is, effectively, a closed surface. If we use $\mathbf{u}(\mathbf{r}')$ to represent the displacements of all points inside of ∂E_{Jo} (*i.e.*, we do not give special labels to the displacements in the fluid, solid, and jacketed regions and we exclude the displacements in the small tube), then we have that $\mathbf{e} = V_o^{-1} \int_{\Omega_o + \Omega_{Jo}} \nabla' \mathbf{u} \, d^3\mathbf{r}'$ where Ω_{Jo} represents the region occupied by the jacket material alone. Because the volume of the jacket material is assumed to be much smaller than the volume of the porous material, the integral over Ω_{Jo} can be dropped and $\mathbf{e}$ is thus seen to represent the average gradient of displacement throughout the porous sample.

Also note that the trace of $\mathbf{e}$

$$\mathrm{tr}(\mathbf{e}) = \frac{1}{V_o} \int_{\partial E_{Jo}} \mathbf{n}_J \cdot \mathbf{u}_J \, d^2\mathbf{r}' = \frac{\delta V}{V_o} \tag{39}$$

exactly represents the overall volumetric strain of the sample (for small strains). As usual, it is instructive to write the tensor $\mathbf{e}$ as

$$\mathbf{e} = \frac{1}{3}\mathrm{tr}(\mathbf{e})\, \mathbf{I} + \mathbf{e}^D + \mathbf{e}^R \tag{40}$$

where

$$\mathbf{e}^D = \frac{1}{2V_o} \int_{\partial E_{Jo}} \left(\mathbf{n}_J \mathbf{u}_J + \mathbf{u}_J \mathbf{n}_J - \frac{2}{3} \mathbf{n}_J \cdot \mathbf{u}_J \, \mathbf{I} \right) d^2\mathbf{r}' \tag{41}$$

$$\mathbf{e}^R = \frac{1}{2V_o} \int_{\partial E_{Jo}} \left(\mathbf{n}_J \mathbf{u}_J - \mathbf{u}_J \mathbf{n}_J \right) d^2\mathbf{r}', \tag{42}$$

i.e., $\mathbf{e}^D$ quantifies the overall deviatoric (shear) deformation while $\mathbf{e}^R$ quantifies the overall rotation of a sample. The rotational contribution $\mathbf{e}^R$ will not contribute to the work required to deform the sample (no stress moments are being applied) so that only $\delta V/V_o$ and $\mathbf{e}^D$ are used in defining laboratory elastic moduli.

4 Mechanical Equilibrium

In this section, the implications of mechanical equilibrium are explored. The principal objectives are to relate the confining stress of the jacketed experiment to the volume-averaged stress of poroelasticity theory and to obtain the porous-continuum statement of the conservation of linear momentum. The conservation of angular momentum is not explicitly considered since effective stress moments are not acting on our sample in the long-wavelength limit of interest (as discussed in the introduction). Thus, we take the local stress tensor $\delta\boldsymbol{\tau}_s(\mathbf{r}')$ in the grains to be symmetric from the outset.

The conservation of linear momentum requires that for static loading, $\nabla'\delta p_f(\mathbf{r}') = 0$ in the fluid and $\nabla' \cdot \delta\boldsymbol{\tau}_s(\mathbf{r}') = 0$ in the solid. Thus, the fluid pressure is uniform $\delta p_f(\mathbf{r}') = \delta\overline{p}_f(\mathbf{r})$ throughout each averaging region $\Omega_{fo}(\mathbf{r})$. To investigate the implications of $\nabla' \cdot \delta\boldsymbol{\tau}_s(\mathbf{r}') = 0$ on the volume-averaged stress, we follow Landau and Lifshitz (1986, p. 6) and integrate $\nabla' \cdot (\delta\boldsymbol{\tau}_s\mathbf{r}') = (\nabla' \cdot \delta\boldsymbol{\tau}_s)\mathbf{r}' + \delta\boldsymbol{\tau}_s^T \cdot \nabla'\mathbf{r}' = \delta\boldsymbol{\tau}_s$ over the region $\Omega_{so}(\mathbf{r})$ to obtain

$$(1-\phi_o)\delta\overline{\boldsymbol{\tau}}_s = \frac{1}{V_o}\int_{\partial E_{so}} \mathbf{n}_s \cdot \delta\boldsymbol{\tau}_s\mathbf{r}'\, d^2\mathbf{r}' - \frac{\delta\overline{p}_f}{V_o}\int_{\partial G_o} \mathbf{n}_s\mathbf{r}'\, d^2\mathbf{r}'. \tag{43}$$

Recall that ∂E_{so} is the solid part of the sample's exterior surface and ∂G_o is the internal grain-fluid interface. We have used the fact that $\mathbf{n}_s \cdot \delta\boldsymbol{\tau}_s = -\delta\overline{p}_f\mathbf{n}_s$ on ∂G_o. It is also useful to integrate $\nabla'(\delta\overline{p}_f\mathbf{r}') = \delta\overline{p}_f\,\mathbf{I}$ over the region Ω_{fo} to obtain the equivalent relation for the fluid

$$-\phi_o\delta\overline{p}_f\,\mathbf{I} = -\frac{\delta\overline{p}_f}{V_o}\int_{\partial E_{fo}} \mathbf{n}_f\mathbf{r}'\, d^2\mathbf{r}' + \frac{\delta\overline{p}_f}{V_o}\int_{\partial G_o} \mathbf{n}_s\mathbf{r}'\, d^2\mathbf{r}', \tag{44}$$

where we have used that $\mathbf{n}_f = -\mathbf{n}_s$ on ∂G_o. Adding these results then gives the total average stress $\delta\overline{\boldsymbol{\tau}}$ acting through all of Ω_o

$$\begin{aligned}
\delta\overline{\boldsymbol{\tau}} &\equiv (1-\phi_o)\delta\overline{\boldsymbol{\tau}}_s - \phi_o\delta\overline{p}_f\mathbf{I} &\quad (45)\\
&= \frac{1}{V_o}\int_{\partial E_{so}} \mathbf{n}_s \cdot \delta\boldsymbol{\tau}_s\mathbf{r}'\, d^2\mathbf{r}' - \frac{\delta\overline{p}_f}{V_o}\int_{\partial E_{fo}} \mathbf{n}_f\mathbf{r}'\, d^2\mathbf{r}'. &\quad (46)
\end{aligned}$$

These relations hold regardless of how $\mathbf{n}_s \cdot \delta\boldsymbol{\tau}_s(\mathbf{r}')$ is distributed over ∂E_{so}.

We now make connection to the jacketed experiment. Recall that the region Ω_{Jo} is that occupied by the jacket material prior to deformation. It has a measure V_{Jo} and does not include the points of the small-radius tube piercing the jacket. The outside surface of the jacket material is ∂E_{Jo}, while its inside surface is $\partial E_{so} + \partial E_{fJo}$. The sample's external fluid surface is being partitioned as $\partial E_{fo} = \partial E_{fJo} + \partial E_{To}$ where, as before, ∂E_{To} is the portion coincident with the small-radius tube and ∂E_{fJo} is the remainder that is in contact with the jacket material. All of ∂E_{fo} is maintained at the pressure $\delta\overline{p}_f$. The surface ∂E_{so} experiences the traction field $\mathbf{n}_s \cdot \delta\boldsymbol{\tau}_s(\mathbf{r}')$ where $\delta\boldsymbol{\tau}_s(\mathbf{r}')$ is nonuniform in general, while the external surface ∂E_{Jo} experiences $\mathbf{n}_J \cdot \delta\boldsymbol{\tau}_c$ where $\delta\boldsymbol{\tau}_c$ is the spatially uniform confining stress of the surrounding reservoir. Upon integrating $\nabla' \cdot (\delta\boldsymbol{\tau}_J \mathbf{r}') = \delta\boldsymbol{\tau}_J$ over the jacket region and dividing by V_o, we obtain

$$\begin{aligned} \frac{V_{Jo}}{V_o}\left[\frac{1}{V_{Jo}}\int_{\Omega_{Jo}} \boldsymbol{\tau}_J \, d^3\mathbf{r}'\right] &= \frac{1}{V_o}\int_{\partial E_{Jo}} \mathbf{n}_J \cdot \delta\boldsymbol{\tau}_c \mathbf{r}' \, d^2\mathbf{r}' \\ &\quad -\frac{1}{V_o}\int_{\partial E_{so}} \mathbf{n}_s \cdot \delta\boldsymbol{\tau}_s \mathbf{r}' \, d^2\mathbf{r}' \\ &\quad +\frac{\delta\overline{p}_f}{V_o}\int_{\partial E_{fJo}} \mathbf{n}_f \mathbf{r}' \, d^2\mathbf{r}'. \end{aligned} \tag{47}$$

Although the average stress tensor in the jacket is of the same order of magnitude as the average stress in the sample (*i.e.*, the second and third terms on the right-hand side), we assume that the jacket is very thin relative to sample dimensions so that $V_{Jo} \ll V_o$. Thus, the left-hand side is negligible. Note as well that $\mathbf{n}_J \cdot \delta\boldsymbol{\tau}_c \mathbf{r}' = \delta\boldsymbol{\tau}_c \cdot \mathbf{n}_J \mathbf{r}'$ ($\delta\boldsymbol{\tau}_c$ is taken as symmetric) and that because $|\partial E_{To}| \ll |\partial E_{Jo}|$, we can take ∂E_{Jo} to be a closed surface to give $V_o^{-1}\int_{\partial E_{Jo}} \mathbf{n}_J \mathbf{r}' \, d^2\mathbf{r}' = V_o^{-1}(V_o + V_{Jo})\,\mathbf{I} = \mathbf{I}$. Thus, we find that

$$\boldsymbol{\tau}_c = \frac{1}{V_o}\int_{\partial E_{so}} \mathbf{n}_s \cdot \delta\boldsymbol{\tau}_s \mathbf{r}' \, d^2\mathbf{r}' - \frac{\delta\overline{p}_f}{V_o}\int_{\partial E_{fJo}} \mathbf{n}_f \mathbf{r}' \, d^2\mathbf{r}' \tag{48}$$

$$= \overline{\boldsymbol{\tau}}. \tag{49}$$

The second line follows from Eq. (46) and the fact that $|\partial E_{To}| \ll |\partial E_{Jo}|$. The commonsense relation $\boldsymbol{\tau}_c = \overline{\boldsymbol{\tau}}$ relating the confining stress of the laboratory to the volume-averaged stress of poroelastic theory is not new; however, the argument just given, that is free of any material property constraints, is new. Equation (49) is exact only to the extent that $|\Omega_{Jo}| \ll |\Omega_o|$ and $|\partial E_{To}| \ll |\partial E_{Jo}|$.

We now demonstrate a natural way that a material-property constraint enters the analysis. If $\nabla' \cdot \boldsymbol{\tau}_s(\mathbf{r}') = 0$ is integrated over the grain space Ω_{so} using both $\mathbf{n}_s \cdot \delta\boldsymbol{\tau}_s = -\delta\overline{p}_f \mathbf{n}_s$ on ∂G_o and the definition (20) of the porosity gradient, we obtain

$$\frac{1}{V_o}\int_{\partial E_{so}} \mathbf{n}_s \cdot \delta\boldsymbol{\tau}_s \, d^2\mathbf{r}' = \delta\overline{p}_f \nabla\phi_o. \tag{50}$$

For the special case where $\delta\boldsymbol{\tau}_s(\mathbf{r}')$ is taken to be uniform over ∂E_{so} [*i.e.*, on ∂E_{so} we write

$\delta\boldsymbol{\tau}_s(\mathbf{r}') = -\delta P_s \mathbf{I} + \delta \mathbf{T}^D$ where δP_s and $\delta\mathbf{T}^D$ are spatially constant] we obtain

$$\left[-(\delta P_s - \delta\overline{p}_f)\,\mathbf{I} + \delta\mathbf{T}^D\right] \cdot \nabla\phi_o = 0, \tag{51}$$

where definition (18) of the porosity gradient was used. In the presence of both a porosity gradient and a uniform stress-tensor over ∂E_{so}, mechanical equilibrium can only be maintained if $\delta\mathbf{T}^D = 0$ and $\delta P_s = \delta\overline{p}_f$. This result is not surprising since a porosity gradient is present when there is more grain area on one side of the sample compared to the other. Thus, if $\delta P_s \neq \overline{p}_f$, there is necessarily a net force on the sample in the direction of $\nabla\phi$ and the sample will accelerate. Note, however, that if the sample possesses a porosity gradient in a jacketed experiment, it will not accelerate. This is because although $\delta\boldsymbol{\tau}_c$ is uniform over the outside jacket surface, mechanical equilibrium requires there to be a significant variation of $\delta\boldsymbol{\tau}_J(\mathbf{r}')$ throughout the thin jacket. In particular, because the fluid pressure is uniform over ∂E_{fJo}, there must be a systematic variation of $\delta\boldsymbol{\tau}_s(\mathbf{r}')$ over ∂E_{so} from one side of the sample to the other (recall that ∂E_{fJo} and ∂E_{so} define the inside jacket surface). Whether the resulting effective stress moments on ∂E_{so} play a significant role in the deformation in this case is something to be considered in future work—we ignore the effect here. These points are emphasized because several authors (Brown and Korringa, 1975; Pride *et al.*, 1992; Zimmerman *et al.*, 1994) have taken $\delta\boldsymbol{\tau}_s(\mathbf{r}')$ as uniform over ∂E_{so} in their arguments without noting the restriction ($\nabla\phi_o = 0$) that this places on the material. In this work, we never use the boundary condition that $\delta\boldsymbol{\tau}_s$ is constant on ∂E_{so}.

We conclude with the macroscopic statements of the conservation of linear momentum. From the definition (12) of a macroscopic gradient, Eq. (50) can be written

$$\nabla \cdot [(1-\phi_o)\delta\overline{\boldsymbol{\tau}}_s] = \delta\overline{p}_f \nabla\phi_o. \tag{52}$$

From Eq. (11), we find the analogous result in the fluid

$$\nabla[\phi_o \delta\overline{p}_f] = \delta\overline{p}_f \nabla\phi_o. \tag{53}$$

Thus, in terms of the total average stress tensor $\delta\overline{\boldsymbol{\tau}} = (1-\phi_o)\delta\overline{\boldsymbol{\tau}}_s - \phi_o\delta\overline{p}_f\,\mathbf{I}$, these two equations can be written in the final form

$$\begin{aligned} \nabla \cdot \delta\overline{\boldsymbol{\tau}} &= 0 &(54)\\ \nabla \delta\overline{p}_f &= 0. &(55) \end{aligned}$$

Of course, to solve an elastostatics problem, these equations must be supplemented by constitutive laws (and boundary conditions) which we will now consider.

5 Strain-Energy Density and Constitutive Laws

We now derive the exact statement of the poroelastic strain-energy density appropriate for long-wavelength applied stress fields. Given the form of the strain-energy density, the poroelastic constitutive laws immediately follow.

The poroelastic strain-energy density is obtained by a direct volume averaging of the local strain-energy densities in the solid grains $\delta\boldsymbol{\tau}_s : \nabla'\mathbf{u}_s(\mathbf{r}')$ and fluid $-\delta p_f \nabla' \cdot \mathbf{u}_f(\mathbf{r}')$. No strain-energy is stored in the form of localized vorticity gradients because such gradients are not excited in the long-wavelength limit. Thus, at each point $\mathbf{r}$ in the material, we situate an averaging volume $\Omega_o(\mathbf{r})$ and define

$$\begin{aligned} \delta\mathcal{W}_s(\mathbf{r}) &= \frac{1}{V_o}\int_{\Omega_{so}(\mathbf{r})} \delta\boldsymbol{\tau}_s(\mathbf{r}') : \nabla'\mathbf{u}_s(\mathbf{r}')\, d^3\mathbf{r}' &(56)\\ \delta\mathcal{W}_f(\mathbf{r}) &= -\frac{1}{V_o}\int_{\Omega_{fo}(\mathbf{r})} \delta p_f(\mathbf{r}')\nabla' \cdot \mathbf{u}_f(\mathbf{r}')\, d^3\mathbf{r}', &(57) \end{aligned}$$

so that the total poroelastic strain-energy density is the sum

$$\delta\mathcal{W} = \delta\mathcal{W}_s + \delta\mathcal{W}_f. \tag{58}$$

Because $\delta p_f(\mathbf{r}') = \delta\overline{p}_f(\mathbf{r})$ throughout Ω_{fo}, we have

$$\delta\mathcal{W}_f = -\delta\overline{p}_f \left[\frac{1}{V_o}\int_{\partial E_{fo}} \mathbf{n}_f \cdot \mathbf{u}_f\, d^2\mathbf{r}' - \frac{1}{V_o}\int_{\partial G_o} \mathbf{n}_s \cdot \mathbf{u}_s\, d^2\mathbf{r}'\right], \tag{59}$$

where the boundary condition $\mathbf{n}_f \cdot \mathbf{u}_f = -\mathbf{n}_s \cdot \mathbf{u}_s$ was used on ∂G_o. Similarly, because $\delta\boldsymbol{\tau}_s : \nabla'\mathbf{u}_s = \nabla' \cdot (\delta\boldsymbol{\tau}_s \cdot \mathbf{u}_s)$ throughout the solid, we have

$$\delta\mathcal{W}_s = \frac{1}{V_o}\int_{\partial E_{so}} \mathbf{n}_s \cdot \delta\boldsymbol{\tau}_s \cdot \mathbf{u}_s\, d^2\mathbf{r}' - \frac{\delta\overline{p}_f}{V_o}\int_{\partial G_o} \mathbf{n}_s \cdot \mathbf{u}_s\, d^2\mathbf{r}', \tag{60}$$

where the boundary condition $\mathbf{n}_s \cdot \delta\boldsymbol{\tau}_s = -\mathbf{n}_s \delta\overline{p}_f$ was used on ∂G_o. Adding then gives $\delta\mathcal{W}$ in the useful form

$$\delta\mathcal{W} = \frac{1}{V_o}\left[\int_{\partial E_{so}} \mathbf{n}_s \cdot \delta\boldsymbol{\tau}_s \cdot \mathbf{u}_s \, d^2\mathbf{r}' - \delta\overline{p}_f \int_{\partial E_{fo}} \mathbf{n}_f \cdot \mathbf{u}_f \, d^2\mathbf{r}'\right], \tag{61}$$

i.e., $\delta\mathcal{W}$ is the total work performed in deforming the external surface of the averaging region divided by the volume of the averaging region. It is important to note that only the normal component $\mathbf{n}_f \cdot \mathbf{u}_f$ of the fluid displacement on ∂E_{fo} has an influence on $\delta\mathcal{W}$.

We now claim that the form of $\delta\mathcal{W}$, when expressed in terms of macroscopic variables, is an intrinsic property of the system Ω_o and is independent of the particular boundary conditions that are being imposed on ∂E_o so long as these boundary conditions do not violate the fundamental restrictions that: 1) only long wavelengths of applied stress or applied deformation are acting on the sample; and 2) mechanical equilibrium is maintained [*e.g.,* when porosity gradients are present, we cannot take $\boldsymbol{\tau}_s$ to be a constant over ∂E_{so} as discussed in the previous section]. To obtain the macroscopic form of $\delta\mathcal{W}$ using Eq. (61), we thus can employ any boundary conditions on ∂E_o deemed convenient that are consistent with the two restrictions just stated. This is an important claim that will be justified in Sec. 7. In what follows, we adopt to use the boundary conditions consistent with the jacketed experiment.

The strain-energy density stored in each volume element of the jacket material is $\delta\boldsymbol{\tau}_J : \nabla'\mathbf{u}_J$. If the identity $\delta\boldsymbol{\tau}_J : \nabla'\mathbf{u}_J = \nabla' \cdot [\delta\boldsymbol{\tau}_J \cdot \mathbf{u}_J]$ is integrated over the jacket volume, we find

$$\begin{aligned} \frac{V_{Jo}}{V_o}\left[\frac{1}{V_{Jo}}\int_{\Omega_{Jo}} \delta\boldsymbol{\tau}_J : \nabla'\mathbf{u}_J \, d^3\mathbf{r}'\right] &= \frac{1}{V_o}\int_{\partial E_{Jo}} \mathbf{n}_J \cdot \delta\boldsymbol{\tau}_c \cdot \mathbf{u}_J \, d^2\mathbf{r}' \\ &\quad -\frac{1}{V_o}\int_{\partial E_{so}} \mathbf{n}_s \cdot \delta\boldsymbol{\tau}_s \cdot \mathbf{u}_s \, d^2\mathbf{r}' \\ &\quad +\frac{1}{V_o}\int_{\partial E_{fJo}} \mathbf{n}_f \, \delta p_f \cdot \mathbf{u}_f \, d^2\mathbf{r}'. \end{aligned} \tag{62}$$

The average strain-energy density in the jacket is of the same order of magnitude as that in the sample (the second and third terms on the right-hand side). Thus, because $V_{Jo} \ll V_o$, the left-hand side is negligible. Because both $\mathbf{n}_J \cdot \delta\boldsymbol{\tau}_c \cdot \mathbf{u}_J = \delta\boldsymbol{\tau}_c : \mathbf{n}_J\mathbf{u}_J$ and $\delta\boldsymbol{\tau}_c$ is spatially uniform, we can use the general expression (61) of $\delta\mathcal{W}$ along with the partitioning $\partial E_{fo} = \partial E_{fJo} + \partial E_{To}$

of the fluid's external surface to write

$$
\begin{aligned}
\delta\mathcal{W} &= \mathbf{e} : \delta\boldsymbol{\tau}_c - \frac{\delta\overline{p}_f}{V_o}\int_{\partial E_{To}} \mathbf{n}_T \cdot \mathbf{u}_f \, d^2\mathbf{r}' && (63)\\
&= \mathbf{e} : \delta\overline{\boldsymbol{\tau}} + \zeta \, \delta\overline{p}_f, && (64)
\end{aligned}
$$

where $\mathbf{e} \equiv V_o^{-1} \int_{\partial E_{Jo}} \mathbf{n}_J \mathbf{u}_J \, d^2\mathbf{r}'$ and $\zeta \equiv -V_o^{-1} \int_{\partial E_{To}} \mathbf{n}_T \cdot \mathbf{u}_f \, d^2\mathbf{r}'$ as proposed in Sec. 3. This is the fundamental form of $\delta\mathcal{W}$ in terms of macroscopic quantities. As promised, the strains $\mathbf{e}$ and ζ are seen to be the exact conjugates to $\delta\boldsymbol{\tau}_c$ and $\delta\bar{p}_f$. In the next section, these laboratory strain measures will be related to the gradients of the averaged displacements that are the strain measures of poroelastic theory.

We now consider the constitutive equations implicit in $\delta\mathcal{W} = \mathbf{e} : \delta\overline{\boldsymbol{\tau}} + \zeta\,\delta\overline{p}_f$. First, let us write the tensors $\delta\overline{\boldsymbol{\tau}}$ and $\mathbf{e}$ in the separated forms $\delta\overline{\boldsymbol{\tau}} = -\delta\overline{P}\,\mathbf{I} + \delta\overline{\boldsymbol{\tau}}^D$ and $\mathbf{e} = \mathrm{tr}(\mathbf{e})\,\mathbf{I}/3 + \mathbf{e}^D + \mathbf{e}^R$ so that

$$
\delta\mathcal{W} = \zeta\,\delta\overline{p}_f - \frac{\delta V}{V_o}\,\delta\overline{P} + \mathbf{e}^D : \delta\overline{\boldsymbol{\tau}}^D. \tag{65}
$$

We have used the fact that $\delta V/V_o = \mathrm{tr}(\mathbf{e})$. Because $\delta\overline{\boldsymbol{\tau}}$ is symmetric, $\delta\mathcal{W}$ is invariant to the presence of the antisymmetric rotation tensor $\mathbf{e}^R$. Equation (65) tells us that $\mathcal{W}$ has the functional dependence $\mathcal{W} = \mathcal{W}(\overline{p}_f, \overline{P}, \overline{\boldsymbol{\tau}}^D)$ because, upon taking the total derivative, we have

$$
d\mathcal{W} = \frac{\partial\mathcal{W}}{\partial\overline{p}_f}\,d\overline{p}_f + \frac{\partial\mathcal{W}}{\partial\overline{P}}\,d\overline{P} + \frac{\partial\mathcal{W}}{\partial\overline{\boldsymbol{\tau}}^D} : d\overline{\boldsymbol{\tau}}^D, \tag{66}
$$

which is the same form as Eq. (65). By ζ we have always meant $-(V_\zeta - V_{\zeta o})/V_o$ and by $\delta V/V_o$ we have meant $(V - V_o)/V_o$. Thus we can identify

$$
\begin{aligned}
\frac{V_\zeta(\overline{p}_f, \overline{P}, \boldsymbol{\tau}^D) - V_{\zeta o}}{V_o} &= \frac{\partial\mathcal{W}(\overline{p}_f, \overline{P}, \overline{\boldsymbol{\tau}}^D)}{\partial\overline{p}_f} && (67)\\
\frac{V(\overline{p}_f, \overline{P}, \overline{\boldsymbol{\tau}}^D) - V_o}{V_o} &= \frac{\partial\mathcal{W}(\overline{p}_f, \overline{P}, \overline{\boldsymbol{\tau}}^D)}{\partial\overline{P}} && (68)\\
\mathbf{e}^D(\overline{p}_f, \overline{P}, \overline{\boldsymbol{\tau}}^D) &= \frac{\partial\mathcal{W}(\overline{p}_f, \overline{P}, \overline{\boldsymbol{\tau}}^D)}{\partial\overline{\boldsymbol{\tau}}^D}. && (69)
\end{aligned}
$$

If we further take the total derivatives of these three equations, the poroelastic constitutive laws are obtained in the form

$$
\begin{bmatrix} -\zeta \\ dV/V_o \\ d\mathbf{e}^D \end{bmatrix} = \begin{bmatrix} \mathcal{S}_w & \mathcal{S}_{wb} & \mathbf{S}^w : \\ \mathcal{S}_{wb} & \mathcal{S}_b & \mathbf{S}^b : \\ \mathbf{S}^w & \mathbf{S}^b & {}_4\mathbf{S}^d : \end{bmatrix} \begin{bmatrix} -d\overline{p}_f \\ -d\overline{P} \\ d\overline{\boldsymbol{\tau}}^D \end{bmatrix}, \tag{70}
$$

where the various compliances are defined as the second derivative of $\mathcal{W}$ with respect to the appropriate stresses. As usual, the off-diagonal terms are symmetric because $\partial^2\mathcal{W}(x,y)/\partial x\partial y = \partial^2\mathcal{W}(x,y)/\partial y\partial x$. For this same reason, the fourth-order deviatoric-compliance tensor ${}_4\mathbf{S}^d = S^d_{ijk\ell}\hat{\mathbf{x}}_i\hat{\mathbf{x}}_j\hat{\mathbf{x}}_k\hat{\mathbf{x}}_\ell$ possesses the symmetry $S^d_{ijk\ell} = S^d_{k\ell ij}$.

Other symmetries follow from the symmetry of $d\mathbf{e}^D$ ($S^w_{ij} = S^w_{ji}$, $S^b_{ij} = S^b_{ji}$, $S^d_{ijk\ell} = S^d_{jik\ell}$) and the symmetry of $d\overline{\boldsymbol{\tau}}^D$ ($S^d_{ijk\ell} = S^d_{ij\ell k}$). Note as well that because $d\mathbf{e}^D$ has zero trace, the second-order tensors $\mathbf{S}^w$ and $\mathbf{S}^b$ must also possess zero trace as well as the first two components of ${}_4\mathbf{S}^d$ (*i.e.*, $S^d_{iik\ell} = 0$ for all k and ℓ). This means there are at most 28 independent poroelastic compliances for an arbitrarily anisotropic material (these are the three compressibilities $\mathcal{S}_w$, $\mathcal{S}_{wb}$, and $\mathcal{S}_b$; the five independent components in each of $\mathbf{S}^w$ and $\mathbf{S}^b$; and the 15 independent components in ${}_4\mathbf{S}^d$).

Let us write the constitutive law of Eq. (70) in the form $d\mathbf{X} = \mathbf{L}\cdot d\mathbf{F}$ where $d\mathbf{X} = [-\zeta,\ dV/V_o,\ d\mathbf{e}^D]^T$ and $d\mathbf{F} = [-d\overline{p}_f,\ -d\overline{P},\ d\overline{\boldsymbol{\tau}}^D]^T$. The standard condition for the system to be in a stable equilibrium (*e.g.*, Landau and Lifshitz, 1980) is that the curvature $d^2\mathcal{W} = d\mathbf{F}^T\cdot\mathbf{L}\cdot d\mathbf{F}$ of the strain-energy density must be positive. Thus, the coefficient matrix $\mathbf{L}$ of Eq. (70) must be positive definite which means that the determinants of its minors are all positive. As concerns the compressibilities, this gives that $\mathcal{S}_w \geq 0$, $\mathcal{S}_b \geq 0$, and $\mathcal{S}_{wb}^2 \leq \mathcal{S}_w\mathcal{S}_b$. Note that these inequalities in no way prevent $\mathcal{S}_{wb}$ from being negative. Indeed, because we expect the overall volume of a sample to increase when the fluid pressure is increased at constant $\overline{P}$ and $\overline{\boldsymbol{\tau}}^D$, we normally expect that $\mathcal{S}_{wb} < 0$ (again, the requirement of thermodynamic stability only requires that $\mathcal{S}_{wb}^2 \leq \mathcal{S}_w\mathcal{S}_b$).

Equation (70) represents the principle deformation of a porous sample. Other volumetric deformation measures (such as those defined in Sec. 3) can be derived from it. In particular, if the change in porosity is desired, we use Eq. (70) in the identity $d\phi = -\phi_o dV/V_o + \zeta - \phi_o\mathcal{S}_f\, d\overline{p}_f$ where $\mathcal{S}_f = 1/K_f$ is the fluid compressibility. The result is

$$d\phi = (\mathcal{S}_w + \phi_o\mathcal{S}_{wb} - \phi_o\mathcal{S}_f)\, d\overline{p}_f + (\mathcal{S}_{wb} + \phi_o\mathcal{S}_b)\, d\overline{P} - (\mathbf{S}^w + \phi_o\mathbf{S}^b) : d\overline{\boldsymbol{\tau}}^D, \tag{71}$$

which is the generalization to the anisotropic case of Eq. (36). We expect that the porosity will, most normally, be reduced as the confining pressure is increased at constant $\overline{p}_f$ and $\overline{\boldsymbol{\tau}}^D$

so that $\mathcal{S}_{wb} < -\phi_o \mathcal{S}_b$. We repeat that such qualitative "expectations" are not requirements of thermodynamic stability.

The three compressibilities $\mathcal{S}_w$, $\mathcal{S}_{wb}$, and $\mathcal{S}_b$ can be related to the moduli of Brown and Korringa (1975) using Eq. (34). Note that no restrictions on material properties are required to arrive at the symmetry of the compliance matrix in Eq. (70). Brown and Korringa, however, derived the symmetry using an argument in which the external solid faces of a sample were bathed in a uniform pressure (different than the fluid pressure). We have seen that mechanical equilibrium then demands that $\nabla \phi_o = 0$ so that their argument is limited to the case in which porosity gradients are absent.

In an isotropic material, the form of the compliance matrix in Eq. (70) is greatly simplified because the scalar responses ζ and dV/V_o decouple from the deviatoric stress tensor $d\overline{\boldsymbol{\tau}}^D$, *i.e.,* for isotropic materials, we have $\mathbf{S}^w = \mathbf{S}^b = 0$. This is a consequence of "Curie's principle" (*e.g.,* deGroot and Mazur, 1984), which states that in isotropic media, only responses and forces of the same tensorial order are coupled.

In order to see this for our problem, consider an orthogonal coordinate transformation $\mathbf{A}$ that has the defining property $\mathbf{A} \cdot \mathbf{A}^T = \mathbf{I}$ (*e.g.,* rotation of the coordinate axes). Since the tensor $\mathbf{S}^w$ transforms as $\mathbf{S}^{w\prime} = \mathbf{A} \cdot \mathbf{S}^w \cdot \mathbf{A}^T$, while a vector $\mathbf{a}$ transforms as $\mathbf{a}' = \mathbf{A} \cdot \mathbf{a}$, we can construct the scalar identity $\mathbf{a} \cdot \mathbf{S}^w \cdot \mathbf{b} = \mathbf{a}' \cdot \mathbf{S}^{w\prime} \cdot \mathbf{b}'$, where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary vectors. Furthermore, we must have $\mathbf{S}^w = \mathbf{S}^{w\prime}$ in an isotropic material so that $\mathbf{a} \cdot \mathbf{S}^w \cdot \mathbf{b} = \mathbf{a}' \cdot \mathbf{S}^w \cdot \mathbf{b}'$. The two sides of this scalar identity involve identical linear products of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}'$, $\mathbf{b}'$ components. The only such linear product that is invariant to arbitrary orthogonal transformations is the dot product $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}' \cdot \mathbf{b}'$. Thus, $\mathbf{S}^w$ must take the form $\mathbf{S}^w = s_w \mathbf{I}$ in an isotropic material. But, as discussed above, $\mathbf{S}^w$ must also have zero trace. Thus, $s_w = 0$ and we can conclude that $\mathbf{S}^w = \mathbf{S}^b = 0$ in isotropic media.

In an isotropic material, the constitutive laws then take the simple form

$$\begin{bmatrix} -\zeta \\ dV/V_o \\ d\mathbf{e}^D \end{bmatrix} = \begin{bmatrix} \mathcal{S}_w & \mathcal{S}_{wb} & 0 \\ \mathcal{S}_{wb} & \mathcal{S}_b & 0 \\ 0 & 0 & (2G)^{-1} \end{bmatrix} \begin{bmatrix} -d\overline{p}_f \\ -d\overline{P} \\ d\overline{\boldsymbol{\tau}}^D \end{bmatrix}, \tag{72}$$

where it is a standard exercise to show that for isotropic media, double-dot multiplication with ${}_4\mathbf{S}^d$ gets replaced by a simple scalar multiplication. The stiffness G is the rigidity (shear modulus) of the isotropic porous material.

6 Poroelastic Deformation Measures

We next consider the connection between the laboratory strain measures $\mathbf{e}$ and ζ and the gradients of the averaged displacements. Following Biot (1962), the relative-displacement vector $\overline{\mathbf{w}}$ is introduced as

$$\overline{\mathbf{w}} \equiv \phi_o(\overline{\mathbf{u}}_f - \overline{\mathbf{u}}_s), \tag{73}$$

where $\overline{\mathbf{u}}_f$ and $\overline{\mathbf{u}}_s$ are the average fluid and solid displacement vectors. What we are seeking is the linear transformation $\mathbf{G}$ defined by

$$\begin{bmatrix} \nabla \cdot \overline{\mathbf{w}} \\ \nabla \cdot \overline{\mathbf{u}}_s \\ (\nabla \overline{\mathbf{u}}_s)^D \end{bmatrix} = \mathbf{G} \cdot \begin{bmatrix} -\zeta \\ \delta V/V_o \\ \mathbf{e}^D \end{bmatrix}, \tag{74}$$

where $(\nabla \overline{\mathbf{u}}_s)^D \equiv [\nabla \overline{\mathbf{u}}_s + (\nabla \overline{\mathbf{u}}_s)^T - 2\nabla \cdot \overline{\mathbf{u}}_s \, \mathbf{I}/3]/2$ denotes the deviatoric part of $\nabla \overline{\mathbf{u}}_s$. Biot and Willis (1957) simply postulated that $\mathbf{G}$ is the identity matrix $\mathbf{I}$. We will show here, however, that when either porosity gradients are present or the surface porosity defined on a sample's external surface is different from the volume porosity, then $\mathbf{G} \neq \mathbf{I}$. The matrix $\mathbf{G}$ depends, in general, on the spatial distribution of the grains in the sample.

In the averaging relations (11) and (12), we replace ψ_s and ψ_f by $\mathbf{u}_s$ and $\mathbf{u}_f$ to obtain the general results

$$\nabla \left[(1-\phi_o)\overline{\mathbf{u}}_s\right] = \frac{1}{V_o} \int_{\partial E_{so}} \mathbf{n}_s \mathbf{u}_s \, d^2\mathbf{r}' \tag{75}$$

$$\nabla \left[\phi_o \overline{\mathbf{u}}_f\right] = \frac{1}{V_o} \int_{\partial E_{fo}} \mathbf{n}_f \mathbf{u}_f \, d^2\mathbf{r}'. \tag{76}$$

To connect to the laboratory measures, we introduce again the partitioning $\partial E_{fo} = \partial E_{fJo} + \partial E_{To}$ and define the tensor $\mathbf{Z}$ as

$$\mathbf{Z} \equiv -\frac{1}{V_o} \int_{\partial E_{To}} \mathbf{n}_T \mathbf{u}_f \, d^2\mathbf{r}', \tag{77}$$

so that Eq. (76) may be written

$$\nabla \left[\phi_o \overline{\mathbf{u}}_f\right] = \frac{1}{V_o} \int_{\partial E_{fJo}} \mathbf{n}_f \mathbf{u}_f \, d^2\mathbf{r}' - \mathbf{Z}. \tag{78}$$

Recalling Eq. (37), we have that $\mathrm{tr}(\mathbf{Z}) = \zeta$. Indeed, it is only the trace of $\mathbf{Z}$ that has meaning as a deformation measure. This is clear from the fundamental form (64) for $\delta\mathcal{W}$ that depends only on ζ. As Eq. (61) makes clear, the reason for this is that only the normal component of $\mathbf{u}_f$ on ∂E_{fo} has an influence on $\delta\mathcal{W}$ [recall that only those changes in the material-particle distribution that require work to be performed are defined as "deformations"]. This can also be seen directly from the definition (77) of $\mathbf{Z}$. If we assume that $\mathbf{u}_f$ is normal to the tube cross section and if we assume that the tube cross section is so small (the size of a pore) that the unit normal $\mathbf{n}_T$ is uniform over ∂E_{To}, then we have $\mathbf{Z} = \zeta\, \mathbf{n}_T\mathbf{n}_T$. The normal $\mathbf{n}_T$ is defined prior to deformation. It depends on the orientation of the overall coordinate system as well as where the tube is attached to the sample, but it is independent of the deformation and, thus, only the trace of $\mathbf{Z}$ has meaning as a deformation measure.

If we now consider the deformation in the jacketing material alone, and integrate $\nabla'\mathbf{u}_J(\mathbf{r}')$ over Ω_{Jo}, we obtain (in analogy with previous sections)

$$\begin{aligned} \frac{V_{Jo}}{V_o}\left[\frac{1}{V_{Jo}}\int_{\Omega_{Jo}} \nabla'\mathbf{u}_J\, d^3\mathbf{r}'\right] &= \frac{1}{V_o}\int_{\partial E_{Jo}} \mathbf{n}_J\mathbf{u}_J\, d^2\mathbf{r}' \\ &\quad -\frac{1}{V_o}\int_{\partial E_{so}} \mathbf{n}_s\mathbf{u}_s\, d^2\mathbf{r}' \\ &\quad +\frac{1}{V_o}\int_{\partial E_{fJo}} \mathbf{n}_f\mathbf{u}_f\, d^2\mathbf{r}'. \end{aligned} \tag{79}$$

Since $V_{Jo} \ll V_o$ and $\mathbf{e} = V_o^{-1}\int_{\partial E_{Jo}} \mathbf{n}_J\mathbf{u}_J\, d^2\mathbf{r}'$, we then have

$$\mathbf{e} = \nabla\left[(1-\phi_o)\overline{\mathbf{u}}_s\right] + \nabla\left[\phi_o\overline{\mathbf{u}}_f\right] + \mathbf{Z} \tag{80}$$

$$= \nabla\overline{\mathbf{u}}_s + \nabla\overline{\mathbf{w}} + \mathbf{Z}. \tag{81}$$

Equation (81) provides a fundamental constraint between the quantities defined in a jacketed experiment $\mathbf{e}$ and $\mathbf{Z}$ and the poroelastic quantities $\nabla\overline{\mathbf{u}}_s$ and $\nabla\overline{\mathbf{w}}$.

To define the transformation matrix $\mathbf{G}$, we consider the (long-wavelength) deformation defined by the boundary conditions

$$\mathbf{u}_s(\mathbf{r}') = \overline{\mathbf{u}}_s(\mathbf{r}) + \mathbf{e}(\mathbf{r})\cdot(\mathbf{r}'-\mathbf{r}) \qquad \text{on } \partial E_{so} \tag{82}$$

$$\mathbf{u}_f(\mathbf{r}') = \overline{\mathbf{u}}_s(\mathbf{r}) + \mathbf{e}(\mathbf{r})\cdot(\mathbf{r}'-\mathbf{r}) \qquad \text{on } \partial E_{fJo}. \tag{83}$$

In this deformation, all points $\mathbf{r}'$ on the exterior faces of the porous sample—except those coincident with the small-tube cross section—displace as $\overline{\mathbf{u}}_s(\mathbf{r}) + \mathbf{e}(\mathbf{r}) \cdot (\mathbf{r}' - \mathbf{r})$. If we would include higher-order powers of $\mathbf{r}' - \mathbf{r}$ in a general expansion of these surface displacements, such higher-order terms would integrate to zero in the definitions (39) and (40) of $\delta V/V_o$ and $\mathbf{e}^D$ [the even-ordered powers of $\mathbf{r}' - \mathbf{r}$ directly integrate to zero, while the odd-ordered powers possess deviatoric tensorial coefficients such that scalar products between the coefficients and the integrated powers of $\mathbf{r}' - \mathbf{r}$ are zero]. Such higher-order terms in the boundary conditions allow for overall sample deformation beyond that of volume changes and deviatoric shear and are related to the gradient(s) of the overall strain. In this work, we are not attempting to allow for strain gradients and thus do not consider further such higher-order terms [see as well the discussion at the end of Sec. 7].

Carrying out the surface integrals in Eqs. (75) and (78) in this case gives

$$\nabla\left[(1-\phi_o)\overline{\mathbf{u}}_s\right] = -\nabla\phi_o\overline{\mathbf{u}}_s + \left\{(1-\phi_o)\,\mathbf{I} + \nabla\left[(1-\phi_o)(\mathbf{r}_s - \mathbf{r})\right]\right\}\cdot\mathbf{e} \tag{84}$$

$$\nabla\left[\phi_o\overline{\mathbf{u}}_f\right] + \mathbf{Z} = \nabla\phi_o\overline{\mathbf{u}}_s + \left\{\phi_o\,\mathbf{I} - \nabla\left[(1-\phi_o)(\mathbf{r}_s - \mathbf{r})\right]\right\}\cdot\mathbf{e}, \tag{85}$$

where the definitions (17) and (18) of the porosity gradients have been used and a new parameter $\mathbf{r}_s(\mathbf{r})$ defined as

$$\mathbf{r}_s(\mathbf{r}) \equiv \frac{1}{V_{so}(\mathbf{r})}\int_{\Omega_{so}(\mathbf{r})} \mathbf{r}'\, d^3\mathbf{r}' \tag{86}$$

has been introduced. The point $\mathbf{r}_s(\mathbf{r})$ is defining the center point of the grain distribution $\Omega_{so}(\mathbf{r})$ just as $\mathbf{r}$ defines the center point of the entire averaging region $\Omega_o(\mathbf{r})$ [*i.e.*, $\mathbf{r} \equiv V_o^{-1}\int_{\Omega_o(\mathbf{r})} \mathbf{r}'\, d^3\mathbf{r}'$]. From Eq. (12), we have that $\mathbf{r}_s(\mathbf{r})$ satisfies

$$\nabla\left[(1-\phi_o)\mathbf{r}_s\right] = \frac{1}{V_o}\int_{\partial E_{so}} \mathbf{n}_s\mathbf{r}'\, d^2\mathbf{r}', \tag{87}$$

which was used in writing Eqs. (84) and (85). It is important that we do not arbitrarily take $\mathbf{r}_s = \mathbf{r}$ (which is the implicit assumption of Biot). Upon using the definition $\overline{\mathbf{w}} = \phi_o(\overline{\mathbf{u}}_f - \overline{\mathbf{u}}_s)$, Eqs. (84) and (85) then yield

$$\nabla\overline{\mathbf{u}}_s = (\mathbf{I} - \mathbf{B}_o)\cdot\mathbf{e} \tag{88}$$

$$\nabla\overline{\mathbf{w}} = -\mathbf{Z} + \mathbf{B}_o\cdot\mathbf{e}, \tag{89}$$

where the tensor $\mathbf{B}_o$ is defined

$$\mathbf{B}_o \equiv \frac{\nabla\left[(1-\phi_o)(\mathbf{r}-\mathbf{r}_s)\right]}{1-\phi_o} \tag{90}$$

$$= \mathbf{I} - \frac{1}{V_{so}}\int_{\partial E_{so}} \mathbf{n}_s(\mathbf{r}'-\mathbf{r})\, d^2\mathbf{r}'. \tag{91}$$

This tensor is the new grain topology term introduced in this analysis. As throughout this work, the subscript "o" indicates that this tensor is defined prior to deformation. The results (88) and (89) are seen to be consistent with the constraint of Eq. (81). Because only the trace of $\mathbf{Z}$ has meaning as a deformation measure, the same holds true for $\nabla\overline{\mathbf{w}}$ and we obtain

$$\nabla\cdot\overline{\mathbf{w}} = -\zeta + \mathbf{B}_o : \mathbf{e} \tag{92}$$

as the deformation relation of interest. Equations (88) and (92) are the key relations we have been seeking. In the special case where $\mathbf{r}_s = \mathbf{r}$ [the center of the grain distribution Ω_{so} is at the center of Ω_o], we have that $\mathbf{B}_o = 0$ and $\nabla\overline{\mathbf{u}}_s = \mathbf{e}$ and $\nabla\cdot\overline{\mathbf{w}} = -\zeta$, which are precisely the postulates of Biot and Willis (1957). It should be pointed out that at the start of his analysis of porous-media acoustics, Biot (1956) carefully stated that he was assuming that surface porosity defined on an arbitrary cross section of a sample and the volume porosity of the sample were the same. As will be discussed shortly, this assumption is almost (but not exactly) equivalent to the constraint that $\mathbf{r}_s = \mathbf{r}$ in an isotropic material.

For a fully anisotropic material, it seems impossible to relate $\mathbf{B}_o$ in an exact way to any "standard" laboratory measurements; however, using the definition (91), $\mathbf{B}_o$ could certainly be measured. Furthermore, although the difference vector $\mathbf{r}-\mathbf{r}_s$ must be related to the presence of porosity gradients, defining the exact proportionality between $\mathbf{r}-\mathbf{r}_s$ and $\nabla\phi_o$ without introducing new parameters is not obvious (and probably not possible). In general, $\mathbf{B}_o$ will not be symmetric. However, for the special case where Ω_o is a sphere, it is easy to show that $\mathbf{B}_o = \mathbf{B}_o^T$ exactly. Finally, in the anisotropic case, the transformation matrix $\mathbf{G}$ of Eq. (74) linking $\left[\nabla\cdot\overline{\mathbf{w}},\, \nabla\cdot\overline{\mathbf{u}}_s,\, (\nabla\overline{\mathbf{u}}_s)^D\right]$ to the experimental observables $[-\zeta,\, \mathrm{tr}(\mathbf{e}),\, \mathbf{e}^D]$ is

$$\mathbf{G} = \begin{bmatrix} 1 & \mathrm{tr}(\mathbf{B}_o)/3 & \mathbf{B}_o : \\ 0 & 1-\mathrm{tr}(\mathbf{B}_o)/3 & -\mathbf{B}_o : \\ 0 & -\frac{1}{3}\left(\mathbf{B}_o - \frac{\mathrm{tr}(\mathbf{B}_o)}{3}\mathbf{I}\right) & (\mathbf{I}-\mathbf{B}_o)\cdot + \frac{\mathbf{I}\mathbf{B}_o}{3} : \end{bmatrix}, \tag{93}$$

where the expression for G_{33} means that there is the sum of 1) a dot product involving the second-order tensor $\mathbf{I} - \mathbf{B}_o$, and 2) a double-dot product involving the fourth-order tensor $\mathbf{I}\,\mathbf{B}_o/3$.

We now consider the case where the material is isotropic and, therefore, only the isotropic part of $\mathbf{B}_o$ is nonzero

$$\mathbf{B}_o = \frac{\mathrm{tr}(\mathbf{B}_o)}{3}\mathbf{I}. \tag{94}$$

In this case, the transformation $\mathbf{G}$ reduces to the relatively simple form

$$\mathbf{G} = \begin{bmatrix} 1 & \mathrm{tr}(\mathbf{B}_o)/3 & 0 \\ 0 & 1 - \mathrm{tr}(\mathbf{B}_o)/3 & 0 \\ 0 & 0 & 1 - \mathrm{tr}(\mathbf{B}_o)/3 \end{bmatrix}. \tag{95}$$

The terms G_{13} and G_{23} vanish because $\mathbf{e}^D$ has zero trace. Some physical insight into the nature of the scalar material property $\mathrm{tr}(\mathbf{B}_o)/3$ can be extracted as follows:

$$\frac{\mathrm{tr}(\mathbf{B}_o)}{3} \equiv 1 - \frac{1}{3(1-\phi_o)V_o}\int_{\partial E_{so}} \mathbf{n}_s \cdot (\mathbf{r}' - \mathbf{r})\, d^2\mathbf{r}' \tag{96}$$

$$= \frac{\beta_o - \phi_o}{1 - \phi_o}, \tag{97}$$

where the parameter β_o is defined

$$1 - \beta_o = \frac{\int_{\partial E_{so}} \mathbf{n}_s \cdot (\mathbf{r}' - \mathbf{r})\, d^2\mathbf{r}'}{\int_{\partial E_o} \mathbf{n} \cdot (\mathbf{r}' - \mathbf{r})\, d^2\mathbf{r}'}. \tag{98}$$

We have used the fact that $\int_{\partial E_o} \mathbf{n} \cdot (\mathbf{r}' - \mathbf{r})\, d^2\mathbf{r}' = \int_{\Omega_o} \nabla' \cdot \mathbf{r}'\, d^3\mathbf{r}' = 3V_o$. Clearly, β_o is a type of weighted surface porosity defined on the external surface ∂E_o of a sample. Indeed, for the special cases where Ω_o is a sphere, a cube, or certain regular polyhedra, the weight function $\mathbf{n} \cdot (\mathbf{r}' - \mathbf{r})$ is constant over ∂E_o and $1 - \beta_o = \int_{\partial E_{so}} d^2\mathbf{r}' / \int_{\partial E_o} d^2\mathbf{r}'$; *i.e.*, for these special averaging volumes, β_o is exactly the surface porosity.

We conclude this section by writing out the transformation (74) for an isotropic material

$$\begin{bmatrix} \nabla \cdot \overline{\mathbf{w}} \\ \nabla \cdot \overline{\mathbf{u}}_s \\ (\nabla \overline{\mathbf{u}}_s)^D \end{bmatrix} = \frac{1}{1-\phi_o} \begin{bmatrix} 1 - \phi_o & \beta_o - \phi_o & 0 \\ 0 & 1 - \beta_o & 0 \\ 0 & 0 & 1 - \beta_o \end{bmatrix} \begin{bmatrix} -\zeta \\ \delta V / V_o \\ \mathbf{e}^D \end{bmatrix}. \tag{99}$$

We see that when $\beta_o = \phi_o$ (weighted surface porosity equals the usual volume porosity), the relations reduce to those of Biot and Willis. These isotropic relations do not depend on the presence of a porosity gradient [qualitatively, one might think that $\nabla\phi_o$ defines an aeolotropic symmetry axis within Ω_o and, thus, precludes the isotropic case; however, this need not be true in general]. A final interpretation of β_o follows from the identification $(1-\phi_o)\nabla\cdot\overline{\mathbf{u}}_s = (1-\beta_o)\delta V/V_o$ in Eq. (99). If the displacements on ∂E_{so} are written as

$$\mathbf{u}_s(\mathbf{r}') = \overline{\mathbf{u}}_s(\mathbf{r}) + \Delta\mathbf{u}_s(\mathbf{r}', \mathbf{r}), \tag{100}$$

where $\overline{\mathbf{u}}_s(\mathbf{r})$ corresponds to a rigid-body translation in the $\mathbf{r}'$ coordinates, then the trace of Eq. (75) and the definition (18) of the porosity gradient can be used to write

$$1 - \beta_o = \frac{1}{\delta V}\int_{\partial E_{so}} \mathbf{n}_s \cdot \Delta\mathbf{u}_s \, d^2\mathbf{r}'. \tag{101}$$

Physically, the integral $\int_{\partial E_{so}} \mathbf{n}_s \cdot \Delta\mathbf{u}_s \, d^2\mathbf{r}'$ corresponds to the volume of grains that have expanded (or contracted) into the region $\Omega - \Omega_o$ due to deformation. Since δV is the measure of $\Omega - \Omega_o$, we see that β_o is equivalently interpreted as the volume porosity within the deformed region $\Omega - \Omega_o$. Although this provides physical insight, it is important to remember that β_o is defined prior to the deformation and is purely a function of the grain-area distribution on ∂E_o as defined by Eq. (98)

7 Beyond the Jacketed Experiment

The derivations leading to our key results [$\delta\mathcal{W} = \mathbf{e} : \delta\overline{\boldsymbol{\tau}} + \zeta\,\delta\overline{p}_f$, $\nabla\overline{\mathbf{u}}_s = (\mathbf{I} - \mathbf{B}_o)\cdot\mathbf{e}$, $\nabla\cdot\overline{\mathbf{w}} = -\zeta + \mathbf{B}_o : \mathbf{e}$] have, in each case, exploited the specific nature of the jacketed experiment. In particular, the partitioning $\partial E_{fo} = \partial E_{fJo} + \partial E_{To}$ of a sample's external fluid surface has been a key step in easily obtaining these results. In Sec. 5, we claimed that the fundamental form $\delta\mathcal{W} = \mathbf{e} : \delta\overline{\boldsymbol{\tau}} + \zeta\,\delta\overline{p}_f$ was generally valid and did not depend on the boundary conditions over ∂E_o (subject to the two important restrictions given there). We now consider the more general case where Ω_o corresponds to a non-isolated volume embedded within a larger porous body (like the earth) and where stressing occurs due to some long-wavelength event (like a seismic wave with sufficiently low frequency content). We now show that the key results are also obtained in this more general setting.

What the jacket and tube were allowing us to clearly define was the distinction between the overall deformation of the sample (defined by $\mathbf{e}$) and the volume of fluid that displaced completely outside of the sample (defined by $-\zeta\, V_o$). When the jacket and tube are not present, in order to define $\mathbf{e}$ and ζ, we must imagine fictitious surfaces stretched over each of the pore openings on ∂E_o. These imaginary surfaces must be completely permeable to the water molecules and must smoothly expand or contract as the framework of grains expands or contracts. In this case, the volume $\zeta\, V_o$ quantifies the total volume of fluid that traverses these imaginary surfaces stretched over the pores during the course of a deformation, while $\mathbf{e}$ corresponds to the sum of the deformation of both the external grain surfaces and the imaginary surfaces stretched over the pores. Thus, when no jacket and tube are present, $\mathbf{e}$ and ζ become rather artificial quantities that, in a final theory of poroelasticity, should be replaced by $\nabla\overline{\mathbf{u}}_s$ and $\nabla \cdot \overline{\mathbf{w}}$. Indeed, when geophones are used to measure the material response for poroelastic problems in the earth, the averaged displacement fields $\overline{\mathbf{u}}_s$ and $\overline{\mathbf{w}}$ become the natural response observables (*i.e.*, $\mathbf{e}$ and ζ are the natural response observables only in laboratory deformation experiments where jackets and tubes are present).

Let us return to the general deformation relations of Eqs. (75) and (76). Consider the long-wavelength deformation defined by

$$\mathbf{u}_s(\mathbf{r}') = \overline{\mathbf{u}}_s(\mathbf{r}) + \mathbf{e}(\mathbf{r}) \cdot (\mathbf{r}' - \mathbf{r}) \qquad \text{on } \partial E_{so} \tag{102}$$

$$\mathbf{u}_f(\mathbf{r}') = \overline{\mathbf{u}}_s(\mathbf{r}) + \mathbf{e}(\mathbf{r}) \cdot (\mathbf{r}' - \mathbf{r}) + \Delta\mathbf{u}_f(\mathbf{r}', \mathbf{r}) \quad \text{on } \partial E_{fo}, \tag{103}$$

where $\mathbf{e}(\mathbf{r})$ thus satisfies

$$\mathbf{e}(\mathbf{r}) = \frac{1}{V_o}\int_{\partial E_{so}} \mathbf{n}_s \mathbf{u}_s \, d^2\mathbf{r}' + \frac{1}{V_o}\int_{\partial E_{fo}} \mathbf{n}_f(\mathbf{u}_f - \Delta\mathbf{u}_f)\, d^2\mathbf{r}'. \tag{104}$$

The displacement $\mathbf{u}_f(\mathbf{r}') - \Delta\mathbf{u}_f(\mathbf{r}', \mathbf{r})$ corresponds to the displacement of the imaginary surfaces stretched over the pores on ∂E_o. If we now define

$$\mathbf{Z} = -\frac{1}{V_o}\int_{\partial E_{fo}} \mathbf{n}_f \Delta\mathbf{u}_f \, d^2\mathbf{r}' \tag{105}$$

[*c.f.*, Eq. (77)] and substitute Eqs. (102) and (103) into the general relations (75) and (76), all the deformation relations of the previous section are again obtained. We use the word "general" here to describe an expression that has assumed no particular boundary conditions on ∂E_o. As in the previous section, only $\zeta \equiv \mathrm{tr}(\mathbf{Z})$ has meaning as a deformation measure due to the general expression (61) for $\delta\mathcal{W}$.

If we next substitute the boundary values of Eqs. (102) and (103) into the generally valid Eq. (61) and use the generally valid equilibrium conditions (43)–(46) as well as the deformation relations (88) and (92) which have just been shown to be general, then, after some manipulation, the fundamental form $\delta\mathcal{W} = \mathbf{e} : \delta\overline{\boldsymbol{\tau}} + \zeta\, \delta\overline{p}_f$ is exactly recovered with no reference whatsoever having been made to the jacketed experiment. Thus, all of our key results are obtained even when the jacket and tube are absent and Ω_o corresponds to a volume embedded within a larger porous body.

In summary, whether we take the "reservoir" surrounding our porous sample to be defined by 1) a uniform stress tensor acting on a jacket that seals the sample or 2) a uniform displacement $\mathbf{e} \cdot (\mathbf{r}' - \mathbf{r})$ acting directly on the surface of an unjacketed sample, the same fundamental relation appears and, thus, the same constitutive laws are obtained. For the first type of boundary condition, we must also have a tube present to allow for fluid to pass through the jacket surface while in the second the fluid can pass freely through all of the pores.

In both of these boundary conditions, we have taken the applied field to be uniform at the contact between the sample and the "reservoir". In the earth, we imagine the averaging volume to be embedded within a larger porous body (the reservoir) that might have material properties that are highly variable over the averaging-volume surface. Thus, even if a uniform field is applied at the limits of the porous body (*e.g.*, at the earth's surface), the fields that actually act on any given averaging-volume surface within the body might be highly variable over the sample surface. The actual deformation of the averaging volume in such a case can still be resolved into a volume change and a shear (as the boundary-conditions (102) and (103) allow for); however, there is also some elastic-energy stored in higher-order moments of deformation and this energy should be represented in the poroelastic strain-energy function and, thus, in the final constitutive laws.

As discussed in Sec. 6, these effects can be theoretically allowed for by adding to the boundary-conditions (102) and (103) the higher-order displacements

$$ {}_3\mathbf{F}(\mathbf{r}) : (\mathbf{r}' - \mathbf{r})(\mathbf{r}' - \mathbf{r}) + {}_4\mathbf{G}(\mathbf{r}) \overset{\cdot}{_3} (\mathbf{r}' - \mathbf{r})(\mathbf{r}' - \mathbf{r})(\mathbf{r}' - \mathbf{r}) + \text{etc.}, $$

where the third-order tensor ${}_3\mathbf{F}(\mathbf{r})$ is related to the average gradient of strain throughout the sample, the fourth-order tensor ${}_4\mathbf{G}(\mathbf{r})$ to the average double-gradient of strain, and so on, while $\overset{\cdot}{_3}$ is representing the triple dot product. Such higher-order deformation is in linear addition to the volume changes and shear allowed for in this paper. For example, if one imagines a spherical sample, the volumetric change corresponds to a uniform change in radius, the shear strain might correspond to an inward contraction of the poles and an outward expansion of the equator, and the higher-order moments (not allowed for in this paper) correspond to any higher-order modes of "bumpiness" that develop on the sample's external surface due to material heterogeneity.

The presence of such higher-order strain moments corresponds to the presence of higher-order stress moments. Such stress moments are added linearly to the (symmetric) stress allowed for in this paper. As part of a future study, it will be shown that each order of strain moment is the conjugate tensor to the corresponding order of stress moment in the expression for the strain-energy density. Thus, the elastic moduli defined in Eq. (70) are not changed by such higher-order deformation. However, the constitutive laws will contain additional terms corresponding to the higher-order stress and strain moments.

In anisotropic materials, such higher-order deformation could couple to the volumetric and shear deformations allowed for in this paper. However, in an isotropic material, Curie's principle guarantees that only stress and strain moments of the same tensorial order are coupled. Thus, if we are interested in the value of a certain volumetric change in an isotropic material (*e.g.*, the change in porosity at some point), we need only know the average confining pressure and fluid pressure at that point even if shear and higher-order deformation is present.

8 Isotropic Poroelasticity Theory

We now put our results into a form where they may actually be used to solve a poroelasticity boundary-value problem. The goal of such a problem is to find the volume-averaged displacement fields $\overline{\mathbf{u}}_s$ and $\overline{\mathbf{w}}$ throughout a porous continuum given boundary conditions on the limits. In order to do this, the macroscopic statements of momentum conservation $\nabla \cdot \overline{\boldsymbol{\tau}} = 0$ and $\nabla \overline{p}_f = 0$ must be supplemented with constitutive laws relating $\overline{\boldsymbol{\tau}}$ and $\overline{p}_f$ to the gradients of the volume-averaged displacements. Our final results here will be limited to the case of isotropic porous continua for the reason that working with the general anisotropic transformation matrix $\mathbf{G}$ of Eq. (93) is cumbersome. Throughout this last section, we will, for convenience, drop the "δ" in front of the variables that has been indicating "increment".

It is usually convenient when the coefficient matrix of a constitutive law [such as in Eq. (70)] is symmetric. We will require that the coefficient matrix $\mathbf{L}'$ relating the response vector

$$\mathbf{X}' = [\nabla \cdot \overline{\mathbf{w}},\, \nabla \cdot \overline{\mathbf{u}}_s,\, (\nabla \mathbf{u}_s)^D]^T \tag{106}$$

to the force vector

$$\mathbf{F}' = [-\overline{p}_f{}',\, -\overline{P}\,',\, \overline{\boldsymbol{\tau}}\,'^D]^T \tag{107}$$

is symmetric, *i.e.,* we require $\mathbf{L}' = \mathbf{L}'^T$ in the constitutive law $\mathbf{X}' = \mathbf{L}' \cdot \mathbf{F}'$. Such symmetry is guaranteed if we can define the components of $\mathbf{F}'$ such that the strain-energy density is of the form

$$\mathcal{W} = \mathbf{X} \cdot \mathbf{F} = \mathbf{X}' \cdot \mathbf{F}', \tag{108}$$

where $\mathbf{X} = [-\zeta,\, \mathrm{tr}(\mathbf{e}),\, \mathbf{e}^D]^T$ and $\mathbf{F} = [-\overline{p}_f,\, -\overline{P},\, \overline{\boldsymbol{\tau}}^D]^T$ as defined by Eq. (65). In Sec. 6, we have already determined the transformation matrix $\mathbf{G}$ in $\mathbf{X}' = \mathbf{G} \cdot \mathbf{X}$. It remains to determine the transformation matrix $\mathbf{H}$ in

$$\mathbf{F}' = \mathbf{H} \cdot \mathbf{F} \tag{109}$$

such that Eq. (108) is satisfied. It is trivial to see that the matrix $\mathbf{H}$ is given by

$$\mathbf{H} = \left(\mathbf{G}^T\right)^{-1} \tag{110}$$

$$= \frac{1}{1-\beta_o} \begin{bmatrix} 1-\beta_o & 0 & 0 \\ -(\beta_o - \phi_o) & 1-\phi_o & 0 \\ 0 & 0 & 1-\phi_o \end{bmatrix}, \tag{111}$$

where we have written out the components of $\mathbf{H}$ for the case of an isotropic material. This defines the components of $\mathbf{F}'$ as

$$\mathbf{F}' = \left[-\overline{p}_f,\ \frac{-(1-\phi_o)\overline{P} + (\beta_o - \phi_o)\overline{p}_f}{1-\beta_o},\ \frac{1-\phi_o}{1-\beta_o}\overline{\boldsymbol{\tau}}^D \right]^T. \tag{112}$$

If $\mathbf{X} = \mathbf{L} \cdot \mathbf{F}$ represents the constitutive law of Eq. (72), then $\mathbf{L}'$ in $\mathbf{X}' = \mathbf{L}' \cdot \mathbf{F}'$ is simply $\mathbf{L}' = \mathbf{G} \cdot \mathbf{L} \cdot \mathbf{G}^T$ or

$$\mathbf{X}' = \begin{bmatrix} \mathcal{S}_w{}' & \mathcal{S}_{wb}{}' & 0 \\ \mathcal{S}_{wb}{}' & \mathcal{S}_b{}' & 0 \\ 0 & 0 & (2G')^{-1} \end{bmatrix} \cdot \mathbf{F}', \tag{113}$$

where the primed moduli are defined in terms of those of Sec. 5 as

$$\mathcal{S}_w{}' = \frac{(1-\phi_o)^2 \mathcal{S}_w + 2(1-\phi_o)(\beta_o-\phi_o)\mathcal{S}_{wb} + (\beta_o-\phi_o)^2\mathcal{S}_b}{(1-\phi_o)^2} \tag{114}$$

$$\mathcal{S}_{wb}{}' = \frac{1-\beta_o}{(1-\phi_o)^2}\left[(1-\phi_o)\mathcal{S}_{wb} + (\beta_o-\phi_o)\mathcal{S}_b\right] \tag{115}$$

$$\mathcal{S}_b{}' = \left(\frac{1-\beta_o}{1-\phi_o}\right)^2 \mathcal{S}_b \tag{116}$$

$$G' = \left(\frac{1-\phi_o}{1-\beta_o}\right)^2 G. \tag{117}$$

If so desired, these moduli can be further expressed in terms of the moduli of Brown and Korringa (1975). Equation (113) is the symmetric form of the poroelastic constitutive laws we have been seeking.

Finally, to solve a boundary-value problem for the averaged displacement fields, we need to express the conservation of momentum $\nabla \cdot \overline{\boldsymbol{\tau}} = 0$ in terms of the primed stresses of Eq. (107). This is accomplished using Eq. (112), so that the final complete set of governing equations is

$$\nabla \cdot \left[\frac{1-\beta_o}{1-\phi_o}\left(-\overline{P}'\,\mathbf{I} + \overline{\boldsymbol{\tau}}'^D\right) - \frac{(\beta_o-\phi_o)}{1-\phi_o}\overline{p}_f\,\mathbf{I}\right] = 0 \tag{118}$$

$$\nabla \overline{p}_f = 0 \tag{119}$$

$$-\begin{bmatrix} \overline{p}_f \\ \overline{P}' \end{bmatrix} = \frac{1}{\mathcal{S}_w{}'\mathcal{S}_b{}' - \mathcal{S}_{wb}{}'^2} \begin{bmatrix} \mathcal{S}_b{}' & -\mathcal{S}_{wb}{}' \\ -\mathcal{S}_{wb}{}' & \mathcal{S}_w{}' \end{bmatrix} \begin{bmatrix} \nabla \cdot \overline{\mathbf{w}} \\ \nabla \cdot \overline{\mathbf{u}} \end{bmatrix} \tag{120}$$

$$\overline{\boldsymbol{\tau}}'^D = G'\left[\nabla \overline{\mathbf{u}}_s + (\nabla \overline{\mathbf{u}}_s)^T - \frac{2}{3}\nabla \cdot \overline{\mathbf{u}}_s\,\mathbf{I}\right], \tag{121}$$

where we have inverted the constitutive law to give (primed) stress in terms of strain. The primed moduli are defined in Eqs. (114)–(117). Following standard arguments (*e.g.,* Love, 1944, Sec. 118) it is easy to show that unique solutions of Eqs. (118)–(121) are obtained when the boundary conditions on the surface enclosing the domain of interest (with outward normal $\mathbf{n}$) are the specification of both

$$\begin{array}{lccc} (i) & \mathbf{n}\cdot\left[\frac{1-\beta_o}{1-\phi_o}\left(-\overline{P}'\,\mathbf{I}+\overline{\boldsymbol{\tau}}'^{D}\right)-\frac{(\beta_o-\phi_o)}{1-\phi_o}\,\mathbf{I}\,\overline{p}_f\right] & \text{or} & \overline{\mathbf{u}}_s \\ (ii) & \overline{p}_f & \text{or} & \mathbf{n}\cdot\overline{\mathbf{w}}. \end{array}$$

The terms from both (i) and (ii) must be specified on all points of the bounding surface for unique solutions to be obtained. Deresiewicz and Skalak (1963) gave this same condition but using the unprimed stress. Equations (118)–(121) are exact results and only reduce to the standard form of Biot (1962) in the special case where $\beta_o = \phi_o$. Using Eq. (112), the primed stress variables could be replaced by the usual (unprimed) stress; however, this operation would render the compressibility law (120) unsymmetric.

9 Discussion

We now summarize the essence of what has been done in this paper. For the three physical quantities of stress, strain-energy density, and deformation, we have expressed the volume average of the quantity throughout a sample in terms of surface integrals over the sample's external surface. In laboratory experiments, we can only observe (or control) the response (or the force) on the external surface of the sample. Thus, the exact connection between the volume-averaged fields and the laboratory measures has been obtained through the intermediary of what happens on the sample's surface. If a smoothly-weighted volume average is used, the volume-averaged values of stress, strain-energy density, and deformation can no longer be expressed as surface integrals on the sample's exterior surface and exact relations between the laboratory measures and the averaged field quantities cannot be defined.

In order to search for a possible use of smoothly-weighted averages in poroelasticity, consider the following. If there were sufficient heterogeneity in the porous material that averaged fields were changing significantly over length scales much smaller than the applied wavelengths of stress, one might think to perform a second volume averaging over the porous-continuum variables. A hierarchy of such averagings is one way to scale-up through the many possible length scales of variation that exist between the grains and the applied wavelengths. Such a doubly averaged field is defined

$$\overline{\overline{\psi}}_\xi(\mathbf{r}) = \frac{1}{V_c}\int_{\Omega_\infty} d^3\mathbf{r}''\, W_c(\mathbf{r}''-\mathbf{r})\left[\frac{1}{V_\xi(\mathbf{r}'')}\int_{\Omega_{\xi\infty}} d^3\mathbf{r}'\, W_o(\mathbf{r}'-\mathbf{r}'')\psi_\xi(\mathbf{r}')\right], \tag{122}$$

where the weight function W_c (with measure V_c) is the one used when averaging the porous-continuum fields, while the weight W_o is used to perform the initial averages over the actual grains and pores and could be taken to be the step function. Since the porous-continuum fields (either "fluid" or "solid") are defined at each point of a porous continuum, the distinction between the grain space $\Omega_{s\infty}$ and pore space $\Omega_{f\infty}$ disappears after the initial averaging over the grains and pores (*i.e.*, we average over the entire set of points Ω_∞ using the weight W_c).

It is natural to ask whether such a double average can be realized by a single smoothly-weighted average over the grains and pores, *i.e.*, can we find a smooth weight function $W_s(\mathbf{r}', \mathbf{r}) = W_s(\mathbf{r}'-\mathbf{r})$ such that

$$\overline{\overline{\psi}}_\xi(\mathbf{r}) = \frac{1}{V_\xi(\mathbf{r})}\int_{\Omega_{\xi\infty}} d^3\mathbf{r}'\, W_s(\mathbf{r}', \mathbf{r})\psi_\xi(\mathbf{r}'). \tag{123}$$

It is straightforward to see that the required weight function is given by

$$W_s(\mathbf{r}', \mathbf{r}) = \frac{V_\xi(\mathbf{r})}{V_c}\int_{\Omega_\infty} d^3\mathbf{r}''\frac{W_c(\mathbf{r}''-\mathbf{r})W_o(\mathbf{r}'-\mathbf{r}'')}{V_\xi(\mathbf{r}'')}. \tag{124}$$

However, this weight function is not a simple function of $\mathbf{r}'-\mathbf{r}$. For each point $\mathbf{r}$, the functional dependence on $\mathbf{r}'$ will be different. Thus, it is not possible to accomplish the double averaging with a single smoothly-weighted averaging involving a weight function $W_s(\mathbf{r}', \mathbf{r}) = W_s(\mathbf{r}'-\mathbf{r})$. It seems that in poroelasticity, weighted averages are useful only as an easy means for deriving Eqs. (11) and (12). The entire question of upscaling through many length scales of variation is a subject to be treated in future studies.

Finally, we address the related question as to what size the averaging volume should be. It has long been realized (*e.g.*, Hubbert, 1956) that both the porous-material properties and the averaged fields themselves are functions of the size of the averaging volume. In this work, we have, up to now, only assumed that the volume should be much larger than the grains but much smaller than the applied wavelengths of stress (and, of course, exactly equal to the sample dimension when connecting to laboratory measures). We have also carefully allowed for the possibility that the center of the grain distribution within a sample is different from the center of the sample (*i.e.*, $\mathbf{r} \neq \mathbf{r}_s$) so that the analysis would be as free of material-property constraints as possible. However, in poroelasticity problems where laboratory measures will not be used to directly fix material properties, it is now suggested that for an isotropic material a reasonable criterion for the minimum size of the averaging volume is one that gives $\beta_o \simeq \phi_o$, as was assumed on statistical grounds by Mandl (1964), among others.

Imagine the process of determining β_o on a series of concentric averaging spheres. When the sphere radius is small (on the order of the grain radii or less), we expect there to be strong variations in β_o as the sphere size grows. But when the averaging sphere begins to contain several grains, one may expect that the value of β_o will begin to stabilize. If β_o remains stable about a certain value for a range of sphere radii many grain radii in width (say ten), then this stable value of β_o must begin to approximate the volume porosity ϕ_o fairly well. The size of the averaging sphere for which we may first take $\beta_o \simeq \phi_o$ is a good choice for the size of the averaging volume in general since it eliminates an extra parameter (β_o) from the anlysis.

However, it is certainly possible to imagine an isotropic material so heterogeneous that, at least for some points in the material, the value of β_o never stabilizes as the averaging-sphere radii increases. In this case, β_o becomes an important additional parameter required in the poroelastic theory and cannot simply be equated to ϕ_o. In the previous section, we have given the necessary modifications of the standard theory required in this case. However, in this case, it is also likely that higher-order stress and strain moments would need to be allowed for as discussed at the end of Sec. 7. The careful treatment of such higher-order deformation is the subject of a future study.

For anisotropic materials, it is possible that the new term $\mathbf{B}_o$ must always be included, *i.e.*, surface porosity and volume porosity may never become equivalent regardless of the averaging-volume size and regardless of the degree of heterogeneity in the porous material.

Acknowledgments

The initial work of JGB was performed under the auspices of the U. S. Department of Energy by the Lawrence Livermore National Laboratory under contract No. W-7405-ENG-48, while on sabbatical at IPGP. The concluding work of JGB was supported by the Environmental Management Science Program, a joint effort of the Office of Environmental Management and the Office of Energy Research within the Department of Energy, USA. The support of the French government for both authors is also gratefully acknowledged. The comments of both reviewers are appreciated.

References

Berryman, J. G., and Milton, G. W. (1991) Exact results for generalized Gassmann's equations in composite porous media with two constituents. *Geophysics*, **56**, 1950–1960.

Berryman, J. G., and Wang, H. F. (1995) The elastic coefficients of double-porosity models for fluid transport in jointed rock. *J. Geophys. Res.*, **100**, 24611–24627.

Biot, M. A. (1941) General theory of three-dimensional consolidation. *J. Appl. Phys.*, **12**, 155–164.

Biot, M. A., (1956) Theory of propagation of elastic waves in a fluid-saturated porous solid. I. Low-frequency range. *J. Acoust. Soc. Am.*, **28**, 168–178.

Biot, M. A. (1962) Mechanics of deformation and acoustic propagation in porous media. *J. Appl. Phys.*, **33**, 1482–1498.

Biot, M. A., and Willis, D. G. (1957) The elastic coefficients of the theory of consolidation. *J. Appl. Mech.*, **24**, 594–601.

Brown, R. J. S. and Korringa, J. (1975) On the dependence of the elastic properties of a porous rock on the compressibility of the pore fluid. *Geophysics*, **40**, 608–616.

Burridge, R., and Keller, J. B. (1981) Poroelasticity equations derived from microstructure. *J. Acoust. Soc. Am.* **70**, 1140–1146.

deGroot, S. R., and Mazur, P. (1984) *Non-equilibrium Thermodynamics.* Dover, New York.

Deresiewicz, H., and Skalak, R. (1963) On uniqueness in dynamic poroelasticity. *Bull. Seismol. Soc. Am.* **53**, 783–788.

Eringen, A.C. (1968) Theory of micropolar elasticity. *Fracture, vol. 2* (ed. H. Liebowitz), pp.622–736. Academic Press, New York.

Gassmann, F. (1951) Über die elastizität poröser medien. *Veirteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, **96**, 1–23.

Hubbert, M. K. (1956) Darcy's law and the field equations of the flow of underground fluids. *J. Petroleum Technology*, **207**, 222–239.

Joseph, D.D., Lundgren, T.S., Jackson, R., and Saville, D.A. (1990) Ensemble averaged and mixture theory equations for incompressible fluid-particle suspensions. *Int. J. Multiphase Flow*, **16**, 35–42.

Landau, L. D. and Lifshitz, E. M. (1980) *Statistical Physics, 3rd Edition, Part 1.* Pergamon, Oxford.

Landau, L. D. and Lifshitz, E. M. (1986) *Theory of Elasticity, 3rd Edition.* Pergamon, Oxford.

Love, A.E.H. (1944) *A Treatise on the Mathematical Theory of Elasticity, 4th Edition.* Dover, New York.

Mandl, G. (1964) Change in skeletal volume of a fluid-filled porous body under stress. *J. Mech. Phys. Solids*, **12**, 299–315.

Nur, A. and Byerlee, J. D. (1971) An exact effective stress law for elastic deformation of rock with fluids. *J. Geophys. Res.*, **76**, 6414–6419.

Nye, J. F. (1985) *Physical Properties of Crystals.* Oxford University Press, New York.

Pride, S. R., Gangi, A. F., and Morgan, F. D. (1992) Deriving the equations of motion for porous isotropic media. *J. Acoust. Soc. Am.*, **92**, 3278–3290.

Rice, J. R., and Cleary, M. P. (1976) Some basic stress diffusion solutions for fluid-saturated elastic porous media with compressible constituents. *Rev. Geophys. Space Phys.*, **14**, 227-241.

Saffman, P.G. (1971) On the boundary condition at the surface of a porous medium. *Studies in Appl. Math.*, **1**, 93–101.

Slattery, J. C. (1967) Flow of viscoelastic fluids through porous media. *Am. Inst. Chem. Eng. J.*, **13**, 1066–1071.

Slattery, J. C. (1981) *Momentum, energy, and mass transfer in continua, 2nd Edition.* Krieger, New York.

Thompson, M., and Willis, J.R. (1991) A reformation of the equations of anisotropic poroelasticity. *J. Appl. Mech.*, **58**, 612-616.

Tuncay, K., and Corapcioglu, M. Y. (1995) Effective stress principle for saturated fractured porous media. *Water Resources Res.*, **31**, 3103–3106.

Walton, K. (1988) Wave propagation within random packings of spheres. *Geophysical Journal*, **92**, 89–97.

Zimmerman, R. W., Somerton, W. H., and King, M. S. (1986) Compressibility of porous rocks. *J. Geophys. Res.*, **91**, 12765–12777.

Zimmerman, R. W., Myer, L. R., and Cook, N. G. W. (1994) Grain and void compression in fractured and porous rock. *Int. J. Rock Mech. Min. Sci. and Geomech. Abst.*, **31**, 179–184.