EXTENSIONS OF BIOT'S THEORY OF POROELASTICITY TO COMPLEX POROUS MEDIA

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ABSTRACT

We summarize recent efforts to extend the theory of poroelasticity to semilinear and nonlinear elastic response, to partially saturated pores, to inhomogeneous frame materials, and to viscous losses due to localized flow effects. The prospects for a comprehensive theory of wave propagation in porous media are also discussed.

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INTRODUCTION

In his original derivation, Biot constructed a theory of poroelasticity that was quite limited in scope.[1,2] He assumed linear, isotropic elastic response on the macroscopic scale for porous media composed of homogeneous frame materials and fully saturated pores. The principal attenuation mechanism of this theory was viscous attenuation due to shear induced during macroscopic flow of the single-phase fluid filling the pores. Even with these simplifications, the resulting theory has remained a scientific oddity for 30 years: (1) It is relatively hard to analyze the predictions of this theory[3] because it involves two coupled wave equations forming a system somewhat more complex than the equations of viscoelasticity[4] – which are nontrivial to analyze themselves! (2) The most startling predictions of the theory – such as the existence of a slow bulk compressional wave[1] or slow surface[5] and extensional waves[6] – are often very hard to verify in the laboratory.[7–11] (3) Even the physical interpretation of many of the coefficients in the equations remained unclear for 25 years,[8,12–14] and in some cases are still in dispute today.[15] It is therefore understandable that significant progress towards eliminating the many simplifying assumptions contained in the original work had not been made prior to the present decade. Indeed, why complicate a subject which is already so difficult?

The reason of course is "realism." For many of the geophysical applications of most interest, the pertinent geological materials are anisotropic and very heterogeneous, composed of multiple frame materials and multiple pore fluids. In some applications, the exciting waves are of large amplitude so that linear equations of motion are simply inadequate to describe the phenomena we want to study. Often we argue that the elementary theory should suffice to explain the gross behavior of such materials, justifying our approximations with the comparative simplicity and elegance of the resulting theory. If the theory is really successful at explaining the preponderance of experimental data, then of course our arguments are justified and it would appear to be of only academic interest to expend such effort as would be required to construct a truly comprehensive theory. On the other hand, since the theory to date has been unable to explain some of the most elementary experimental results for waves in geological materials, it seems to us essential to proceed with a program of research designed to produce a sophisticated theory capable of treating most of the complications encountered in practice.

Other researchers have also produced extensions of the elementary theory. Biot himself had generalized the theory to include anisotropic effects for dynamic problems[16] and nonlinear effects for quasistatic problems.[17] Various other authors have treated the generalization to partial saturation at very low frequencies in an intuitively appealing manner,[18–22] but without having any clear procedure for generalizing their results for higher frequencies. The form of the equations for the elastic coefficients when the frame material is composed of two or more constituents has been known for some time,[22] but no method for obtaining the required data has been suggested.

The authors have initiated a general program of research aimed at producing a comprehensive theory of dynamic poroelasticity. Although irreversible pore collapse[23] is important in some of our applications, we have neglected such effects initially in order to construct what is otherwise a quite general Lagrangian variational principle[24] for nonlinear and semilinear (reversible) deformations of dry and fluid saturated porous solids. This approach is very closely related to an Eulerian variational formulation by Drumheller and Bedford[25] for flow of complex mixtures of fluids and solids. We have shown that our theory reduces correctly to Biot's equations of poroelasticity[1] for small

Fig. 1. The Expanding Microcosm of Poroelasticity – Schematic diagram of the many directions research must take to produce a comprehensive theory of wave propagation in porous media.

amplitude wave propagation and that it also reduces correctly to Biot's theory[17] of nonlinear and semilinear rheology for porous solids when the deformations are sufficiently slow. The resulting theory is a nontrivial generalization of Biot's ideas including explicit equations of motion for changes of solid and fluid density. Furthermore, if we assume that capillary pressure effects may be neglected, then the linear theory also shows that calculations on problems with only partially saturated pores may be reduced to computations of the same level of difficulty as those for fully saturated pores.[26] We expect the general theory to give a very good account of the behavior of wet porous materials during elastic deformations. Figure 1 provides a schematic illustration of the goals for this program, but of course much of this work has been done by others and much of it remains to be done.

In the presentation that follows, we will concentrate on three recent extensions of the theory of poroelasticity that tend to make the theory more realistic for applications to rocks. First, we show how the theory may be generalized to partially saturated porous media. Then, we use an effective medium approach to find estimates of the coefficients in the equations when the frame material is inhomogeneous. Finally, we analyze the attenuation of the fast compressional wave in heterogeneous media and show that the physically correct damping coefficient depends not on the global permeability, but on a simple spatial average of the local permeability.

SIMPLIFIED EQUATIONS FOR PARTIAL SATURATION

When the mechanical and thermodynamical processes set in motion by a deformation are reversible, an energy functional which includes all the important effects involved in the motion may be constructed. Equations of motion may then be found by an application of Hamilton's principle. Such variational methods based on energy functionals are well-known in continuum mechanics.[27] Thus, the only really new feature in the present context is the degree of complexity; porous earth may be composed of many types of solid constituents and the pore space may be filled with a mixture of water and air. Some irreversible effects may also be included in the variational method (e.g., losses of energy due to drag between constituents) when they may be analyzed in terms of a dissipation functional. Other irreversible effects such as those associated with collapse of the pore space lie outside the scope of the traditional variational approaches; the forms normally used for the energy functionals are quadratic with constant coefficients in the linear problems or simply positive definite polynomials with constant coefficients for nonlinear problems. During pore collapse, the usual assumptions about the form of the energy functionals are violated, so the usefulness of the variational method is questionable. However, if we restrict discussion to linear processes, the variational methods are entirely adequate.

Using these variational methods, we have shown[26] that the general equations of motion for linear elastic wave propagation through a porous medium containing both liquid and gas in the pores are given by

$$\Lambda_{(\xi)0}\ddot{\bar{\rho}}_{(\xi)} = -\frac{\partial E_{(\xi)}}{\partial \bar{\rho}_{(\xi)}} - \frac{\lambda_\phi}{\bar{\rho}^2_{(\xi)0}}, \tag{1}$$

$$\rho_{(s)0}\ddot{u}_{(s)i} + \sum_{\gamma=g,l}\rho_{(s\gamma)0}(\ddot{u}_{(s)i} - \ddot{u}_{(\gamma)i}) = \left[\rho_{(s)0}\frac{\partial E_{(s)}}{\partial u_{(s)i,j}} + \lambda_\phi\phi_{(s)0}\delta_{ij}\right]_{,j} + d_{(s)i} + \rho_{(s)0}b_{(s)i}, \tag{2}$$

and

$$\rho_{(\gamma)0}\ddot{u}_{(\gamma)i} + \sum_{\xi\neq\gamma}\rho_{(\gamma\xi)0}(\ddot{u}_{(\gamma)i} - \ddot{u}_{(\xi)i}) = \phi_{(\gamma)0}(\lambda_\phi)_{,i} + d_{(\gamma)i} + \rho_{(\gamma)0}b_{(\gamma)i} \tag{3}$$

where $\gamma = g$ or l and $\xi = g, l$, or s. The displacements are $u_{(\xi)i}$; the local densities (mass per unit volume of constituent) are $\bar{\rho}_{(\xi)}$; and the partial densities (mass per unit total volume) are $\rho_{(\xi)} = \phi_{(\xi)}\bar{\rho}_{(\xi)}$. The internal energies of these immiscible constituents are $E_{(\xi)}$. The induced mass coefficients are $\rho_{(s\gamma)0}$. The body forces are given by $b_{(\xi)i}$ and the drag forces by $d_{(\xi)i}$. The authors have shown elsewhere[28] that the drag forces may be written in the form $d_{(\gamma)i} = -\sum_\xi D_{(\gamma\xi)}(\dot{u}_{(\xi)i} - \dot{u}_{(s)i})$ where $D_{(\gamma\xi)}$ is a symmetric, positive semidefinite matrix whose matrix elements satisfy $\sum_\gamma D_{(\gamma\xi)} = 0$ for $\xi = g$ or l.

One major simplification that occurs in the equations for partial saturation follows from (1) and the approximation $\Lambda_{(\xi)0} = 0$. We find that

$$\lambda_\phi = -\bar{\rho}^2_{(\xi)0}\frac{\partial E_{(\xi)}}{\partial \bar{\rho}_{(\xi)}} \equiv -p_{(\xi)} \tag{4}$$

where $p_{(\xi)}$ is the pressure for constituent ξ. Eq. (4) implies that all the pressures are equal – which is consistent with an assumption that capillary pressure effects are negligible. The subscript may subsequently be dropped from p. If $2e_{(\xi)ij} \equiv u_{(\xi)i,j} + u_{(\xi)j,i}$, then the first two strain invariants are defined by $I_{(\xi)1} = e_{(\xi)ii}$ and $I_{(\xi)2} = \frac{1}{2}[I^2_{(\xi)1} - e_{(\xi)ij}e_{(\xi)ji}]$. The changes in density are defined by $\Delta\bar{\rho}_{(\xi)} = \bar{\rho}_{(\xi)} - \bar{\rho}_{(\xi)0}$. In terms of these invariants, the standard definitions of the internal energies are

$$\rho_{(s)0}E_{(s)} = \frac{1}{2}aI_{(s)1}^2 + bI_{(s)2} + cI_{(s)1}\Delta\bar{\rho}_{(s)} + \frac{1}{2}d\Delta\bar{\rho}_{(s)}^2 \tag{5}$$

and

$$\rho_{(\gamma)0}E_{(\gamma)} = \frac{1}{2}h_{(\gamma)}\Delta\bar{\rho}_{(\gamma)}^2 \tag{6}$$

for $\gamma = g, l$. Applying (4) to (5) and (6), we find

$$\begin{aligned} p &= \frac{\bar{\rho}_{(s)0}}{\phi_{(s)0}}(cI_{(s)1} + d\Delta\bar{\rho}_{(s)}) \\ &= \frac{\bar{\rho}_{(l)0}}{\phi_{(l)0}}h_{(l)}\Delta\bar{\rho}_{(l)} \\ &= \frac{\bar{\rho}_{(g)0}}{\phi_{(g)0}}h_{(g)}\Delta\bar{\rho}_{(g)}. \end{aligned} \tag{7}$$

The coefficients in (5) have been shown elsewhere[24] to be related to known quantities: $a = \phi_{(s)0}K^*/(\sigma - \phi_{(f)0}) + \frac{4}{3}\mu^*$, $b = -2\mu^*$, $c = \phi_{(s)0}K^*/\bar{\rho}_{(s)0}(\sigma - \phi_{(f)0})$, and $d = [\phi_{(s)0}/\bar{\rho}_{(s)0}]^2K_{(s)}/(\sigma - \phi_{(f)0})$ where $\sigma = 1 - K^*/K_{(s)}$. The bulk and shear moduli of the drained porous solid frame are K^* and μ^*. The bulk modulus of the (assumed) single constituent composing the microscopically homogeneous frame is $K_{(s)}$. If the solid frame is composed of two or more constituents, then these formulas must be modified. The coefficient $h_{(\gamma)}$ is related to the bulk modulus $K_{(\gamma)}$ of the γ-th fluid constituent by

$$h_{(\gamma)} = \frac{\phi_{(\gamma)0}}{\bar{\rho}_{(\gamma)0}^2}K_{(\gamma)}. \tag{8}$$

Now we define the linearized increment of fluid content for partial saturation to be

$$\zeta \equiv I_{(s)1} + \sum_{\xi}\frac{\phi_{(\xi)0}}{\bar{\rho}_{(\xi)0}}\Delta\bar{\rho}_{(\xi)}. \tag{9}$$

If only one fluid phase is present, (9) reduces to the exact result obtained previously by Berryman and Thigpen.[24] If more than one fluid phase is present, then we observe that by defining an effective total fluid density change according to

$$\frac{\phi_{(f)0}}{\bar{\rho}_{(f)0}}\Delta\bar{\rho}_{(f)} \equiv \frac{\phi_{(g)0}}{\bar{\rho}_{(g)0}}\Delta\bar{\rho}_{(g)} + \frac{\phi_{(l)0}}{\bar{\rho}_{(l)0}}\Delta\bar{\rho}_{(l)} \tag{10}$$

with $\phi_{(f)0} \equiv \sum_\gamma \phi_{(\gamma)0}$ and we find that (9) reduces again to the exact result. Furthermore, applying (8), it is straight forward to show that (4) implies that

$$\frac{\Delta\bar{\rho}_{(\gamma)}}{\bar{\rho}_{(\gamma)0}} = \frac{p}{K_{(\gamma)}} \tag{11}$$

for $\gamma = g$ or l. Substituting (11) into both sides of (10) shows that the effective bulk modulus of the multiphase fluid is given by

$$\frac{\phi_{(f)0}}{K_{(f)}} \equiv \sum_{\gamma} \frac{\phi_{(\gamma)0}}{K_{(\gamma)}} \tag{12}$$

which is just the harmonic mean or Reuss average of the constituents' bulk moduli.

To check the consistency of our definition of ζ, we can show easily that

$$\zeta = \sum_{\gamma} \phi_{(\gamma)0}[I_{(s)1} - I_{(\gamma)1}]. \tag{13}$$

If we define the average displacement of a fluid relative to the solid frame by

$$w_{(\gamma)i} = \phi_{(\gamma)0}[u_{(\gamma)i} - u_{(s)i}] \tag{14}$$

for $\gamma = g$ or l and the total relative fluid displacement by

$$w_i = \sum_{\gamma} w_{(\gamma)i}, \tag{15}$$

then (13) becomes

$$\zeta = -w_{i,i}. \tag{16}$$

Eq. (16) reduces to the standard definition for full saturation when only one fluid saturates the pore space and is a natural generalization of this definition for partially saturated materials.

The total relative fluid displacement w_i defined by (15) is important in partial saturation problems not only because of the analogy just developed with the fully saturated problems, but also for convenience in applying boundary conditions in practical problems. We have shown previously[26] that uniqueness of the solutions to the equations (1) – (3) demands the specification of either p or the normal component of this same w_i on the boundaries of the porous material. Therefore, it proves most convenient to combine these equations so that $u_{(s)i}$ and w_i are the dependent variables. We will subsequently drop the subscript (s) on u_i since no confusion will arise and also define $e \equiv I_{(s)1}$. In addition, the zero subscripts on density and volume fraction may also be dropped in the remainder of the analysis.

To determine the relations among p, ζ, and e, substitute (11) and the first equation of (4) into (8) to eliminate $\Delta\bar{\rho}_{(\xi)}$ for all ξ. Using known identities and rearranging terms, we find easily that

$$p = M\zeta - Ce \tag{17}$$

where the coefficients C and M are given by

$$C = \left\{\left[(\sigma - \phi_{(g)} - \phi_{(l)})/K_{(s)} + \phi_{(g)}/K_{(g)} + \phi_{(l)}/K_{(l)}\right]/\sigma\right\}^{-1}, \tag{18}$$

and

$$M = C/\sigma \tag{19}$$

with

$$\sigma = 1 - K^*/K_{(s)}. \tag{20}$$

Substituting (12) into (18) gives

$$C = \left\{\left[(\sigma - \phi_{(f)0})/K_{(s)} + \phi_{(f)0}/K_{(f)}\right]/\sigma\right\}^{-1} \tag{21}$$

which is the standard result for single-phase saturation.[29]

Next we suppose the body forces vanish and sum the equations (2) and (3) to obtain

$$\rho\ddot{u}_i + \bar{\rho}_{(g)}\ddot{w}_{(g)i} + \bar{\rho}_{(l)}\ddot{w}_{(l)i} = \left[\rho_{(s)}\frac{\partial E_{(s)}}{\partial u_{i,j}} - p\delta_{ij}\right]_{,j} \tag{22}$$

where $\rho = \sum_\xi \rho_{(\xi)}$. Dividing (3) through by $\phi_{(f)0}$ and rearranging terms, we find

$$\bar{\rho}_{(g)}\ddot{u}_i + \frac{\alpha_{(g)}\bar{\rho}_{(g)}}{\phi_{(g)}}\ddot{w}_{(g)i} + \frac{D_{(gg)}}{\phi^2_{(g)}}\dot{w}_{(g)i} - \frac{\rho_{(gl)}}{\phi^2_{(g)}}\ddot{w}_{(l)i} = -p_{,i} \tag{23}$$

and

$$\bar{\rho}_{(l)}\ddot{u}_i + \frac{\alpha_{(l)}\bar{\rho}_{(l)}}{\phi_{(l)}}\ddot{w}_{(l)i} + \frac{D_{(ll)}}{\phi^2_{(l)}}\dot{w}_{(l)i} - \frac{\rho_{(lg)}}{\phi^2_{(l)}}\ddot{w}_{(g)i} = -p_{,i}. \tag{24}$$

In (23) and (24), $\alpha_{(g)}$ is the electrical tortuosity of the pore space occupied only by the gas, while $\alpha_{(l)}$ is the electrical tortuosity of the pore space occupied only by the liquid. Introducing a Fourier time dependence of the form $exp(-i\omega t)$ into (23) and (24), combining, rearranging terms, and keeping the same names for the transformed and untransformed variables, we have

$$-\omega^2\begin{pmatrix} q_{(g)} & -r_{(g)} \\ -r_{(l)} & q_{(l)} \end{pmatrix}\begin{pmatrix} w_{(g)i} \\ w_{(l)i} \end{pmatrix} = \begin{pmatrix} -p_{,i} + \omega^2\bar{\rho}_{(g)0}u_i \\ -p_{,i} + \omega^2\bar{\rho}_{(l)0}u_i \end{pmatrix} \tag{25}$$

where

$$\phi^2_{(\gamma)0}q_{(\gamma)} \equiv \alpha_{(\gamma)}\rho_{(\gamma)0} + iD_{(\bar{\gamma}\gamma)}/\omega \tag{26}$$

and

$$\phi^2_{(\gamma)0}r_{(\gamma)} \equiv \rho_{(gl)0}. \tag{27}$$

In (26), $\bar{\gamma} \neq \gamma$ so $\bar{\gamma} = l$ or g as $\gamma = g$ or l. Inverting the matrix in (25) and summing the results gives

$$-\omega^2[q_{(g)}q_{(l)} - r_{(g)}r_{(l)}]w_i = -(s_{(g)} + s_{(l)})p_{,i} + \omega^2[s_{(g)}\bar{\rho}_{(l)} + s_{(l)}\bar{\rho}_{(g)}]u_i \tag{28}$$

where

$$s_{(\gamma)} = q_{(\gamma)} + r_{(\gamma)}. \tag{29}$$

Using the expressions for $w_{(\gamma)i}$ from (25) again, we find

$$\begin{aligned}\sum_\gamma \bar{\rho}_{(\gamma)}w_{(\gamma)i} = \{&[\bar{\rho}_{(g)}(q_{(l)} + r_{(g)}) + \bar{\rho}_{(l)}(q_{(g)} + r_{(l)})]p_{,i} \\ &- \omega^2[q_{(l)}\bar{\rho}^2_{(g)} + (r_{(g)} + r_{(l)})\bar{\rho}_{(g)}\bar{\rho}_{(l)} + q_{(g)}\bar{\rho}^2_{(l)}]u_i\}/\omega^2[q_{(g)}q_{(l)} - r_{(g)}r_{(l)}] \\ &= \frac{\bar{\rho}_{(g)}(q_{(l)} + r_{(g)}) + \bar{\rho}_{(l)}(q_{(g)} + r_{(l)})}{s_{(g)} + s_{(l)}}w_i - \frac{(\bar{\rho}_{(g)} - \bar{\rho}_{(l)})^2}{s_{(g)} + s_{(l)}}u_i\end{aligned} \tag{30}$$

where $p_{,i}$ has been eliminated in the second step of (30) using (28).

The final form of these equations is found by substituting (30) into (22), using (18) in the result and also in (28), and finally rearranging terms. The equations then take the familiar form

$$\mu\nabla^2\vec{u} + (H-\mu)\nabla e - C\nabla\zeta + \omega^2(\rho_{uu}\vec{u} + \rho_{uw}\vec{w}) = 0, \tag{31}$$

$$C\nabla e - M\nabla\zeta + \omega^2(\rho_{wu}\vec{u} + \rho_{ww}\vec{w}) = 0, \tag{32}$$

where the inertial coefficients are given by

$$\rho_{uu} = \rho - \frac{(\bar{\rho}_{(g)} - \bar{\rho}_{(l)})^2}{s_{(g)} + s_{(l)}}, \tag{33}$$

$$\rho_{wu} = \frac{\bar{\rho}_{(g)}s_{(l)} + \bar{\rho}_{(l)}s_{(g)}}{s_{(g)} + s_{(l)}} = \rho_{uw} + \frac{(r_{(l)} - r_{(g)})(\bar{\rho}_{(g)} - \bar{\rho}_{(l)})}{s_{(g)} + s_{(l)}}, \tag{34}$$

and

$$\rho_{ww} = \frac{q_{(g)}q_{(l)} - r_{(g)}r_{(l)}}{s_{(g)} + s_{(l)}}. \tag{35}$$

The coefficient H is given by

$$H = K^* + \frac{4}{3}\mu^* + \sigma C \tag{36}$$

while C and M are given by (18) and (19). Thus, we find the remarkable result that the equations of motion for partial saturation and for full saturation are the same – the only difference being that the inertial coefficients are more complex when the porous solid is only partially saturated.

BIOT'S THEORY OF POROELASTICITY

Now we will change notation somewhat and consider two porous media (i.e., host and inclusion) each of whose connected pore space is saturated with a single-phase viscous fluid. The fraction of the total volume occupied by the fluid is the void volume fraction or porosity ϕ, which is assumed to be uniform within a constituent but which may vary between the the host and inclusion. The bulk modulus and density of the fluid are K_f and ρ_f, respectively, in the host. The bulk and shear moduli of the drained porous frame for the host are K and μ. For now we assume the frame of the host is composed of a single constituent whose bulk and shear moduli and density are K_m, μ_m, and ρ_m. Corresponding parameters for the inclusion will be distinguished by adding a prime superscript. The frame moduli may be measured directly or they may be estimated using one of the many methods developed to estimate elastic constants of composites.[30]

For long-wavelength disturbances ($\lambda > h$, where h is a typical pore size) propagating through such a porous medium, we define average values of the (local) displacements in the solid and also in the saturating fluid. The average displacement vector for the solid frame is $\vec{u}$ while that for the pore fluids is $\vec{u}_f$. The average displacement of the fluid relative to the frame is $\vec{w} = \phi(\vec{u}_f - \vec{u})$. For small strains, the frame dilatation is

$$e = e_x + e_y + e_z = \nabla\cdot\vec{u}, \tag{37}$$

where e_x, e_y, e_z are the Cartesian strain components. Similarly, the average fluid dilatation is

$$e_f = \nabla \cdot \vec{u}_f \tag{38}$$

(e_f also includes flow terms as well as dilatation) and the increment of fluid content is defined by

$$\zeta = \nabla \cdot \vec{w} = \phi(e - e_f). \tag{39}$$

With these definitions, Biot[1,2,31] shows that the strain-energy functional for an isotropic, linear medium is a quadratic function of the strain invariants[32] $I_1 = e, I_2$, and of ζ having the form

$$2E = He^2 - 2Ce\zeta + M\zeta^2 - 4\mu I_2, \tag{40}$$

where

$$I_2 = e_y e_z + e_z e_x + e_x e_y - \frac{1}{4}(\gamma_x^2 + \gamma_y^2 + \gamma_z^2), \tag{41}$$

and $\gamma_x, \gamma_y, \gamma_z$ are the shear strain components. Our earlier definitions (5) and (6) for partial saturation are completely consistent[24,33] with these definitions.

With time dependence of the form $exp(-i\omega t)$, the Fourier transformed version of the coupled wave equations of poroelasticity in the presence of dissipation take the form

$$\mu\nabla^2\vec{u} + (H - \mu)\nabla e - C\nabla\zeta + \omega^2(\rho\vec{u} + \rho_f\vec{w}) = 0, \tag{42}$$

$$C\nabla e - M\nabla\zeta + \omega^2(\rho_f\vec{u} + q\vec{w}) = 0, \tag{43}$$

where

$$\rho = \phi\rho_f + (1 - \phi)\rho_m \tag{44}$$

and

$$q = \rho_f[\alpha/\phi + iF(\xi)\eta/\kappa\omega]. \tag{45}$$

The kinematic viscosity of the liquid is η, the permeability of the porous frame is κ, and the dynamic viscosity factor[2] is given (for our present choice of sign for the frequency dependence) by

$$F(\xi) = \frac{1}{4}\xi T(\xi)/[1 + 2T(\xi)/i\xi], \tag{46}$$

where

$$T(\xi) = \frac{ber'(\xi) - ibei'(\xi)}{ber(\xi) - ibei(\xi)} \tag{47}$$

and

$$\xi = (\omega h^2/\eta)^{\frac{1}{2}}. \tag{48}$$

The functions $ber(\xi)$ and $bei(\xi)$ are the real and imaginary parts of the Kelvin function. The dynamic parameter h is a characteristic length generally associated with (and comparable in magnitude to) the steady-flow hydraulic radius. The electrical tortuosity α is a pure number related to the frame inertia which has been measured recently[14] for porous glass bead samples and has also been estimated theoretically.[8,12] The electrical tortuosity α and the fluid flow tortuosity τ are related by $\alpha = \tau^2 = \phi F$, where F is the electrical formation factor.

The coefficients H, C, and M are given by[22,29]

$$H = K + \frac{4}{3}\mu + \sigma C, \tag{49}$$

$$C = \left\{[(\sigma - \phi)/K_m + \phi/K_f]/\sigma\right\}^{-1}, \tag{50}$$

$$M = C/\sigma, \tag{51}$$

where

$$\sigma = 1 - K/K_m. \tag{52}$$

To decouple the wave equations (42) and (43) into Helmholtz equations for three modes of propagation, we note that the displacements $\vec{u}$ and $\vec{w}$ can be decomposed as

$$\vec{u} = \nabla\Upsilon + \nabla \times \vec{\beta},\ \vec{w} = \nabla\psi + \nabla \times \vec{\chi}, \tag{53}$$

where Υ,ψ are scalar potentials and $\vec{\beta}$,$\vec{\chi}$ are vector potentials. Substituting (53) into Biot's equations (42) and (43), we find they are satisfied if two pairs of equations hold:

$$(\nabla^2 + k_s^2)\vec{\beta} = 0,\ \vec{\chi} = -\Gamma_s\vec{\beta}, \tag{54}$$

where $\Gamma_s = \rho_f/q$ and

$$(\nabla^2 + k_\pm^2)A_\pm = 0. \tag{55}$$

In this notation, the subscripts $+$,$-$, and s refer respectively to the fast and slow compressional waves and the shear wave. The wave vectors in (54) and (55) are defined by

$$k_s^2 = \omega^2(\rho - \rho_f\Gamma_s)\mu \tag{56}$$

and

$$k_\pm^2 = (\omega^2/2\Delta)(b + f \mp [(b-f)^2 + 4cd]^{\frac{1}{2}}), \tag{57}$$

where

$$b = \rho M - \rho_f C,\ c = \rho_f M - qC,\ d = \rho_f H - \rho C,\ f = qH - \rho_f C, \tag{58}$$

with

$$\Delta = MH - C^2. \tag{59}$$

The linear combination of scalar potentials has been chosen to be

$$A_\pm = \Gamma_\pm\Upsilon + \psi, \tag{60}$$

where

$$\Gamma_\pm = d/[(k_\pm\Delta/\omega^2)^2 - b] = [(k_\pm\Delta/\omega^2)^2 - f]/c. \tag{61}$$

With the identification (61), the decoupling is complete.

Since (54) and (55) are valid for any choice of coordinate system, they may be applied to boundary value problems with arbitrary symmetry. Biot's theory has therefore been applied to the scattering of elastic waves from a spherical inhomogeneity by Berryman.[34] The results of that calculation will be summarized in the next section.

SCATTERING FROM A SPHERICAL INHOMOGENEITY

The full analysis of scattering from a spherical inhomogeneity in a fluid-saturated porous medium is quite tedious. Fortunately, much of this work has already been done[34] and we may therefore merely quote the pertinent results here.

Let the spherical inhomogeneity have radius a. For the moment, we will place no restrictions on the properties of the inhomogeneous region. Thus the frame bulk and shear moduli, the grain bulk modulus, the density, the porosity, and the permeability of a solid inclusion may all be different from those of the host. Furthermore, the bulk modulus, density, and viscosity of the fluid in an inhomogeneous region may also all be different from those of the host fluid. Suppose now that a plane fast compressional wave is generated at a free surface far from the inclusion. Then, if the incident fast compressional wave has the form

$$\vec{u} = \hat{z}\frac{A_0}{ik_+}\, exp\, i(k_+z - \omega t), \tag{62}$$

the radial component of the scattered compressional wave contains both fast and slow parts in the far field and is given by

$$\begin{aligned} u_{1r} = {} & (ik_+)^{-1} exp\, i(k_+r - \omega t)/k_+r[B_0^{(+)} - B_1^{(+)} \cos\theta - B_2^{(+)}(3\cos 2\theta + 1)/4] \\ & - (ik_-)^{-1} exp\, i(k_-r - \omega t)/k_-r[B_0^{(-)} - B_1^{(-)} \cos\theta - B_2^{(-)}(3\cos 2\theta + 1)/4]. \end{aligned} \tag{63}$$

Then, with the definitions $\kappa_\pm = k_\pm a$ and $\kappa_s = k_s a$ and with no restrictions on the materials, we find that

$$\begin{aligned} B_0^{(-)} = {} & \frac{i\kappa_-^3 A_0}{3M'(\Gamma_+ - \Gamma_-)(K' + \frac{4}{3}\mu)}\Big[(C - M\Gamma_-)(K' + \frac{4}{3}\mu) \\ & - (C' - M'\Gamma_-)(K + \frac{4}{3}\mu) + (C - M\Gamma_-)(C' - M'\Gamma_-)\Big(\frac{C'}{M'} - \frac{C}{M}\Big)\Big], \end{aligned} \tag{64}$$

and

$$B_0^{(+)} = \frac{\kappa_+^3 A_0}{3i}\frac{[K' - K + (C - M\Gamma_-)(C'/M' - C/M)]}{K' + \frac{4}{3}\mu} + (\kappa_+/\kappa_-)^3 B_0^{(-)}. \tag{65}$$

Expansions of the other coefficients in the small parameter $\epsilon = C/K$ have been given in the reference.[34] However, for the present application, only the first two coefficients are needed and these happen to be the only ones known exactly at present. Of course, the full scattered wave also contains transverse components of the compressional wave, relative fluid/solid displacement, and mode converted shear waves. However, the scattering coefficients for these contributions are linearly dependent on the the coefficients in (63) and therefore contain no new information. It is sufficient then to base our discussion on the expression (63).

As an elementary check on our analysis, we should first consider the limit in which the porosity ϕ vanishes. Then the fluid effects disappear from the equations and only the first line of (63) survives. Furthermore, it is not difficult to check[34] that the coefficients $B_n^{(+)}$ for $n = 0, 1, 2$ reduce to the well-known results for scattering from a spherical elastic inclusion in an infinite elastic medium.[30] For example,

$$B_0^{(+)} = -i\kappa_+^3 A_0(K' - K)/(3K' + 4\mu) \tag{66}$$

in this limit as expected.

These results have a multitude of potential uses. One straightforward application is the calculation of energy losses from elastic wave scattering by randomly distributed particles. A second important application is to use these results as the basis for an effective medium approximation for the effective constants of complex porous media. The second application is the one we will address in the next section.

Fig. 2. A spherical inhomogeneity in a porous medium could be caused by local variations in fluid content, grain composition, porosity, permeability, etc.

MICROSCOPIC HETEROGENEITY

As we have noted previously, the equations of poroelasticity have several significant limitations. For example, these equations were derived with an explicit long-wavelength (low-frequency) assumption and also with strong implicit assumptions of homogeneity and isotropy on the macroscopic scale. Another restriction assumes that the pore fluid is uniform and that it fully saturates the pore space. For the present application, we will assume that a single fluid saturates all the pore space for host as well as inclusion and the scattering is caused by microscopic heterogeneity in the solid properties.

Before deriving our main results, consider the problem of the porous frame without a saturating fluid (or with a highly compressible saturating gas). Then, since we take $C = M = \rho_f = 0$ in this limit, each term of Eq. (50) vanishes identically and the fluid dependent terms of Eq. (49) also vanish, leaving only the terms for the elastic behavior of the porous frame remaining. Since no slow wave can propagate under these circumstances, the second line of Eq. (63) disappears and only the fast wave terms contribute to the scattering. This limit is formally equivalent to the problem of elastic wave scattering from a spherical inclusion which has been treated in detail previously (see Ref. 30 and other references therein). The effective medium approximation requires the weighted average of the single-scattering results to vanish. This approach simulates the physical requirement that the forward scattering should vanish at infinity if the impedance of the "effective medium" has been well matched to that of the composite. The resulting condition is that the volume weighted average of each of the $B_n^{(+)}$'s for $n = 0-2$ must vanish. Using the convention that the effective constants for the composite porous medium are distinguished by an asterisk, the formulas for the effective bulk (K^*) and shear(μ^*) moduli for the drained porous frame of a microscopically heterogeneous medium are

$$\frac{1}{K^* + \frac{4}{3}\mu^*} = \left\langle \frac{1}{K(\vec{x}) + \frac{4}{3}\mu^*} \right\rangle \tag{67}$$

and

$$\frac{1}{\mu^* + F^*} = \left\langle \frac{1}{\mu(\vec{x}) + F^*} \right\rangle \tag{68}$$

where

$$F = (\mu/6)(9K + 8\mu)/(K + 2\mu). \tag{69}$$

The spatial($\vec{x}$) average is denoted by $\langle \cdot \rangle$. The remaining constant to be determined is the corresponding effective density which is just given by the average density.[34] Eq. (67) follows easily from the volume average of (66), while Eq. (68) follows similarly from the volume average of B_2^+. Note that the equations for K^* and μ^* are coupled and therefore must be solved iteratively (i.e., self-consistently). Although the form of the equations (67) and (68) is identical to that obtained for elastic composites, it is important to recognize that the results can be quite different since the local constants $K(\vec{x})$ and $\mu(\vec{x})$ appearing in the formulas are frame moduli of the constituent spheres of drained porous material, not (necessarily) the moduli of the individual material grains. Of course, since the formula reduces correctly in the absence of porosity to the corresponding result for the purely elastic limit, the user of Eqs. (67) and (68) has some discretion about conceptually lumping grains together to form a porous frame or treating them as isolated elastic inclusions. For purposes of modelling complex aggregates of grains typical of earth materials, this freedom of choice appears to be a real advantage.

Now we will restrict discussion to the very low frequency limit where

$$\Gamma_+ = H/C \tag{70}$$

and

$$\Gamma_- = 0. \tag{71}$$

With these restrictions, the relevant scattering coefficients reduce to

$$\begin{aligned} B_0^{(-)} = & \frac{i\kappa_-^3 C A_0}{3HM'(K' + \frac{4}{3}\mu)} \Big[C(K' + \frac{4}{3}\mu + \sigma' C') \\ & - C'(K + \frac{4}{3}\mu + \sigma C) \Big], \end{aligned} \tag{72}$$

and

$$B_0^{(+)} = \frac{\kappa_+^3 A_0}{3i} \frac{[K' - K + (\sigma' - \sigma)C]}{K' + \frac{4}{3}\mu} + (\kappa_+/\kappa_-)^3 B_0^{(-)}. \tag{73}$$

The resulting conditions on the effective constants are

$$\left\langle \frac{C^* \big[K(\vec{x}) + \frac{4}{3}\mu^* + \sigma(\vec{x})C(\vec{x})\big] - C(\vec{x})\big[K^* + \frac{4}{3}\mu^* + \sigma^* C^*\big]}{M(\vec{x})\big[K(\vec{x}) + \frac{4}{3}\mu^*\big]} \right\rangle = 0 \tag{74}$$

and

$$\left\langle \frac{K(\vec{x}) - K^* + \big[\sigma(\vec{x}) - \sigma^*\big]C^*}{K(\vec{x}) + \frac{4}{3}\mu^*} \right\rangle = 0. \tag{75}$$

Recall that the averages in (74) and (75), as elsewhere in this paper, refer to spatial averages over (possibly) porous constituents of the overall porous aggregate. The limitations on the assumed geometry of the resulting aggregate have been discussed previously.[35] Note that (74) and (75) depend on the effective medium frame moduli K^* and μ^* determined by (67) and (68). The new constants determined by (74) and (75) are C^* and σ^*. The expressions for C^* and σ^* are coupled as written but may be uncoupled after some algebra. The final expressions for these constants are

$$C^* = \sigma^* \Big/ \Big[\Big\langle \frac{1}{M(\vec{x})} \Big\rangle + \Big\langle \frac{\sigma^2(\vec{x}) - (\sigma^*)^2}{K(\vec{x}) + \frac{4}{3}\mu^*} \Big\rangle \Big] \tag{76}$$

and

$$\sigma^* = \Big\langle \frac{\sigma(\vec{x})}{K(\vec{x}) + \frac{4}{3}\mu^*} \Big\rangle \Big/ \Big\langle \frac{1}{K(\vec{x}) + \frac{4}{3}\mu^*} \Big\rangle. \tag{77}$$

Notice that both constants are determined explicitly by the formulas, in contrast to the frame moduli K^* and μ^* which are determined only implicitly by (67) and (68). It has also been shown[35] that (76) and (77) are completely consistent with all known constraints[22,29] on the form of these coefficients.

The same idea used to derive (76) and (77) was also used to show[36] that the speed of waves propagating through a mixture of liquid and gas in the low frequency limit is given by Wood's formula[37] as expected.[20,21]

LOCAL-FLOW BIOT THEORY

A convincing demonstration has been given[38] that, if we assume global fluid-flow effects dominate the viscous dissipation, Biot's theory of poroelasticity cannot explain the observed magnitude of wave attenuation in partially saturated rocks. Since the same theory explains the wave speeds quite well, it is reasonable to suppose that a small change in the theory may be adequate to repair this flaw. Many possible explanations are possible of course, but within the context of Biot's theory the simplest postulate is to suppose that local – rather than global – fluid-flow effects dominate the dissipation.[15] We will distinguish two related issues in this section which are summarized in the following questions: (1) Does the physics of wave propagation require that the value of the permeability κ appearing in the Biot's equations should be that for global flow or for local flow? Then, if we can show that the value should be that for local flow, (2) does this change in the interpretation make enough difference so that the theory can explain the correct magnitude for the attenuation?

To address the first question, we explore the consequences of assuming that Biot's theory should be applied at the local flow level rather than at the global flow level. This assessment is easily done by examining the dispersion relations. When the Fourier time dependence is $e^{-i\omega t}$ with angular frequency ω sufficiently low, Biot's theory predicts [see Eq. (57)] the dispersion relations for the fast (+) and slow (−) compressional modes in any homogeneous porous material to be

$$k_+^2 \simeq \frac{\omega^2}{v_+^2}[1 + i\omega\frac{\rho_f^2}{\rho q_0}(1 - v_0^2/v_+^2)^2] \tag{78}$$

and

$$k_-^2 \simeq \frac{i\omega q_0 H}{MH - C^2} \tag{79}$$

where

$$v_+^2 = H/\rho, v_0^2 = C/\rho_f, q_0 = \rho_f\eta/\kappa. \tag{80}$$

The fraction of the total volume occupied by the fluid is the void volume fraction or porosity ϕ, which is assumed to be uniform. The bulk modulus and density of the fluid are K_f and ρ_f. The bulk and shear moduli of the drained porous frame are K and μ. For simplicity we assume the frame is composed of a single constituent whose bulk and shear moduli and density are K_m, μ_m, and ρ_m. Then the coefficients H, C, and M are given by (49)-(52). The overall density is

$$\rho = \phi\rho_f + (1 - \phi)\rho_m. \tag{81}$$

The kinematic viscosity of the fluid is η and the permeability of the porous frame is κ.

Defining the quality factor for the fast compressional wave Q_+ by

$$k_+^2 = \frac{\omega^2}{v_+^2}[1 + i/Q_+], \tag{82}$$

we find that Q_+ is given by

$$1/Q_+ = \omega\frac{\kappa\rho_f}{\eta\rho}(1 - v_0^2/v_+^2)^2. \tag{83}$$

Since $1/Q_+$ is proportional to the permeability, the attenuation is therefore greatest in regions of high permeability. Thus, we might say that the regions of high permeability control the attenuation.

In the very low frequency limit, the slow compressional mode is known to reduce to Darcy flow with slowly changing magnitude and direction as the driving potential gradient oscillates sinusoidally.[39] Now consider a layered porous material (whose constants depend only on depth z) with constituents having identical physical constants except for the permeability κ which varies widely from layer to layer but which has a constant value κ_n within the n-th layer (lying in the range $z_{n-1} \le z \le z_n$ with $z_0 = 0$). Thus, the permeability is a piece-wise constant function of z. The thickness of the n-th layer is given by $l_n = z_n - z_{n-1}$. If we impose a potential gradient along the z-direction in such a layered material, it is well-known that the effective permeability for fluid flow is found by taking the harmonic mean of the constituent permeabilities, *i.e.*,

$$1/\kappa_f = \frac{1}{L}\sum_n \frac{l_n}{\kappa_n}, \tag{84}$$

where the total sample length L is given by the sum of the layer thicknesses

$$L = \sum_n l_n. \tag{85}$$

From (84), we can conclude that the regions of lowest permeability dominate the effective overall permeability for fluid flow through a porous layered medium. Thus, we might say that the regions of low permeability control the fluid flow – at least for this special choice of geometry.

The apparent attenuation of a fast compressional mode at normal incidence on such a structure has two distinct components: (1) Reflection and mode conversion at layer interfaces will have a tendency to degrade the fast wave, but this effect will be quite small at low frequencies for the model structure we are considering. (2) The attenuation within a layer is determined by the quality factor for that layer, as shown by Eq. (82). Assuming the attenuation is small enough, we may approximate (82) within any layer by $k_+(z) = \frac{\omega}{v_+}[1 + i/2Q_+(z)]$, where the functions $k_+(z)$ and $Q_+(z)$ take the piece-wise constant values appropriate for the depth argument z. Neglecting the small effects of reflection and mode conversion, the behavior of the fast compressional wave at normal incidence is then easily seen to be given by

$$A_+ exp[i\int_0^z dz k_+(z) - i\omega t] \simeq A_+ exp[i\frac{\omega}{v_+}z - i\omega t - \frac{\omega}{v_+}\int_0^z dz\frac{1}{2Q_+(z)}], \tag{86}$$

where A_+ is the amplitude of the wave at $z = 0$. In writing (86), we have used the piece-wise constant property of the functions. The integral in the exponent is given by

$$\int_0^z dz\frac{1}{2Q_+(z)} = \frac{\omega\rho_f}{2\eta\rho}(1 - v_0^2/v_+^2)^2\int_0^z dz\kappa(z). \tag{87}$$

At the $z = L$ boundary of the material, we have

$$\int_0^L dz\kappa(z) = \sum_n l_n\kappa_n. \tag{88}$$

If the layering is periodic with period much less than either z or L or if it is statistically homogeneous on this length scale, then we may approximate the integral in the exponent of (86) using (87) and

$$\int_0^z dz\kappa(z) \simeq \kappa_a z, \tag{89}$$

where the effective permeability for attenuation measurements is given by the mean

$$\kappa_a = \frac{1}{L}\sum_n l_n \kappa_n. \tag{90}$$

It is well-known that the mean is always greater than or equal to the harmonic mean of any function; thus,

$$\kappa_f \leq \kappa_a. \tag{91}$$

The answer to our first question is therefore that the physics of wave propagation dictates that local-flow effects must dominate the attenuation of the fast compressional wave. The necessity of this conclusion is nicely illustrated in Figure 3. Suppose that a fast compressional wave is incident on a layered material with alternating permeable and impermeable layers. If the impermeable layers are very thin and have an acoustic impedance closely matching that of the permeable layers, their presence has a negligible effect on the propagating fast wave. The viscous attenuation of the fast wave occurs solely in the permeable layers and magnitude of that attenuation is completely determined by the permeability of these layers. By contrast, the global permeability of this material in the direction normal to the layering vanishes identically. If this zero value were used in our predictions, the magnitude of the attenuation would be grossly under estimated. Although this choice of geometry is extreme, it clearly shows that errors in estimates of attenuation will arise if the value of permeability for global flow is used.

Now, can the theory predict the correct magnitude for the attenuation even with this change in the interpretation of the permeability factor? To use our conclusions in a predictive mode – *i.e.*, to predict the wave attenuation from measurements of permeability – we need some independent means of measuring the local permeability distribution. Normal laboratory flow experiments will not suffice, because they necessarily measure the global permeability. One promising method of estimating the local permeability uses image processing techniques to measure pertinent statistical properties of rock topology from pictures of cross sections.[40] This approach is still under development and we will not attempt to describe it in detail here.

Another approach, which is ultimately much less satisfactory than the image processing method but much easier to use at present, is to suppose that we can obtain reasonable estimates of the local permeability κ_L from the known values of the global permeability κ_G, the tortuosity $\tau = (\phi F)^{\frac{1}{2}}$, and the porosity ϕ. To do so requires a formula, so we will use a form of the Kozeny-Carman relation derived recently by Walsh and Brace.[41] For tubes of arbitrary ellipsoidal (major and minor axes a,b) cross-section the effective permeability of straight sections of such tubes is given by $\kappa = (\pi/4A)[a^3b^3/(a^2+b^2)]$. The porosity for an ellipsoidal tube is $\phi = \pi ab/A$ and the specific surface area is well approximated by $s \simeq 2\pi[(a^2+b^2)/2]^{\frac{1}{2}}/A$. Then, a Kozeny-Carman relation satisfied by κ, ϕ, and s can be shown to be

$$\kappa = \frac{1}{2}\frac{\phi^3}{s^2} \tag{92}$$

Fig. 3. Illustration of a simple experiment to prove that the attenuation of the fast wave depends on the local – not the global – value of permeability κ. A fast wave incident normal to the impermeable partitions will experience a small but finite attenuation even though the global permeability in this direction vanishes identically.

for the effective permeability of a single tube oriented along the pressure gradient. If the tube is at an angle θ to the this gradient, then it is not hard to show[41] that

$$\kappa = \frac{1}{2}\frac{\phi^3}{s^2\tau^2}, \tag{93}$$

where $\tau = 1/cos\theta$. If we suppose that (92) and (93) are fairly representative of the material of interest, then (92) describes the maximum local permeability κ_L and (93) the effective global permeability κ_G. We then conclude that

$$\kappa_L = \tau^2\kappa_G = \phi F\kappa_G. \tag{94}$$

The tortuosity τ has been measured for many sandstones; the values for samples studied by Simmons et al.[42] lie in the range $1.5 \leq \tau \leq 5$, with most values $\tau \simeq 2$. To obtain estimates of attenuation close to experiment,[20] we need to increase the value of permeability used in Mochizuki's calculations[38] by a factor of $\tau^2 \simeq 10$. This requirement implies a tortuosity of $\tau \simeq 3$, which is clearly well within the established experimental bounds. This argument provides strong evidence for the plausibility of a local-flow explanation of the observed discrepancies. However, a completely satisfying demonstration must await the collection of the required data on local-flow permeability.

One unfortunate consequence of the observation that local permeability controls attenuation is that measured attenuation in wet rocks cannot be used as a diagnostic of the global fluid-flow permeability. Nevertheless, since the mean of the local permeabilities will always be greater than the true fluid-flow permeability regardless of the actual spatial distribution of the constituent κ's, the effective permeability obtained from attenuation measurements can be used to provide an upper bound on the desired global permeability.

DISCUSSION

What then are the prospects for a comprehensive theory of poroelasticity? It appears likely at this point that we will soon have a completely satisfactory linear theory of bulk waves including effects of partial saturation and inhomogeneous frame materials. A satisfactory nonlinear theory of bulk waves including effects of fracture, plastic flow, and pore collapse is at a more elementary stage, but is still likely to be achieved in the next decade. At present it appears that the problems most likely to cause real trouble are those involving surface waves rather than the bulk waves. Surface waves depend critically on the nature of the equations of motion near interfaces. Using the standard boundary conditions of poroelasticity,[26,43] it has been shown that a slow surface wave[5] or slow extensional wave[6] is expected only when a closed-pore boundary condition applies at the porous surface. Yet, the experimental data to date seems to show that such slow surface waves[10] do in fact propagate when the open-pore boundary condition applies. It is possible that the presence of a thin damage region close to the surface has a major effect on the conclusions of the theory regarding the propagation of the surface waves. However, it could also be that these experiments are pointing out still another subtle deficiency of the equations we use to describe wave propagation in porous media.

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