Exact results for generalized Gassmann's equations in composite porous media with two constituents

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Abstract

Wave propagation in fluid-filled porous media is governed by Biot's equations of poroelasticity. Gassmann's relation gives an exact formula for the poroelastic parameters when the porous medium contains only one type of solid constituent. The present paper generalizes Gassmann's relation and derives exact formulas for two elastic parameters needed to describe wave propagation in a conglomerate of two porous phases. The parameters were first introduced by Brown and Korringa when they derived a generalized form of Gassmann's equation for conglomerates. These elastic parameters are the bulk modulus K_s associated with changes in the overall volume of the conglomerate and the bulk modulus K_ϕ associated with the pore volume when the fluid pressure (p_f) and confining pressure (p) are increased, keeping the differential pressure ($p_d = p - p_f$) fixed. These moduli are properties of the composite solid frame (drained of fluid) and are shown here to be completely determined in terms of the bulk moduli associated with the two solid constituents, the bulk moduli of the drained conglomerate and the drained phases, and the porosities in each phase. The pore structure of each phase is assumed uniform and smaller than the grain size in the conglomerate. The relations found are completely independent of the pore microstructure and provide a means of analyzing experimental data.

The key idea leading to the exact results is this: Whenever two scalar fields (in our problem p_f and p_d) can be independently varied in a linear composite containing only two constituents, there exists a special value γ of the increment ratio for these two fields corresponding to an overall expansion or contraction of the medium with no change of relative shape. This fact guarantees that a set of consistency relations exists among the constituent moduli and the effective moduli, which then determine all but one of the effective constants. Thus, K_s and K_ϕ are determined in terms of the drained frame modulus K and the constituents' moduli. Because the composite is linear, the coefficients found for the special value of the increment ratio are also the exact coefficients for an arbitrary ratio. Since modulus K is commonly measured while the other two are not, these exact relations provide a significant advance in our ability to predict the response of porous materials to pressure changes.

It is also shown that additional results (such as rigorous bounds on the parameters) may be easily obtained by exploiting an analogy between the equations of thermoelasticity and those of poroelasticity. The method used to derive these results may also be used to find exact expressions for three component composite porous materials when thermoelastic constants of the components and the composite are known.

1 Introduction

The equations for elastic wave propagation through fluid-saturated porous media were first derived by *Biot (1956a)*. The main limitation to the use of these equations is that relationships between material properties of the rock constituents and coefficients appearing in the equations are still not well understood. The fundamental result of *Gassmann (1951)* shows how the coefficients depend on compressibility of the saturating pore fluid. But rigorous application of Gassmann's result is limited by the assumption in its derivation that the porous frame is composed of only one type of solid constituent. This situation seldom applies to natural materials like rocks (where a sandstone, for example, often contains a substantial proportion of clay in its pores), but it does apply to the behavior of artificial materials like sinteredglass-bead packs. When applying Gassmann's equation in situations where too many phases are present, users must be creative in choosing appropriate effective moduli; this undesirable freedom (imposed by the lack of a more versatile theory) therefore leads to some uncertainty in the results. *Brown and Korringa (1975)* have generalized Gassmann's result to show how the coefficients must depend on fluid compressibility when more than one constituent is present, but their result contains two new compressibilities that are unknown and have unknown dependence on the constituent's properties. Their result is therefore useful for analyzing experimental data, but has not yet found use in predicting behavior of porous materials based on a knowledge of the constituents' properties.

Berryman (1992) has used single-scattering approximations to derive estimates of the coefficients in Biot's equations of poroelasticity *(1956a)*. These results are obtained by using exact calculations of scattering coefficients for a spherical inclusion of one type of Biot material imbedded in another *(Berryman 1985)*. Three types of approximations were considered: an average T-matrix approximation (ATA), a coherent potential approximation (CPA), and a differential effective medium (DEM) approach. All three methods produce results consistent with both the work of Gassmann and that of Brown and Korringa. They extend results obtained earlier by *Berryman (1986)* using only the coherent potential approximation.

Although the approximations generated by *Berryman (1992)* have many appealing features, any nontrivial exact result that could be obtained for the coefficients would be very valuable both for direct applications and also for testing the range of validity of various approximations. We obtain such exact results in this paper.

In Section 2, we present the equations of poroelasticity, including a discussion of differences between Gassmann's result for the coefficients and results of Brown and Korringa. Section 3 presents a simple derivation of exact results for the two parameters of Brown and Korringa when the porous material contains only two types of porous solid. Section 4 shows that an analogy between thermoelasticity and poroelasticity could have been used instead to derive one of the formulas of Section 3, and furthermore that additional exact results such as rigorous bounds on parameters may be obtained by exploiting the analogy. Section 5 examines several limiting cases of the exact results and then summarizes our conclusions.

2 Equations of poroelasticity

Consider a porous medium whose connected pore space is saturated with a single-phase viscous fluid. The fraction of total volume occupied by fluid is the porosity ϕ, which is assumed to be uniform on some appropriate length scale. Bulk modulus and density of the fluid are K_f and ρ_f, respectively. Bulk and shear moduli of the drained porous frame are K and μ. For simplicity, we at first assume the frame is composed of a single granular constituent whose bulk and shear moduli and density are K_m, μ_m, and ρ_m. Frame moduli may be measured on drained samples, or may be estimated using one of a variety of methods from the theory of composites *(Berryman 1980b,c; Berryman and Milton 1988)*.

For long wavelength disturbances ($\lambda >> h$ where h is a typical pore size) propagating through such a porous medium, we define average values of local displacements in the solid and also in the saturating fluid. The average displacement vector in the solid frame is $\mathbf{u}$, while that in the pore fluid is $\mathbf{u}_f$. A more useful way of quantifying fluid displacement is to introduce average displacement of fluid relative to frame which is $\mathbf{w} = \phi(\mathbf{u}_f - \mathbf{u})$. For small strains, frame dilatation is

$$e = \nabla \cdot \mathbf{u}. \tag{1}$$

Similarly, average fluid dilatation is

$$e_f = \nabla \cdot \mathbf{u}_f, \tag{2}$$

which includes fluid flow terms as well as dilatation. The increment of fluid content is defined by

$$\zeta = -\nabla \cdot \mathbf{w} = \phi(e - e_f). \tag{3}$$

With these definitions, *Biot (1956a; 1962)* introduces a quadratic strain-energy functional of the independent variables e and ζ for an isotropic, linear porous medium

$$2E = He^2 - 2Ce\zeta + M\zeta^2 - 4\mu I_2, \tag{4}$$

where, if $e_{xy} = \frac{1}{2}(u_{x,y} + u_{y,x}), \ldots$, then $I_2 = e_{xx}e_{yy} + e_{yy}e_{zz} + e_{zz}e_{xx} - e_{xy}^2 - e_{yz}^2 - e_{zx}^2$ is the second strain invariant *(Berryman and Thigpen 1985)*. Elementary bounds on coefficients in the equations of poroelasticity are presented by *Thigpen and Berryman (1985)*. Thermodynamic and mechanical stability require non-negativity of E, which implies that $H \geq 0$, $M \geq 0$, $HM - C^2 \geq 0$, and $\mu \geq 0$. Then, components of the average stress tensor $\mathcal{T}$ for the saturated porous medium are

$$\mathcal{T}_{ij} = [(H - 2\mu)e - C\zeta]\delta_{ij} + 2\mu e_{ij}, \tag{5}$$

and fluid pressure p_f is

$$p_f = M\zeta - Ce. \tag{6}$$

Two coupled equations of motion for small disturbances in fluid-saturated media may be derived easily from the energy functional with these definitions of stresses and pressures. With time dependence of the form $\exp(-i\omega t)$, Biot's equations of poroelasticity are, using the notation of *Biot (1962)*,

$$\mu\nabla^2\mathbf{u} + (H - \mu)\nabla e - C\nabla\zeta + \omega^2(\rho\mathbf{u} + \rho_f\mathbf{w}) = 0, \tag{7}$$

$$C\nabla e - M\nabla\zeta + \omega^2(\rho_f\mathbf{u} + q\mathbf{w}) = 0, \tag{8}$$

where

$$\rho = \phi\rho_f + (1 - \phi)\rho_m \tag{9}$$

and

$$q(\omega) = \rho_f[\tau/\phi + iF(\xi)\eta/\kappa\omega]. \tag{10}$$

Tortuosity $\tau \geq 1$ is a pure number related to frame inertia which has been measured *(Brown, 1980; Johnson et al., 1982)* and can also be estimated theoretically *(Berryman, 1980a)*. Kinematic viscosity of the saturating fluid is η; permeability of the porous frame is κ; a dynamic viscosity factor proposed by Biot (1956) is (for our choice of sign for frequency dependence)

$$F(\xi) = \frac{1}{4}\{\xi T(\xi)/[1 + 2T(\xi)/i\xi]\}, \tag{11}$$

where

$$T(\xi) = \frac{ber'(\xi) - ibei'(\xi)}{ber(\xi) - ibei(\xi)} \tag{12}$$

and

$$\xi = (\omega h^2/\eta)^{\frac{1}{2}}. \tag{13}$$

Functions $ber(\xi)$ and $bei(\xi)$ are real and imaginary parts of the Kelvin function. The dynamic parameter h is a characteristic length generally associated with steady-flow hydraulic radius of the pores, or with a typical pore size.

Coupled equations (7) and (8) give rise to three distinct modes of wave propagation: two compressional waves (fast and slow) and a single shear wave speed (with two polarizations).

Coefficients appearing in Biot's equations of poroelasticity must be known before quantitative predictions can be made. Results of *Brown and Korringa (1975)* may be used to show (see **Appendix A**) that these coefficients are given for general isotropic porous media by

$$H = K + \sigma C + \frac{4}{3}\mu, \tag{14}$$

$$C = \sigma/\Big[\frac{\sigma}{K_s} + \phi\Big(\frac{1}{K_f} - \frac{1}{K_\phi}\Big)\Big], \tag{15}$$

and

$$M = C/\sigma, \tag{16}$$

where

$$\sigma = 1 - K/K_s. \tag{17}$$

The three bulk moduli characteristic of the porous frame are defined by Brown and Korringa through expressions:

$$\frac{1}{K} = -\frac{1}{V}\Big(\frac{\partial V}{\partial p_d}\Big)_{p_f}, \tag{18}$$

$$\frac{1}{K_s} = -\frac{1}{V}\Big(\frac{\partial V}{\partial p_f}\Big)_{p_d}, \tag{19}$$

and

$$\frac{1}{K_\phi} = -\frac{1}{V_\phi}\Big(\frac{\partial V_\phi}{\partial p_f}\Big)_{p_d}, \tag{20}$$

where V is total sample volume, $V_\phi = \phi V$ is pore volume, $p = -\frac{1}{3}Tr(\mathcal{T}) = -\frac{1}{3}(\mathcal{T}_{xx} + \mathcal{T}_{yy} + \mathcal{T}_{zz})$ is external (confining) pressure, p_f is pore pressure, and $p_d = p - p_f$ is differential pressure. *Brown and Korringa (1975)* state that, although these three bulk moduli have simple physical interpretations, this "does not necessarily help in knowing their values." Observing the change in pore volume required by (20) is clearly more difficult than observing the change in total sample volume required by the other two moduli. Nevertheless, all three moduli may be measured directly, thus determining their values in specific instances. However, general formulas relating these moduli to values of the moduli of constituents of a porous composite would clearly be of great utility.

Constant K is the bulk modulus of the drained porous frame, introduced earlier. However, values of the two remaining constants K_s and K_ϕ are generally not known unless the porous frame is homogeneous on the microscopic scale. For this special circumstance [which is also the only one considered by *Gassmann (1951)*] with a single type of elastic solid composing the frame, these two moduli are both equal to the bulk modulus K_m of the single granular constituent

$$K_s = K_\phi = K_m. \tag{21}$$

Thus, Gassmann's equation is equivalent to

$$\frac{1}{M} = \frac{\phi}{K_f} + \frac{\sigma - \phi}{K_m}, \quad \sigma = 1 - K/K_m, \tag{22}$$

while Brown and Korringa's more general result is equivalent to

$$\frac{1}{M} = \frac{\phi}{K_f} + \frac{\sigma}{K_s} - \frac{\phi}{K_\phi}, \quad \sigma = 1 - K/K_s. \tag{23}$$

Gassmann's result has also been derived within the context of Biot's theory of poroelasticity by *Biot and Willis (1957)* and by *Geertsma (1957)*. *Geertsma and Smit (1962)* discuss practical aspects of applications of the theory to rocks. *Rice and Cleary (1976)* also obtain general results essentially equivalent to those of Brown and Korringa.

The more general constants of Brown and Korringa, K_s and K_ϕ, must somehow be related to material properties of multiple solid constituents of the porous frame. In the next section, we show that exact results are available when the composite porous medium is composed of only two porous constituents, each satisfying Gassmann's equation.

3 Exact results for two porous constituents

The method we will use to obtain an exact expression for K_s^* is completely analogous to methods used by *Cribb (1968)* and by *Schulgasser (1989)* to find relations between the coefficient of thermal expansion and bulk modulus of an isotropic elastic composite containing two constituents. In both problems, coefficients of linear equations are sought and, by a careful choice of the ratio of two macroscopic applied fields, the composite material undergoes a uniform expansion or contraction so global and local fields are identical. This fact leads to simple results relating the two coefficients when only two constituents are present. Similar results relating n coefficients are possible when there are n adjustable macroscopic fields of similar type and when there are at most n isotropic constituents present.

We assume that the two porous constituents fill all the space of the composite (except for the void spaces within themselves), and that they are bonded at points of contact (see Figure 1). There is no porosity in the composite that lies outside of the two porous constituents. If there were such porosity, then the composite would in fact have three distinct component types and we must exclude such possibilities for the present arguments.

There is an implicit assumption in this analysis that each inclusion of a porous constituent is of sufficient size so that it makes sense to speak of effective frame moduli $K^{(1)}$ and $K^{(2)}$ for them. Thus, there are three relevant length scales in our problem, which in order of increasing size are: size of the pores, size of the porous constituents, and macroscopic sample size for which the effective coefficients K^*, K_s^*, and K_ϕ^* must be computed. The analysis clearly cannot be applied to all two-phase porous composites; for example, a gravel containing two types of solid grains randomly mixed may not satisfy the constraints imposed on our derivation, depending on manner of mixing and (lack of) uniformity of porosity throughout the sample. On the other hand, both constituents need not be porous; our analysis applies to one type of solid grain imbedded in another type of porous matrix.

In general, change in overall volume δV as differential pressure p_d and fluid pressure p_f vary is given by

$$\frac{\delta V}{V} = \frac{1}{V}\Big(\frac{\partial V}{\partial p_d}\Big)_{p_f}\delta p_d + \frac{1}{V}\Big(\frac{\partial V}{\partial p_f}\Big)_{p_d}\delta p_f. \tag{24}$$

Similar expressions apply to volumes and changes of volume in constituents of type-1 and type-2.

If the frame and grain moduli of the two constituents are respectively $K^{(1)}, K_m^{(1)}$ and $K^{(2)}, K_m^{(2)}$, while the effective moduli for the composite porous medium are K^*, K_s^*, then there exists a ratio of the two pressure increments $\delta p_d/\delta p_f \equiv \gamma$ such that the relative change in volumes of each constituent equals the volume changes of the equivalent medium. For this choice, local and global pressure increments are identical. Thus,

$$\begin{aligned} -\frac{\delta V}{V} &= \frac{\delta p_d}{K^*} + \frac{\delta p_f}{K_s^*} \\ &= \frac{\delta p_d}{K^{(1)}} + \frac{\delta p_f}{K_m^{(1)}} \\ &= \frac{\delta p_d}{K^{(2)}} + \frac{\delta p_f}{K_m^{(2)}}, \end{aligned} \tag{25}$$

with ratio γ given by

$$\begin{aligned} \gamma &= \frac{1/K_m^{(1)} - 1/K_m^{(2)}}{1/K^{(2)} - 1/K^{(1)}} \\ &= \frac{1/K_m^{(1)} - 1/K_s^*}{1/K^* - 1/K^{(1)}} \end{aligned}$$

$$= \frac{1/K_s^* - 1/K_m^{(2)}}{1/K^{(2)} - 1/K^*}. \tag{26}$$

Relation (25) assumes a uniform swelling or shrinking of the composite so that shapes and relative positions of all porous constituents remain fixed while overall size increases or decreases. It is also possible to analyze (25) by subtracting the first line of (25) from the other two and then setting the 2×2 determinant of the coefficients of the resulting homogeneous equations to zero. That determinant is equivalent to the relations in (26), and for some purposes the analysis of the determinant may be preferred to the approach presented here.

After some algebra (see **Appendix B**), it is possible to express (26) in terms of the variable σ defined in (17) as

$$\frac{\sigma^* - \sigma^{(1)}}{\sigma^{(2)} - \sigma^{(1)}} = \frac{K^* - K^{(1)}}{K^{(2)} - K^{(1)}}. \tag{27}$$

Equations (26) and (27) are our main results. They are exact expressions relating bulk modulus K_s^*, defined by *Brown and Korringa (1975)* for general porous media, to the bulk modulus of the composite porous frame K^* and various bulk moduli of the constituents making up the composite. One important conclusion from (27) is that K_s^* is dependent on constituent volume fractions only implicitly through K^*.

Both the result obtained here and its derivation are very special. This approach cannot be generalized to three or more constituents without requiring that the third constituent obey certain compatibility conditions not needed with only two constituents.

In another sense, there really are three constituents in the problem we have considered: two types of solid constituent and included pore space. In fact, the result we have obtained precludes using the same method to find an exact result for coefficient K_ϕ^*, since it is the effective modulus of this third component. Nevertheless, we can also obtain an exact result for K_ϕ^* by making use of its definition and of the result for K_s^*.

Note that, since $V_\phi = \phi V$,

$$\frac{\delta V_\phi}{V_\phi} = \frac{\delta \phi}{\phi} + \frac{\delta V}{V}. \tag{28}$$

Brown and Korringa (1975) derive a reciprocity relation [their equation (A-6)], from which it can readily be shown that

$$-\frac{\delta V_\phi}{V} = \Big(\frac{1}{K^*} - \frac{1}{K_s^*}\Big)\delta p_d + \frac{\phi^*}{K_\phi^*}\delta p_f. \tag{29}$$

This result can also be seen to follow from the coupled equations for hydrostatic confining pressure p and fluid pressure p_f

$$\begin{pmatrix} -p \\ -p_f \end{pmatrix} = \begin{pmatrix} K + \sigma^2 M & \sigma M \\ \sigma M & M \end{pmatrix} \begin{pmatrix} e \\ -\zeta \end{pmatrix}. \tag{30}$$

Inverting, we find that

$$\begin{pmatrix} e \\ -\zeta \end{pmatrix} = \begin{pmatrix} \frac{1}{K} & -\frac{\sigma}{K} \\ -\frac{\sigma}{K} & \frac{1}{M} + \frac{\sigma^2}{K} \end{pmatrix} \begin{pmatrix} -p \\ -p_f \end{pmatrix}. \tag{31}$$

Thus, reciprocity, or equivalently the existence of the energy functional (4) [see *Biot (1941)*], implies that

$$-\Big(\frac{\partial e}{\partial p_f}\Big)_p = \Big(\frac{\partial \zeta}{\partial p}\Big)_{p_f}, \tag{32}$$

leading to the symmetric matrix in (31). Translating to differential pressure by substituting

$$\begin{pmatrix} -p \\ -p_f \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -p_d \\ -p_f \end{pmatrix} \tag{33}$$

into (31), we find

$$\begin{pmatrix} e \\ -\zeta \end{pmatrix} = \begin{pmatrix} 1/K & 1/K_s \\ -\sigma/K & \phi(1/K_f - 1/K_\phi) \end{pmatrix} \begin{pmatrix} -p_d \\ -p_f \end{pmatrix}. \tag{34}$$

Therefore, the translated matrix in (34) follows from

$$\Big(\frac{\partial e}{\partial p_f}\Big)_p + \Big(\frac{\partial e}{\partial p}\Big)_{p_f} = \Big(\frac{\partial e}{\partial p_f}\Big)_{p_d}, \tag{35}$$

$$\Big(\frac{\partial e}{\partial p}\Big)_{p_f} = \Big(\frac{\partial e}{\partial p_d}\Big)_{p_f}, \tag{36}$$

and similar expressions for variation of ζ. Taking the fluid to be incompressible so $K_f \to \infty$ in (34), then (29) follows directly from (34) using the fact that $\delta\zeta = \delta V_\phi/V$ in this limit.

Having (29), we can use (28) and (25) to find expressions for the change in the overall porosity $\delta\phi^*$ and in the porosities of the two constituents $\delta\phi^{(1)}$ and $\delta\phi^{(2)}$. The results are

$$-\delta\phi^* = \Big(\frac{\sigma^* - \phi^*}{K^*}\Big)\delta p_d + \phi^*\Big(\frac{1}{K_\phi^*} - \frac{1}{K_s^*}\Big)\delta p_f, \tag{37}$$

$$-\delta\phi^{(1)} = \Big(\frac{\sigma^{(1)} - \phi^{(1)}}{K^{(1)}}\Big)\delta p_d, \tag{38}$$

and

$$-\delta\phi^{(2)} = \Big(\frac{\sigma^{(2)} - \phi^{(2)}}{K^{(2)}}\Big)\delta p_d. \tag{39}$$

Equations (37)–(39) are general, but it is only possible to relate these results easily in the same special case that we considered before. When the composite undergoes a uniform expansion or contraction, local pressure increments are the same as global ones. Furthermore, points where constituents touch continue to remain in touch during expansion or contraction, so (by assumption) the only porosity in the composite is the porosity within the porous constituents – no new porosity is created at the contacts. In such circumstances, overall porosity is just the volume average of local porosities $\phi^* = \langle\phi(\mathbf{x})\rangle$, so (37)–(39) may be combined to show that

$$\phi^*\Big(\frac{1}{K_\phi^*} - \frac{1}{K_s^*}\Big) = \Big[\Big\langle\frac{\sigma(\mathbf{x}) - \phi(\mathbf{x})}{K(\mathbf{x})}\Big\rangle - \Big(\frac{\sigma^* - \phi^*}{K^*}\Big)\Big]\delta p_d/\delta p_f, \tag{40}$$

where the ratio $\delta p_d/\delta p_f \equiv \gamma$ with γ given again by (26). With just two constituents, the volume average $\langle\cdot\rangle$ of any material quantity q is $\langle q(\mathbf{x})\rangle = f^{(1)}q^{(1)} + f^{(2)}q^{(2)}$, where fs are volume fractions of constituents with $f^{(1)} + f^{(2)} = 1$.

Exact results (26) [or (27)] and (40) depend on knowledge of K^* as well as material and frame constants of constituents. The composite frame constant K^* may be found through measurement, or it may be estimated theoretically using the theory of composites. In one circumstance, an exact result for K^* is known. *Hill (1963)* showed that if the shear modulus is constant throughout the composite, *i.e.*, $\mu^{(1)} = \mu^{(2)} = \mu^*$, then bulk modulus K^* is given by

$$\frac{1}{K^* + \frac{4}{3}\mu^*} = \Big\langle \frac{1}{K(\mathbf{x}) + \frac{4}{3}\mu^*} \Big\rangle. \tag{41}$$

With this result for K^* or with measured values, we obtain formulas for all bulk poroelastic coefficients of Brown and Korringa and these may then be used to predict speeds of wave propagation.

Now we would like to compare these results with those found in single-scattering approximations *(Berryman, 1986; 1992)*. Using (26), it is possible to derive many expressions for the constant γ (see **Appendix B**). One convenient expression is

$$\begin{aligned} \gamma &= -1 + \sigma^* - K^*\Big(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}\Big) \\ &= -K^*\Big[\frac{1}{K_s^*} + \Big(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}\Big)\Big]. \end{aligned} \tag{42}$$

Similarly, we may use (26) to show that

$$\Big[\Big\langle \frac{\sigma(\mathbf{x}) - \phi(\mathbf{x})}{K(\mathbf{x})}\Big\rangle - \frac{\langle\sigma(\mathbf{x})\rangle - \phi^*}{K^*}\Big]\gamma = \frac{\langle\sigma(\mathbf{x})\rangle - \phi^*}{K_s^*} - \Big\langle \frac{\sigma(\mathbf{x}) - \phi(\mathbf{x})}{K_m(\mathbf{x})}\Big\rangle. \tag{43}$$

The right hand side of (40) becomes

$$\begin{aligned} \Big[\Big\langle \frac{\sigma(\mathbf{x}) - \phi(\mathbf{x})}{K(\mathbf{x})}\Big\rangle - \frac{\sigma^* - \phi^*}{K^*}\Big]\gamma &= \frac{\langle\sigma(\mathbf{x})\rangle - \phi^*}{K_s^*} - \Big\langle \frac{\sigma(\mathbf{x}) - \phi(\mathbf{x})}{K_m(\mathbf{x})}\Big\rangle + \frac{\langle\sigma(\mathbf{x})\rangle - \sigma^*}{K^*}\gamma \\ &= \frac{\sigma^* - \phi^*}{K_s^*} - \Big\langle \frac{\sigma(\mathbf{x}) - \phi(\mathbf{x})}{K_m(\mathbf{x})}\Big\rangle - (\langle\sigma(\mathbf{x})\rangle - \sigma^*)\Big(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}\Big). \end{aligned} \tag{44}$$

using (43) in the first step and (42) in the second. Substituting (44) into (40), the final result for K_ϕ^* is

$$\frac{\phi^*}{K_\phi^*} = \frac{\sigma^*}{K_s^*} - \Big\langle \frac{\sigma(\mathbf{x}) - \phi(\mathbf{x})}{K_m(\mathbf{x})}\Big\rangle - (\langle\sigma(\mathbf{x})\rangle - \sigma^*)\Big(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}\Big). \tag{45}$$

Equations (40) and (45) contain the same information, but (45) is in a form convenient for comparison with the single-scattering approximations derived previously by *Berryman (1986; 1992)*. If quantities K_ϕ^*, K_s^*, and σ^* in (45) are consistently replaced by their values in any one of the approximations considered (ATA, CPA, DEM) by *Berryman (1992)*, the formula for the corresponding approximation to K_ϕ^* is recovered. For example, Eq. (47) of *Berryman (1986)* for the coherent potential approximation (CPA) can be shown to be of the same form as (45) when only two porous constituents are present in the composite. Thus, the single-scattering approximations produce results that agree in form with exact results obtained here.

Recalling that coefficients H and M are coefficients in the quadratic form (4) for internal energy, we have thermodynamic constraints that both of these coefficients must be non-negative in all circumstances. Using (23) and considering the limit $K_f \to \infty$, $M \geq 0$ is sufficient to show that

$$\frac{\sigma^*}{K_s^*} - \frac{\phi^*}{K_\phi^*} \geq 0. \tag{46}$$

To check that (46) is satisfied, we rearrange (45) and use (27) to show that

$$\frac{\sigma^*}{K_s^*} - \frac{\phi^*}{K_\phi^*} = \Big\langle \frac{\sigma(\mathbf{x}) - \phi(\mathbf{x})}{K_m(\mathbf{x})} \Big\rangle + (\langle K(\mathbf{x})\rangle - K^*)\Big(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}\Big)^2 \geq 0. \tag{47}$$

Non-negativity of the right hand side of (47) follows from $K_m(\mathbf{x}) \geq 0$, $\sigma(\mathbf{x}) \geq \phi(\mathbf{x})$, and $\langle K(\mathbf{x})\rangle \geq K^*$ (see **Appendix C**). The non-negativity condition on K_m is another thermodynamic constraint. The other two conditions follow from the fact that the Voigt average is an upper bound on the effective bulk constant. For example, $(1 - \phi(\mathbf{x}))K_m(\mathbf{x}) \geq K(\mathbf{x})$ implies that $\sigma(\mathbf{x}) \geq \phi(\mathbf{x})$. Thus, (47) shows that the exact results found here automatically satisfy thermodynamic constraint (46).

4 Relation to results in thermoelasticity

There is an important connection between results obtained for coefficients of Biot's equations and results for coefficients in equations of thermoelasticity. It was not essential to understand this connection either to obtain or to make use of previously derived formulas. It is very helpful, however, to be aware of this connection, since a significant literature on exact results and rigorous bounds for the coefficient of thermal expansion *(Levin, 1967; Rosen and Hashin, 1970; Hashin, 1984; Schulgasser, 1987; Schulgasser, 1989)* has developed. We need not rediscover these results within the context of Biot's theory if we can show that the mathematical behavior of the two theories is identical.

Biot (1956b), *Rice and Cleary (1976)*, and *Bonnet (1987)* have shown that equations of poroelasticity and of thermoelasticity are mathematically equivalent in both quasistatic and dynamic regimes. *Bonnet (1987)* makes use of this equivalence to find exact expressions for the singular solutions of poroelasticity by taking known results for singular solutions to equations of thermoelasticity *(Kupradze et al., 1979)* and appropriately redefining parameters.

To develop this analogy, Biot's equations (7) and (8) may be rewritten as

$$\mu\nabla^2\mathbf{u} + (K + \mu/3)\nabla e - \sigma\nabla p_f + \omega^2(\rho\mathbf{u} + \rho_f\mathbf{w}) = 0, \tag{48}$$

$$\nabla^2 p_f + \frac{\omega^2 q(\omega)}{M} p_f + \omega^2[\sigma q(\omega) - \rho_f]e = 0, \tag{49}$$

where we have used (14)-(17) and the fact that the fluid pressure is given by

$$p_f = M\zeta - Ce. \tag{50}$$

To first order in ω, (49) becomes

$$\nabla^2 p_f + \frac{i\omega\eta\rho_f}{\kappa M} p_f + \frac{i\omega\eta\rho_f\sigma}{\kappa} e = 0, \tag{51}$$

since $q(\omega) \to i\eta\rho_f/\kappa\omega$ as $\omega \to 0$.

For comparison, the equations of thermoelasticity are

$$\mu\nabla^2\mathbf{u} + (K + \mu/3)\nabla e - 3K\alpha\nabla\theta + \omega^2\rho\mathbf{u} = 0, \tag{52}$$

$$\nabla^2\theta + \frac{i\omega}{D}\theta + i\omega\beta e = 0, \tag{53}$$

where θ is the increment of temperature, α is the (linear) thermal expansion coefficient, D is thermal diffusivity, and $\beta = 3KT_0/\chi$ with T_0 absolute temperature, and χ thermal conductivity.

At low frequencies, analogy between thermoelasticity and poroelasticity is completed by noting the following equivalences:

$$\theta \to -p_f,$$

$$3K\alpha \to -\sigma,$$

$$D \to \kappa M/\eta\rho_f,$$

and

$$\beta \to -\eta\rho_f\sigma/\kappa.$$

Now we can make an argument for thermal expansion coefficient α similar to the one made in the previous section for coefficient K_s^*. Suppose that we have a thermoelastic composite with two constituents. Then the relative change in the volume is given by

$$\begin{aligned} -\frac{\delta V}{V} &= \frac{\delta p}{K^*} - 3\alpha^*\delta\theta \\ &= \frac{\delta p}{K^{(1)}} - 3\alpha^{(1)}\delta\theta \\ &= \frac{\delta p}{K^{(2)}} - 3\alpha^{(2)}\delta\theta \end{aligned} \tag{54}$$

for some ratio $\delta p/\delta\theta = -\epsilon$. The value of ϵ is determined by (54) and is

$$\begin{aligned} \epsilon &= \frac{3\alpha^{(2)} - 3\alpha^{(1)}}{1/K^{(1)} - 1/K^{(2)}} \\ &= \frac{3\alpha^* - 3\alpha^{(1)}}{1/K^{(1)} - 1/K^*} \\ &= \frac{3\alpha^{(2)} - 3\alpha^*}{1/K^* - 1/K^{(2)}}. \end{aligned} \tag{55}$$

Equation (55) implies that

$$\frac{\alpha^* - \alpha^{(1)}}{\alpha^{(2)} - \alpha^{(1)}} = \frac{1/K^* - 1/K^{(1)}}{1/K^{(2)} - 1/K^{(1)}}, \tag{56}$$

which is the exact result for thermal expansion coefficient α^* first derived by *Levin (1967)*. The argument given to derive (56) is essentially the same as one given recently by *Schulgasser (1989)* and earlier by *Cribb (1968)*.

Using the mathematical analogy between thermoelasticity and poroelasticity and specifically using the equivalence relation $3K\alpha \to -\sigma$, we can substitute $-\sigma/3K$ for α everywhere in (56) and then rearrange the resulting formula to find

$$\frac{\sigma^* - \sigma^{(1)}}{\sigma^{(2)} - \sigma^{(1)}} = \frac{K^* - K^{(1)}}{K^{(2)} - K^{(1)}}, \tag{57}$$

which is identical to (27). This analogy can be used to obtain other results for poroelasticity as well. For example, any rigorous bounds available for the thermal expansion coefficient may now be used by analogy to obtain rigorous bounds on coefficients in generalized Gassmann's equations. Such bounds are not restricted to two component composite porous media, and are therefore beyond our present scope. On the other hand, the analogy to thermoelasticity does not help directly with derivation of (40) or (45), since porosity appears in (48) and (49) only through internal structure of the fluid pressure (50). Nothing comparable appears in equations for thermoelasticity.

5 Discussion and conclusions

Several limiting cases of exact formulas (26), (27), (40), and (45) may now be considered. We will discuss five limits and then consider a generalization.

First, to check that exact results reduce correctly to Gassmann's equation, consider the situation with $K_m^{(1)} = K_m^{(2)} \equiv K_m$. Then, (26) shows that $\gamma = 0$, from which it follows immediately that $K_s^* = K_\phi^* = K_m$, using both (26) and (40). Thus, Gassmann's result is recovered.

Second, consider a porous material composed of a single type of solid grain, but with occasional vugs or large pores. The vugs may be treated as a second porous constituent with $\phi^{(2)} = \sigma^{(2)} = 1$ and $K^{(2)} = 0$. Then, $\phi^* = f^{(1)}\phi^{(1)} + f^{(2)}$ and it is not difficult to show that (27) and (40) reduce to $K_s^* = K_\phi^* = K_m^{(1)}$, also in agreement with Gassmann. This result is expected since there is only one type of solid grain and the formulas correctly reflect the flexibility we have in interpreting vugs either as separate entities or as part of the pore space of the type-1 constituent.

Third, consider the physically interesting situation depicted in Fig. 2, where the porous composite is constructed from a uniform porous matrix of one material by imbedding solid grains of another material. This situation is representative of a clayey sandstone [see for example *Han, Nur, and Morgan (1986)*]. If we take the matrix to be type-1 and the imbedded grains as type-2, then we find that $\phi^{(2)} = \sigma^{(2)} = 0$ and $K^{(2)} = K_m^{(2)}$. The result of (27) in this situation becomes

$$\sigma^* = \sigma^{(1)} \frac{K^* - K_m^{(2)}}{K^{(1)} - K_m^{(2)}}, \tag{58}$$

while (40) or (45) become

$$\frac{\phi^*}{K_\phi^*} = \frac{\sigma^*}{K_s^*} - f^{(1)} \frac{\sigma^{(1)} - \phi^{(1)}}{K_m^{(1)}} - (f^{(1)}\sigma^{(1)} - \sigma^*)\Big(\frac{\sigma^{(1)}}{K^{(1)} - K_m^{(2)}}\Big). \tag{59}$$

We expect that (58) and (59) will find considerable use in the analysis of data on natural materials.

Next, we consider a somewhat artificial situation that is easy to analyze. Suppose that the composite porous material contains two porous materials that are identical in every way except that their material bulk moduli differ. Thus, $\phi^{(1)} = \phi^{(2)} = \phi^*$ and $K^{(1)}/K_m^{(1)} = K^{(2)}/K_m^{(2)} = 1 - \sigma^{(1)} = 1 - \sigma^{(2)} = 1 - \sigma^*$, where we have made use of (27). Then, it follows from (26) that

$$\gamma = \sigma^* - 1 \tag{60}$$

and from (40) that

$$\phi^*\Big(\frac{1}{K_\phi^*} - \frac{1}{K_s^*}\Big) = -(1 - \sigma^*)(\sigma^* - \phi^*)\Big(\Big\langle\frac{1}{K(\mathbf{x})}\Big\rangle - \frac{1}{K^*}\Big) \leq 0, \tag{61}$$

using the facts that $\langle 1/K(\mathbf{x})\rangle \geq 1/K^*$ and that $1 \geq \sigma^*$ (see **Appendix C**). Thus, we find for such a material that constants for bulk pore and solid either have the relationship

$$K_s^* \leq K_\phi^*, \tag{62}$$

or

$$K_\phi^* \leq 0. \tag{63}$$

The behavior in (62) was observed in numerical examples by *Berryman (1986)* for calculations based on the coherent potential approximation, and is expected to be the most common situation. The other possibility (63), although it seems odd, does not violate any thermodynamic constraints. In fact, we have shown in (47) that relevant thermodynamic constraints are guaranteed to be satisfied. If the medium has the property that an increase in fluid pressure causes an increase in the pore volume, then $K_\phi^* < 0$. Formula (61) shows that this happens only when the moduli differ substantially in magnitude $K^{(1)} >> K^{(2)}$ so that it is possible to have $\langle 1/K(\mathbf{x})\rangle^{-1} << K^*$ for some intermediate values of $0 \leq f^{(1)}, f^{(2)} \leq 1$. *Berryman (1992)* presents numerical examples showing that $K_\phi^* \geq K_s^*$ for $f^{(1)} \simeq 0$ or $f^{(1)} \simeq 1$, while $K_\phi^* < 0$ for $f^{(1)} \simeq 0.5$ when $K^{(1)}/K^{(2)} \simeq 100$.

Finally, we consider generalizing to materials with more than two constituents. For example, suppose that four constituents (a, b, c, d) are arranged into two types of regions, each containing only two constituents (say, type-1 contains a, b and type-2 contains c, d). Then, (27) may be generalized directly as

$$\frac{\sigma^* - \sigma^{(1)}}{\sigma^{(2)} - \sigma^{(1)}} = \frac{K^* - K^{(1)}}{K^{(2)} - K^{(1)}}, \tag{64}$$

where

$$\frac{\sigma^{(1)} - \sigma^{(a)}}{\sigma^{(b)} - \sigma^{(a)}} = \frac{K^{(1)} - K^{(a)}}{K^{(b)} - K^{(a)}} \tag{65}$$

and

$$\frac{\sigma^{(2)} - \sigma^{(c)}}{\sigma^{(d)} - \sigma^{(c)}} = \frac{K^{(2)} - K^{(c)}}{K^{(d)} - K^{(c)}}. \tag{66}$$

Generalization for K_ϕ^* is not quite so straightforward. Porosity changes for constituents a, b, c, d all continue to satisfy equations of the form (38)-(39), but porosity changes for composite regions of type-1 and type-2 satisfy equations of the form (37). The overall composite also satisfies an equation of the form (37) so (40) must be generalized to

$$\phi^*\Big(\frac{1}{K_\phi^*} - \frac{1}{K_s^*}\Big) = \Big\langle \phi(\mathbf{x})\Big(\frac{1}{K_\phi(\mathbf{x})} - \frac{1}{K_s(\mathbf{x})}\Big)\Big\rangle_{1,2} + \Big[\Big\langle \frac{\sigma(\mathbf{x}) - \phi(\mathbf{x})}{K(\mathbf{x})}\Big\rangle_{1,2} - \Big(\frac{\sigma^* - \phi^*}{K^*}\Big)\Big]\gamma, \tag{67}$$

where $\langle \cdot \rangle_{1,2}$ indicates a volume average taken over composite regions of type-1 and type-2 and γ is still given by (26) with appropriate reinterpretation of constants in the formulas. This procedure is *not* a completely general way of treating three or more constituents because the assumption that the medium can be decomposed into only two types of mixed regions is very restrictive.

The method we used to obtain our exact expression for K_s^* has been shown to be analogous to methods used by *Cribb (1968)* and by *Schulgasser (1989)* to find relations between the coefficient of thermal expansion and bulk modulus of an isotropic elastic composite containing two constituents. The method works when coefficients of linear equations are sought and, by a careful choice of ratios of macroscopic applied fields, the composite material undergoes a uniform expansion or contraction so both global and local fields are the same. This fact led to our simple results relating two coefficients when only two constituents are present. Similar results are also possible relating n coefficients when exactly n constituents are present, as has been shown by *Schulgasser (1989)*. For example, if the thermal expansion coefficients of three types of single grain porous materials are known, then we can obtain exact expressions for the effective value of K_s^* for a composite porous medium containing these three components. To do so, we generalize expression (25) for change in volume to

$$\begin{aligned} -\frac{\delta V}{V} &= \frac{\delta p_d}{K^*} - 3\alpha^*\delta\theta + \frac{\delta p_f}{K_s^*} \\ &= \frac{\delta p_d}{K^{(1)}} - 3\alpha^{(1)}\delta\theta + \frac{\delta p_f}{K_m^{(1)}} \\ &= \frac{\delta p_d}{K^{(2)}} - 3\alpha^{(2)}\delta\theta + \frac{\delta p_f}{K_m^{(2)}} \\ &= \frac{\delta p_d}{K^{(3)}} - 3\alpha^{(3)}\delta\theta + \frac{\delta p_f}{K_m^{(3)}}. \end{aligned} \tag{68}$$

Then, (68) can be solved for the two ratios $\tilde{\gamma} = \delta p_d/\delta p_f$ and $\tilde{\tau} = \delta\theta/\delta p_f$. We can find $\tilde{\tau}$ as a function of $\tilde{\gamma}$. Then, eliminating $\tilde{\tau}$, we find a set of equations similar to (26) and that reduce to it when $\tilde{\tau} = 0$. In this way, exact results may be generated for multiple constituent porous composites when other physical properties of the component materials are known.

To summarize our conclusions, the main results of this paper are contained in (26) [or (27)] and (40). These formulas for K_s^* and K_ϕ^* are exact for composites composed of two porous constituents regardless of inclusion shapes as long as the two constituents are well bonded at points of contact and there is no porosity lying outside the porous constituents but inside the composite. Thus, with a perfect fit between two constituents, it is possible to find a ratio of differential pressure and fluid pressure that maintains that perfect fit as the composite undergoes uniform expansion or contraction. That ratio is given by (26), which also determines K_s^* in terms of K^*. Once we know the pressure ratio that maintains shapes of external bounding surfaces of the porous inclusions, we can use that ratio to compute changes of internal porosity that occur simultaneously. The resulting computation yields (40) or equivalently (45) for K_ϕ^*. Numerical examples and comparisons to the single-scattering approximations appear in *Berryman (1992)*. Discussion of extensions of this approach to anisotropic porous media and other exact results will appear later, as will comparisons with experimental data.

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A Relating results of Brown and Korringa to Biot coefficients

Brown and Korringa (1975) showed that (in our notation)

$$\Big[\frac{1}{K_{sat}} - \frac{1}{K_s}\Big]^{-1} = \Big[\frac{1}{K} - \frac{1}{K_s}\Big]^{-1} + \Big[\phi\Big(\frac{1}{K_f} - \frac{1}{K_\phi}\Big)\Big]^{-1}, \tag{69}$$

where $K_{sat} = H - \frac{4}{3}\mu$ is the effective bulk modulus of the (jacketed) fluid-saturated porous medium. We can rearrange (69) to show that

$$\begin{aligned}\phi\Big(\frac{1}{K_f} - \frac{1}{K_\phi}\Big) = \frac{(1/K - 1/K_s)(1/K_{sat} - 1/K_s)}{1/K - 1/K_{sat}} = \frac{\sigma(1 - K_{sat}/K_s)}{K_{sat} - K} = \\ \frac{\sigma[1 - K/K_s + (K - K_{sat})/K_s]}{K_{sat} - K} = \frac{\sigma^2}{K_{sat} - K} - \frac{\sigma}{K_s},\end{aligned} \tag{70}$$

where $\sigma \equiv 1 - K/K_s$. Then, it follows easily that

$$\frac{\sigma}{K_s} + \phi\Big(\frac{1}{K_f} - \frac{1}{K_\phi}\Big) = \frac{\sigma^2}{K_{sat} - K}. \tag{71}$$

Cross-multiplication gives the final result

$$K_{sat} - K = \frac{\sigma^2}{\sigma/K_s + \phi(1/K_f - 1/K_\phi)}, \tag{72}$$

which may be compared directly with (14). Gassmann's result follows by replacing effective moduli K_s and K_ϕ everywhere by single grain modulus K_m.

B Equivalent expressions for γ

Many equivalent expressions for constant γ may be derived from relations in (26). Those used in the text will be derived here.

The last two identities in (26) may be rewritten as

$$\begin{aligned}\gamma &= K^*K^{(1)}\Big(\frac{1/K_m^{(1)} - 1/K_s^*}{K^{(1)} - K^*}\Big) \\ &= K^*K^{(2)}\Big(\frac{1/K_m^{(2)} - 1/K_s^*}{K^{(2)} - K^*}\Big).\end{aligned} \tag{73}$$

Multiplying through by the denominators, we find that

$$\gamma(K^{(1)} - K^*) = K^*\Big(1 - \sigma^{(1)} - \frac{K^{(1)}}{K_s^*}\Big) \tag{74}$$

and

$$\gamma(K^{(2)} - K^*) = K^*\Big(1 - \sigma^{(2)} - \frac{K^{(2)}}{K_s^*}\Big). \tag{75}$$

Subtracting (75) from (74) and dividing through by the difference $K^{(1)} - K^{(2)}$ gives

$$\begin{aligned}\gamma &= -K^*\Big[\frac{1}{K_s^*} + \Big(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}\Big)\Big] \\ &= -1 + \sigma^* - K^*\Big(\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}\Big),\end{aligned} \tag{76}$$

which is the result quoted in (42).

Using the remaining equality in (26), we have

$$\begin{aligned}\gamma = K^{(1)}K^{(2)}\frac{1/K_m^{(1)} - 1/K_m^{(2)}}{K^{(1)} - K^{(2)}} &= \frac{K^{(2)}(1-\sigma^{(1)}) - K^{(1)}(1-\sigma^{(2)})}{K^{(1)} - K^{(2)}} \\ &= -1 + \frac{K^{(1)}\sigma^{(2)} - K^{(2)}\sigma^{(1)}}{K^{(1)} - K^{(2)}} \\ &= -1 + \sigma^{(1)} - K^{(1)}\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}} \\ &= -1 + \sigma^{(2)} - K^{(2)}\frac{\sigma^{(1)} - \sigma^{(2)}}{K^{(1)} - K^{(2)}}.\end{aligned} \tag{77}$$

Combining these relations with (76) gives our main result (27).

Similarly, if we multiply through by the denominators of the expressions in (26), we find that

$$\gamma\Big(\frac{1}{K^*} - \frac{1}{K^{(1)}}\Big) = \Big(\frac{1}{K_m^{(1)}} - \frac{1}{K_s^*}\Big) \tag{78}$$

and

$$\gamma\Big(\frac{1}{K^*} - \frac{1}{K^{(2)}}\Big) = \Big(\frac{1}{K_m^{(2)}} - \frac{1}{K_s^*}\Big). \tag{79}$$

Multiplying (78) by $f^{(1)}(\sigma^{(1)} - \phi^{(1)})$ and (79) by $f^{(2)}(\sigma^{(2)} - \phi^{(2)})$ and then adding the two results gives

$$\gamma\Big[\frac{\langle\sigma - \phi\rangle}{K^*} - \Big\langle\frac{\sigma - \phi}{K_m}\Big\rangle\Big] = \Big\langle\frac{\sigma - \phi}{K_m}\Big\rangle - \frac{\langle\sigma - \phi\rangle}{K_s^*}, \tag{80}$$

which is the formula quoted in (43) since $\phi^* = \langle\phi\rangle$.

C Some elementary bounds on coefficients

If a porous material is composed of only one solid elastic constituent, then the well-known Voigt bound (Voigt, 1928; Hill, 1963) shows that

$$K^{(i)} \le (1 - \phi^{(i)})K_m^{(i)} \quad \text{for} \quad i = 1, 2. \tag{81}$$

Equation (81) may be rearranged to show that

$$0 \le \phi^{(i)} \le \sigma^{(i)} = 1 - K^{(i)}/K_m^{(i)} \le 1. \tag{82}$$

The lower bound follows from (81) while the upper bound follows from the thermodynamic relation guaranteeing that $K^{(i)} \geq 0$. Then it follows directly from (82), by taking its average, that

$$\phi^* = \langle \phi(\mathbf{x}) \rangle \leq \langle \sigma(\mathbf{x}) \rangle \equiv f^{(1)}\sigma^{(1)} + f^{(2)}\sigma^{(2)} \leq 1. \tag{83}$$

The effective constant σ^* is not necessarily equal to the average $\langle \sigma(\mathbf{x}) \rangle$ for the composite porous medium. Nevertheless, by definition,

$$\sigma^* \equiv 1 - K^*/K_s^* \leq 1, \tag{84}$$

(assuming that $K_s^* \geq 0$) since $K^* \geq 0$ is also required for thermodynamic stability. Positivity of K_s^* for the two component problem follows from the result (27) using the additional fact that $K^{(1)} \leq K^* \leq K^{(2)}$ if $K^{(1)} \leq K^{(2)}$, together with (82).

List of Symbols		
e	=	frame dilatation
e_f	=	average fluid dilatation
$f^{(1)}, f^{(2)}$	=	volume fractions of constituents
h	=	a typical pore size
p	=	confining pressure
p_d	=	$p - p_f$, the differential pressure
p_f	=	fluid pressure
$\mathbf{u}$	=	solid displacement vector
$\mathbf{u}_f$	=	fluid displacement vector
$\mathbf{w}$	=	relative fluid displacement
C	=	Biot cross coefficient
D	=	thermal diffusivity
H	=	$K + \sigma C + \frac{4}{3}\mu$
K	=	bulk modulus of drained porous frame
K_f	=	fluid bulk modulus
K_m	=	material (or grain) bulk modulus
K_s	=	an effective solid bulk modulus
K_ϕ	=	an effective pore bulk modulus
K_{sat}	=	$H - \frac{4}{3}\mu$, the saturated bulk modulus
$K^{(1)}, K^{(2)}$	=	drained frame moduli of porous constituents
$K_m^{(1)}, K_m^{(2)}$	=	material moduli of solid constituents
M	=	C/σ
V	=	total volume
V_s	=	$(1-\phi)V$, the solid volume
V_ϕ	=	ϕV, the pore volume
α	=	thermal expansion coefficient
γ	=	$\delta p_d/\delta p_f$, ratio of pressure increments
ζ	=	increment of fluid content
η	=	kinematic viscosity
θ	=	increment of temperature
κ	=	fluid permeability
λ	=	wavelength
μ	=	shear modulus
ρ	=	density
ρ_f	=	density of a fluid
ρ_m	=	density of a solid material
σ	=	$1 - K/K_s$
τ	=	electrical tortuosity
ϕ	=	porosity
χ	=	thermal conductivity
ω	=	angular frequency
$\mathcal{T}$	=	average stress tensor

Figure captions

Figure 1: Two types of porous constituents completely fill the volume of composite porous material.

Figure 2: This composite porous material contains one granular material imbedded in a matrix of one type of porous material.