ADJOINT-STATE METHOD FOR LEAST-SQUARES MIGRATION IN THE SHOT-PROFILE DOMAIN

Upward-continuation modeling in the shot-profile domain

Starting from the cross-correlation imaging condition in the shot-profile migration

$$r(\mathbf{x},\mathbf{h}) = \sum_{\mathbf{x}_s}\sum_{\omega} p^*(\mathbf{x}-\mathbf{h},\mathbf{x}_s,\omega)u(\mathbf{x}+\mathbf{h},\mathbf{x}_s,\omega), \tag{1}$$

where $p(\mathbf{x},\mathbf{x}_s,\omega)$ is the source wavefield for a single frequency ω at image point $\mathbf{x} = (x, y, z)$ with the source located at $\mathbf{x}_s = (x_s, y_s, 0)$; $u(\mathbf{x},\mathbf{x}_s,\omega)$ is the receiver wavefield and $\mathbf{h} = (h_x, h_y, h_z)$ is the subsurface half-offset, and '*' stands for the complex-conjugate. We can compute the adjoint of equation 1 as:

$$p(\mathbf{x},\mathbf{x}_s,\omega) = \sum_{\mathbf{h}} u(\mathbf{x}+\mathbf{h},\mathbf{x}_s,\omega)r(\mathbf{x},\mathbf{h}) \tag{2a}$$

$$u(\mathbf{x},\mathbf{x}_s,\omega) = \sum_{\mathbf{h}} p(\mathbf{x}-\mathbf{h},\mathbf{x}_s,\omega)r(\mathbf{x},\mathbf{h}). \tag{2b}$$

In a more compact notation, not explicitly writing the dependencies on x, y and $\mathbf{h}$, and for a given depth level, z, equation 2b can be written as:

$$u^z(\omega) = \mathbf{P}^z(\omega)r^z, \tag{3}$$

where $\mathbf{P}^z$ is a convolutional matrix. If $r = r(\mathbf{x}, h_x)$, for the case of 5 x positions and 3 subsurface offsets and subsurface-offset interval equals to the CMP interval, equation 3 can be explicitly written as:

$$\begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} 0 & p_1 & 0 & p_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_2 & 0 & p_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p_1 & 0 & p_3 & 0 & p_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_2 & 0 & p_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 & 0 & p_5 & 0 \end{bmatrix} \begin{bmatrix} r_{1,1} \\ r_{1,2} \\ r_{1,3} \\ r_{2,1} \\ r_{2,2} \\ r_{2,3} \\ r_{3,1} \\ r_{3,2} \\ r_{3,3} \\ r_{4,1} \\ r_{4,2} \\ r_{4,3} \\ r_{5,1} \\ r_{5,2} \\ r_{5,3} \end{bmatrix} \tag{4}$$

where the subscripts of u and p represent x positions. The first and second subscripts of r represent x positions and subsurface offset, respectively.

For subsequent depth levels, $p(\omega)$ is computed by means of the recursive downward propagation

$$\begin{cases} p^{z+1}(\omega) = T^z_\downarrow(\omega, s)p^z \\ p^1(\omega) = f_s(\omega) \end{cases} \tag{5}$$

where $T^z_\downarrow$ is the downward continuation operator, which is function of the slowness, s, and $f_s(\omega)$ is the source signature located at x_s, which explicit dependency has not been showed.

The recursive upward propagation of $u^z(\omega)$ and recording at the surface are given by the following equations

$$\begin{cases} u^{z-1}(\omega) = P^z(\omega)r^z + T^z_\uparrow(\omega, s)u^z \\ d(\omega) = u^1(\omega) \end{cases} \tag{6}$$

where $T^z_\uparrow$ is the upward continuation operator, which is function of the slowness, s.

In a matrix form, the terms in the above equations can be writen as

$$\mathbf{u}(\omega) = \begin{bmatrix} u^1(\omega) \\ u^2(\omega) \\ \vdots \\ u^{n_z-1}(\omega) \\ u^{n_z}(\omega) \end{bmatrix}; \mathbf{T}_\uparrow(\omega) = \begin{bmatrix} 0 & T^2_\uparrow(\omega) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & T^{n_z-1}_\uparrow(\omega) & 0 \\ 0 & 0 & \dots & 0 & T^{n_z-1}_\uparrow(\omega) \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix};$$

$$\mathbf{r} = \begin{bmatrix} r^1 \\ r^2 \\ \vdots \\ r^{n_z-1} \\ r^{n_z} \end{bmatrix}; \mathbf{P}(\omega) = \begin{bmatrix} P^1(\omega) & 0 & \dots & 0 & 0 \\ 0 & P^2(\omega) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P^{n_z-1}(\omega) & 0 \\ 0 & 0 & \dots & 0 & P^{n_z}(\omega) \end{bmatrix}.$$

In a furthermore compact notation, we can write

$$\underline{\mathbf{u}} = \begin{bmatrix} \mathbf{u}(\omega_1) \\ \mathbf{u}(\omega_2) \\ \vdots \\ \mathbf{u}(\omega_{n_\omega-1}) \\ \mathbf{u}(\omega_{n_\omega}) \end{bmatrix}; \mathbf{T}_\uparrow = \begin{bmatrix} \mathbf{T}_\uparrow(\omega_1) & 0 & \dots & 0 & 0 \\ 0 & \mathbf{T}_\uparrow(\omega_2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbf{T}_\uparrow(\omega_{n_\omega-1}) & 0 \\ 0 & 0 & \dots & 0 & \mathbf{T}_\uparrow(\omega_{n_\omega}) \end{bmatrix};$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}(\omega_1) & 0 & \dots & 0 & 0 \\ 0 & \mathbf{P}(\omega_2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbf{P}(\omega_{n_\omega-1}) & 0 \\ 0 & 0 & \dots & 0 & \mathbf{P}(\omega_{n_\omega}) \end{bmatrix};$$

The reflectivity is replicated for every frequency by the spreading matrix, $\mathbf{S}$,

$$\mathbf{S} = \begin{bmatrix} \mathbf{I}(n_x n_z) \\ \mathbf{I}(n_x n_z) \\ \vdots \\ \mathbf{I}(n_x n_z) \\ \mathbf{I}(n_x n_z) \end{bmatrix}, \tag{7}$$

composed of n_ω block-identity matrices of $n_x n_z$-dimension. Finally the restriction matrix , $\mathbf{E}$,

$$\mathbf{E} = \begin{bmatrix} \mathbf{e} & 0 & \dots & 0 & 0 \\ 0 & \mathbf{e} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbf{e} & 0 \\ 0 & 0 & \dots & 0 & \mathbf{e} \end{bmatrix};$$

where

$$\mathbf{e} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix},$$

selects the wavefield $\underline{\mathbf{u}}$ at $z = 0$. The resultant matrix representation for equation 6 is

$$\begin{aligned} \underline{\mathbf{u}} &= \mathbf{PSr} + \mathbf{T}_{\uparrow}\underline{\mathbf{u}} \\ \mathbf{d} &= \mathbf{E}\underline{\mathbf{u}} \end{aligned} \tag{8}$$

Weighted least-squares data fitting

The optimum reflectivity, $\mathbf{r}$, minimizes the objective function

$$J(\mathbf{r}) = h(\underline{\mathbf{u}}(\mathbf{r}), \mathbf{r}) = \frac{1}{2} \left|\left|\mathbf{W}\left(\mathbf{d} - \mathbf{d_{obs}}\right)\right)\right|\right|^2. \tag{9}$$

The forward equation reads:

$$F(\underline{\mathbf{u}}(\mathbf{r}), \mathbf{r}) = \left(\mathbf{I} - \mathbf{T}_{\uparrow}\right)\underline{\mathbf{u}} - \mathbf{PSr} = 0. \tag{10}$$

- Build the augmented functional

$$\begin{aligned} \mathcal{L}(\underline{\lambda}, \underline{\mathbf{u}}, \mathbf{r}) &= h(\underline{\mathbf{u}}, \mathbf{r}) - \langle \underline{\lambda}, F(\underline{\mathbf{u}}, \mathbf{r}) \rangle \\ \mathcal{L}(\underline{\lambda}, \underline{\mathbf{u}}, \mathbf{r}) &= \tfrac{1}{2} \left|\left|\mathbf{W}\left(\mathbf{d} - \mathbf{d_{obs}}\right)\right)\right|\right|^2 - \langle \underline{\lambda}, \left(\mathbf{I} - \mathbf{T}_{\uparrow}\right)\underline{\mathbf{u}} - \mathbf{PSr} \rangle. \end{aligned} \tag{11}$$

- Define the adjoint state equations

$$\frac{\partial \mathcal{L}(\underline{\lambda}, \underline{\mathbf{u}}, \mathbf{r})}{\partial \underline{\mathbf{u}}} = \mathbf{E}^T \mathbf{W}^2 (\mathbf{d} - \mathbf{d_{obs}}) - \left(\mathbf{I} - \mathbf{T}_{\uparrow}\right)^* \lambda = 0. \tag{12}$$

The adjoint state equations define the scalar recursive downward-continuation equations

$$\lambda^{z+1}(\omega) = T^z_{\downarrow}(\omega)\lambda^z(\omega) \tag{13}$$

with the initial conditions

$$\lambda^1(\omega) = w(\omega)\left(d(\omega) - d_{obs}(\omega)\right). \tag{14}$$

In matrix notation, the adjoint state equations are:

$$\underline{\lambda} = \mathbf{T}_{\downarrow}\underline{\lambda} + \mathbf{E}^T\mathbf{W}^2(\mathbf{d} - \mathbf{d_{obs}}). \tag{15}$$

- Compute the gradient with respect to $\mathbf{r}$

$$\frac{\partial\mathcal{L}(\underline{\lambda}, \underline{\mathbf{u}}, \mathbf{r})}{\partial\mathbf{r}} = \frac{\partial\langle\underline{\lambda}, \mathbf{PSr}\rangle}{\partial\mathbf{r}} = \frac{\partial\langle(\mathbf{PS})^*\,\underline{\lambda}, \mathbf{r}\rangle}{\partial\mathbf{r}} = \mathbf{S}^*\mathbf{P}^*\underline{\lambda}. \tag{16}$$

that corresponds to the downward propagation of the residuals and cross-correlation imaging condition of the downward propagated residuals with the source wavefield.