

## APPENDIX A

### RIEMANNIAN GEOMETRY OVERVIEW

Geometry in a generalized 3D Riemannian space is described by a symmetric metric tensor,  $g_{jk} = g_{kj}$ , that relates the geometry in a general non-orthogonal coordinate system,  $\boldsymbol{\xi} = [\xi_1, \xi_2, \xi_3]$ , to an underlying Cartesian mesh,  $\boldsymbol{x} = [x_1, x_2, x_3]$  (?). In matrix form, the metric tensor is written,

$$[g_{jk}] = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \quad (\text{A-1})$$

where  $g_{11}$ ,  $g_{12}$ ,  $g_{22}$ ,  $g_{13}$ ,  $g_{23}$ , and  $g_{33}$  are functions linking the two coordinate systems through,

$$\begin{aligned} g_{11} &= \frac{\partial x_k}{\partial \xi_1} \frac{\partial x_k}{\partial \xi_1}, & g_{12} &= \frac{\partial x_k}{\partial \xi_1} \frac{\partial x_k}{\partial \xi_2}, & g_{22} &= \frac{\partial x_k}{\partial \xi_2} \frac{\partial x_k}{\partial \xi_2}, \\ g_{13} &= \frac{\partial x_k}{\partial \xi_1} \frac{\partial x_k}{\partial \xi_3}, & g_{23} &= \frac{\partial x_k}{\partial \xi_2} \frac{\partial x_k}{\partial \xi_3}, & g_{33} &= \frac{\partial x_k}{\partial \xi_3} \frac{\partial x_k}{\partial \xi_3}. \end{aligned} \quad (\text{A-2})$$

The associated (or inverse) metric tensor,  $g^{jk}$ , is given by,

$$[g^{jk}] = \frac{1}{|\mathbf{g}|} \begin{bmatrix} g_{22}g_{33} - g_{23}^2 & g_{13}g_{23} - g_{12}g_{33} & g_{12}g_{23} - g_{13}g_{22} \\ g_{13}g_{23} - g_{12}g_{33} & g_{11}g_{33} - g_{13}^2 & g_{12}g_{13} - g_{11}g_{23} \\ g_{12}g_{23} - g_{13}g_{22} & g_{12}g_{13} - g_{11}g_{23} & g_{11}g_{22} - g_{12}^2 \end{bmatrix}, \quad (\text{A-3})$$

and has the following metric discriminant,  $|\mathbf{g}|$ ,

$$|\mathbf{g}| = g_{11}g_{22}g_{33} - g_{12}^2g_{33} - g_{23}^2g_{11} - g_{13}^2g_{22} + 2g_{12}g_{13}g_{23}. \quad (\text{A-4})$$

A weighted metric tensor,  $m^{jk} = \sqrt{|\mathbf{g}|} g^{jk}$ , is also used throughout the thesis.

## APPENDIX B

### GENERALIZED PHASE-SHIFT OPERATORS

This appendix specifies a phase-shift approximation to the extrapolation wavenumber used to propagate wavefields described in equation ???. The relationship between the extrapolation wavenumber  $k_{\xi_3}$ , the two other wavenumbers  $k_{\xi_1}$  and  $k_{\xi_2}$ , and the geometry variables is given by

$$k_{\xi_3} = -a_1 k_{\xi_1} - a_2 k_{\xi_2} + ia_3 \pm [a_4^2 \omega^2 - a_5^2 k_{\xi_1}^2 - a_6^2 k_{\xi_2}^2 - a_7 k_{\xi_1} k_{\xi_2} + ia_8 k_{\xi_1} + ia_9 k_{\xi_2} - a_{10}^2]^{\frac{1}{2}}, \quad (\text{B-1})$$

where the non-stationary coefficients,  $a_j$  in equation B-1, are presented in vector  $\mathbf{a}$ ,

$$\mathbf{a} = \begin{bmatrix} \frac{m^{13}}{m^{33}} & \frac{m^{23}}{m^{33}} & \frac{n^3}{2m^{33}} & \frac{\sqrt{|\mathbf{g}|^s}}{\sqrt{m^{33}}} & \sqrt{\frac{m^{11}}{m^{33}} - \left(\frac{m^{13}}{m^{33}}\right)^2} & \sqrt{\frac{m^{22}}{m^{33}} - \left(\frac{m^{23}}{m^{33}}\right)^2} & \dots \\ \dots & \left[ \frac{2m^{12}}{m^{33}} - \frac{2m^{13}m^{23}}{(m^{33})^2} \right] & \left[ \frac{n^1}{m^{33}} - \frac{m^{13}n^3}{(m^{33})^2} \right] & \left[ \frac{n^2}{m^{33}} - \frac{m^{23}n^3}{(m^{33})^2} \right] & \frac{n^3}{2m^{33}} \end{bmatrix}^{\mathbf{T}}. \quad (\text{B-2})$$

The extrapolation wavenumber defined in equations B-1 and B-2 generally cannot be implemented exactly in the Fourier domain due to a simultaneous spatial dependence (i.e. a function of both  $\boldsymbol{\xi}$  and  $\mathbf{k}_{\boldsymbol{\xi}}$ ). This can be addressed using a multi-coefficient version of the split-step Fourier approximation (?) that uses Taylor expansions to separate  $k_{\xi_3}$  into two parts:  $k_{\xi_3} \approx k_{\xi_3}^{PS} + k_{\xi_3}^{SSF}$ . Wavenumbers  $k_{\xi_3}^{PS}$  and  $k_{\xi_3}^{SSF}$  represent a pure Fourier ( $\omega - \mathbf{k}_{\boldsymbol{\xi}}$ ) domain phase-shift and a mixed ( $\omega - \boldsymbol{\xi}$ ) domain split-step correction, respectively.

The phase-shift term is given by,

$$k_{\xi_3}^{PS} = -b_1 k_{\xi_1} - b_2 k_{\xi_2} + ib_3 \pm [b_4^2 \omega^2 - b_5^2 k_{\xi_1}^2 - b_6^2 k_{\xi_2}^2 - b_7 k_{\xi_1} k_{\xi_2} + ib_8 k_{\xi_1} + ib_9 k_{\xi_2} - b_{10}^2]^{\frac{1}{2}}, \quad (\text{B-3})$$

where  $b_j = b_j(\xi_3)$  are reference values of  $a_j = a_j(\xi_1, \xi_2, \xi_3)$ . The split-step approximation is developed by performing a Taylor expansion about each coefficient  $a_j$  and evaluating the results at stationary reference values  $b_j$ . Assuming that the stationary values of  $k_{\xi_1}$  and  $k_{\xi_2}$  are zero, the split-step correction is as follows,

$$k_{\xi_3}^{SSF} = \left. \frac{\partial k_{\xi_3}}{\partial a_3} \right|_0 (a_3 - b_3) + \left. \frac{\partial k_{\xi_3}}{\partial a_4} \right|_0 (a_4 - b_4) + \left. \frac{\partial k_{\xi_3}}{\partial a_{10}} \right|_0 (a_{10} - b_{10}), \quad (\text{B-4})$$

where “0” denotes “with respect to a reference medium”. The partial differential expressions in equation B-4 are,

$$\left. \frac{\partial k_{\xi_3}}{\partial a_3} \right|_0 = b_3, \quad \left. \frac{\partial k_{\xi_3}}{\partial a_4} \right|_0 = \frac{b_4 \omega^2}{\sqrt{b_4^2 \omega^2 - b_{10}^2}}, \quad \left. \frac{\partial k_{\xi_3}}{\partial a_{10}} \right|_0 = -\frac{b_{10}}{\sqrt{b_{10}^2 \omega^2 - b_{10}^2}}, \quad (\text{B-5})$$

resulting in the following split-step Fourier correction wavenumber,

$$k_{\xi_3}^{SSF} = i b_3 (a_3 - b_3) + \frac{b_4 \omega^2 (a_4 - b_4)}{\sqrt{b_4^2 \omega^2 - b_{10}^2}} - \frac{b_{10} (a_{10} - b_{10})}{\sqrt{b_{10}^2 \omega^2 - b_{10}^2}}. \quad (\text{B-6})$$

## APPENDIX C

### RWE WAVENUMBER APPROXIMATIONS

The extrapolation wavenumber developed in equation ?? is appropriate for any non-orthogonal Riemannian geometry. However, there are a number of situations where symmetry or partial orthogonality are present. Moreover, kinematic approximations can be made where one ignores all imaginary wavenumber components. These situations are discussed herein.

**3D Semi-orthogonal coordinate systems** - Semi-orthogonal coordinate systems occur where one coordinate ( $\xi_3$ ) is orthogonal to the other two coordinates ( $\xi_1$  and  $\xi_2$ ) (?). In these cases the  $m^{13}$  and  $m^{23}$  components of the weighted metric tensor are identically zero, which leads to the following extrapolation wavenumber,

$$k_{\xi_3} = ia_3 \pm [a_4^2 \omega^2 - a_5^2 k_{\xi_1}^2 - a_6^2 k_{\xi_2}^2 - a_7 k_{\xi_1} k_{\xi_2} + ia_8 k_{\xi_1} + ia_9 k_{\xi_2} - a_{10}^2]^{\frac{1}{2}}, \quad (\text{C-1})$$

where,

$$\mathbf{a} = \begin{bmatrix} 0 & 0 & \frac{n^3}{2m^{33}} & \frac{\sqrt{|\mathbf{g}|}s}{\sqrt{m^{33}}} & \sqrt{\frac{m^{11}}{m^{33}}} & \sqrt{\frac{m^{22}}{m^{33}}} & \frac{2m^{12}}{m^{33}} & \frac{n^1}{m^{33}} & \frac{n^2}{m^{33}} & \frac{n^3}{2m^{33}} \end{bmatrix}^{\mathbf{T}}. \quad (\text{C-2})$$

which are identical to the coefficients recovered by ?.

**3D kinematic coordinate systems** - One approximation that reduces the computational and memory costs is to consider any term in equation ?? with an imaginary number to be purely an amplitude factor. (Note that this is not purely correct because the square root of an imaginary number generally has a real component.) This 'kinematic approximation' leads to extrapolation wavenumber,

$$\hat{k}_{\xi_3} = -a_1 k_{\xi_1} - a_2 k_{\xi_2} \pm [a_4^2 \omega^2 - a_5^2 k_{\xi_1}^2 - a_6^2 k_{\xi_2}^2 - a_7 k_{\xi_1} k_{\xi_2} - a_{10}^2]^{\frac{1}{2}}, \quad (\text{C-3})$$

where,

$$\mathbf{a} = \begin{bmatrix} \frac{m^{13}}{m^{33}} & \frac{m^{23}}{m^{33}} & 0 & \frac{\sqrt{|\mathbf{g}|}s}{\sqrt{m^{33}}} & \sqrt{\frac{m^{11}}{m^{33}} - \left(\frac{m^{13}}{m^{33}}\right)^2} & \sqrt{\frac{m^{22}}{m^{33}} - \left(\frac{m^{23}}{m^{33}}\right)^2} & \frac{2m^{12}}{m^{33}} - \frac{2m^{13}m^{23}}{(m^{33})^2} & 0 & 0 & \frac{n^3}{2m^{33}} \end{bmatrix}^{\mathbf{T}}. \quad (\text{C-4})$$

**3D kinematic semi-orthogonal coordinate systems** - Combining the two above restrictions yields the following extrapolation wavenumber,

$$\hat{k}_{\xi_3} = \pm [a_4^2 \omega^2 - a_5^2 k_{\xi_1}^2 - a_6^2 k_{\xi_2}^2 - a_7 k_{\xi_1} k_{\xi_2} - a_{10}^2]^{\frac{1}{2}}, \quad (\text{C-5})$$

where,

$$\mathbf{a} = \begin{bmatrix} 0 & 0 & 0 & \frac{\sqrt{|\mathbf{g}|}s}{\sqrt{m^{33}}} & \sqrt{\frac{m^{11}}{m^{33}}} & \sqrt{\frac{m^{22}}{m^{33}}} & \frac{2m^{12}}{m^{33}} & 0 & 0 & \frac{n^3}{2m^{33}} \end{bmatrix}^{\mathbf{T}}. \quad (\text{C-6})$$

**2D non-orthogonal coordinate systems** - Two-dimensional situations are handled by identifying  $\xi_2 = 0$ . All derivatives in the associated metric tensor  $m^{jk}$  with

respect coordinate  $\xi_2$  are identically zero, and the resulting 2D non-orthogonal coordinate system wavenumber is,

$$k_{\xi_3} = -a_1 k_{\xi_1} + ia_3 \pm [a_4^2 \omega^2 - a_5^2 k_{\xi_1}^2 + ia_8 k_{\xi_1} - a_{10}^2]^{\frac{1}{2}}, \quad (\text{C-7})$$

where,

$$\mathbf{a} = \begin{bmatrix} \frac{m^{13}}{m^{33}} & 0 & \frac{n^3}{2m^{33}} & \frac{\sqrt{|\mathbf{g}|}s}{\sqrt{m^{33}}} & \sqrt{\frac{m^{11}}{m^{33}} - \left(\frac{m^{13}}{m^{33}}\right)^2} & 0 & 0 & \frac{n^1}{m^{33}} - \frac{m^{13}n^3}{(m^{33})^2} & 0 & \frac{n^3}{2m^{33}} \end{bmatrix}^{\mathbf{T}}. \quad (\text{C-8})$$

**2D non-orthogonal kinematic coordinate systems** - Two-dimensional kinematic situations are handled through identity  $\xi_2 = 0$ . Again, all derivatives in the associated metric tensor  $m^{jk}$  with respect coordinate  $\xi_2$  are identically zero, and the 2D non-orthogonal kinematic extrapolation wavenumber is

$$\hat{k}_{\xi_3} = -a_1 k_{\xi_1} \pm [a_4^2 \omega^2 - a_5^2 k_{\xi_1}^2 - a_{10}^2]^{\frac{1}{2}}, \quad (\text{C-9})$$

where,

$$\mathbf{a} = \begin{bmatrix} \frac{m^{13}}{m^{33}} & 0 & 0 & \frac{\sqrt{|\mathbf{g}|}s}{\sqrt{m^{33}}} & \sqrt{\frac{m^{11}}{m^{33}} - \left(\frac{m^{13}}{m^{33}}\right)^2} & 0 & 0 & 0 & 0 & \frac{n^3}{2m^{33}} \end{bmatrix}^{\mathbf{T}}. \quad (\text{C-10})$$

**2D orthogonal coordinate systems** - Two-dimensional situations are handled with  $\xi_2 = g_{13} = 0$ . Accordingly, all derivatives in the associated metric tensor  $m^{jk}$  with respect coordinate  $\xi_2$  are identically zero, and the 2D orthogonal coordinate system is represented by

$$k_{\xi_3} = ia_3 \pm [a_4^2 \omega^2 - a_5^2 k_{\xi_1}^2 + ia_8 k_{\xi_1} - a_{10}^2]^{\frac{1}{2}}, \quad (\text{C-11})$$

where,

$$\mathbf{a} = \begin{bmatrix} 0 & 0 & \frac{n_3}{2m^{33}} & \frac{\sqrt{|\mathbf{g}|}s}{\sqrt{m^{33}}} & \sqrt{\frac{m^{11}}{m^{33}}} & 0 & 0 & \frac{n^1}{m^{33}} & 0 & \frac{n^3}{2m^{33}} \end{bmatrix}^{\mathbf{T}}. \quad (\text{C-12})$$

**2D orthogonal kinematic coordinate systems** - The above two approximations can be combined to yield the following extrapolation wavenumber for 2D orthogonal kinematic coordinate systems,

$$\hat{k}_{\xi_3} = \pm [a_4^2 \omega^2 - a_5^2 k_{\xi_1}^2 - a_{10}^2]^{\frac{1}{2}}, \quad (\text{C-13})$$

where,

$$\mathbf{a} = \begin{bmatrix} 0 & 0 & 0 & \frac{\sqrt{|\mathbf{g}|}s}{\sqrt{m^{33}}} & \sqrt{\frac{m^{11}}{m^{33}}} & 0 & 0 & 0 & 0 & \frac{n^3}{2m^{33}} \end{bmatrix}^{\mathbf{T}}. \quad (\text{C-14})$$

## APPENDIX D

### ELLIPTIC COORDINATE SYSTEMS

Exploring constant coordinate surfaces provides additional insight into some characteristics of the elliptic coordinate system. As illustrated by the following trigonometric identities, curves of constant  $\xi_1$  represent hyperbolas

$$\frac{x_1^2}{a^2 \cos^2 \xi_1} - \frac{x_3^2}{a^2 \sin^2 \xi_1} = \cosh^2 \xi_3 - \sinh^2 \xi_3 = 1, \quad (\text{D-1})$$

while curves of constant  $\xi_3$  form ellipses

$$\frac{x_1^2}{a^2 \cosh^2 \xi_3} + \frac{x_3^2}{a^2 \sinh^2 \xi_3} = \cos^2 \xi_1 + \sin^2 \xi_1 = 1. \quad (\text{D-2})$$

Thus, outward extrapolation in the  $\xi_3$  direction would step a wavefield through a family of elliptic surfaces defined by equation D-2.

Equation D-1 may also be used to derive an expression that defines the local extrapolation axis angle,  $\theta(\boldsymbol{\xi})$ , relative to vertical reference. Taking the total derivative of equation D-1,

$$\frac{2 x_1 dx_1}{a^2 \cosh^2 \xi_3} + \frac{2 x_3 dx_3}{a^2 \sinh^2 \xi_3} = 0, \quad (\text{D-3})$$

and further manipulating the result yields the local extrapolation axis angle  $\theta(\boldsymbol{\xi})$

$$\tan \theta = \frac{dx_1}{dx_3} = \frac{x_3 \cos^2 \xi_1}{x_1 \sin^2 \xi_1} = \tanh \xi_3 \cot \xi_1. \quad (\text{D-4})$$

## APPENDIX E

### ADCIG COORDINATE TRANSFORM

This appendix addresses how to express operators  $\frac{\partial}{\partial x_3}$  and  $\frac{\partial}{\partial h_{x_1}}$  in generalized coordinate systems to derive equation ???. I first assume that generalized coordinate systems are related to the Cartesian variables through a bijection (i.e., one-to-one mapping)

$$x_1 = f(\xi_1, \xi_3) \quad \text{and} \quad x_3 = g(\xi_1, \xi_3) \quad (\text{E-1})$$

with a non-vanishing Jacobian of coordinate transformation,  $J_{\boldsymbol{\xi}}$ . The bijection between a generalized and Cartesian coordinate system allows us to rewrite the left-hand-sides of equations ??? as (?)

$$\frac{\partial t}{\partial x_1} = \frac{1}{J_{\boldsymbol{\xi}}} \frac{\partial(t, x_3)}{\partial(\xi_1, \xi_3)} \quad \text{and} \quad \frac{\partial t}{\partial x_3} = \frac{1}{J_{\boldsymbol{\xi}}} \frac{\partial(x_1, t)}{\partial(\xi_1, \xi_3)}. \quad (\text{E-2})$$

Expanding the Jacobian notation leads to

$$\begin{bmatrix} \frac{\partial t}{\partial \xi_1} \frac{\partial x_3}{\partial \xi_3} - \frac{\partial t}{\partial \xi_3} \frac{\partial x_3}{\partial \xi_1} \\ \frac{\partial t}{\partial \xi_3} \frac{\partial x_1}{\partial \xi_1} - \frac{\partial t}{\partial \xi_1} \frac{\partial x_1}{\partial \xi_3} \end{bmatrix} = 2 J_{\xi} s \cos \gamma \begin{bmatrix} \sin \alpha \\ \cos \alpha \end{bmatrix}. \quad (\text{E-3})$$

The right-hand-sides of equations E-3 are analogous to those derived by ?. Cross-multiplying the expressions by factors  $\frac{\partial x_1}{\partial \xi_3}$  and  $\frac{\partial x_3}{\partial \xi_3}$

$$\begin{bmatrix} \frac{\partial x_1}{\partial \xi_3} \left( \frac{\partial t}{\partial \xi_1} \frac{\partial x_3}{\partial \xi_3} - \frac{\partial t}{\partial \xi_3} \frac{\partial x_3}{\partial \xi_1} \right) \\ \frac{\partial x_3}{\partial \xi_3} \left( \frac{\partial t}{\partial \xi_3} \frac{\partial x_1}{\partial \xi_1} - \frac{\partial t}{\partial \xi_1} \frac{\partial x_1}{\partial \xi_3} \right) \end{bmatrix} = 2 J_{\xi} s \cos \gamma \begin{bmatrix} \frac{\partial x_1}{\partial \xi_3} \sin \alpha \\ \frac{\partial x_3}{\partial \xi_3} \cos \alpha \end{bmatrix} \quad (\text{E-4})$$

and adding the two expressions results in

$$\frac{\partial t}{\partial \xi_3} \left( \frac{\partial x_3}{\partial \xi_3} \frac{\partial x_1}{\partial \xi_1} - \frac{\partial x_1}{\partial \xi_3} \frac{\partial x_3}{\partial \xi_1} \right) = 2 J_{\xi} s \cos \gamma \left( \frac{\partial x_1}{\partial \xi_3} \sin \alpha + \frac{\partial x_3}{\partial \xi_3} \cos \alpha \right). \quad (\text{E-5})$$

A similar argument can be used to construct the equations for the subsurface-offset axis. The bijection between the generalized coordinate and Cartesian subsurface-offset axes allows for the left-hand-side of equations ?? to be rewritten as

$$\frac{\partial t}{\partial h_{x_1}} = \frac{1}{J_{\mathbf{h}}} \frac{\partial(t, h_{x_3})}{\partial(h_{\xi_1}, h_{\xi_3})} \quad \text{and} \quad \frac{\partial t}{\partial h_{x_3}} = \frac{1}{J_{\mathbf{h}}} \frac{\partial(h_{x_1}, t)}{\partial(h_{\xi_1}, h_{\xi_3})}, \quad (\text{E-6})$$

where  $J_{\mathbf{h}}$  is the subsurface-offset Jacobian of transformation. Expanding the Jacobian notation leads to

$$\begin{bmatrix} \frac{\partial t}{\partial h_{\xi_1}} \frac{\partial h_{x_3}}{\partial h_{\xi_3}} - \frac{\partial t}{\partial h_{\xi_3}} \frac{\partial h_{x_3}}{\partial h_{\xi_1}} \\ \frac{\partial t}{\partial h_{\xi_3}} \frac{\partial h_{x_1}}{\partial h_{\xi_1}} - \frac{\partial t}{\partial h_{\xi_1}} \frac{\partial h_{x_1}}{\partial h_{\xi_3}} \end{bmatrix} = 2 J_{\mathbf{h}} s \sin \gamma \begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix}. \quad (\text{E-7})$$

The right-hand-side of equations E-7 are again analogous to those given by ?. Cross-multiplying the expressions by factors  $\frac{\partial h_{x_1}}{\partial h_{\xi_1}}$  and  $\frac{\partial h_{x_3}}{\partial h_{\xi_1}}$

$$\begin{bmatrix} \frac{\partial h_{x_1}}{\partial h_{\xi_1}} \left( \frac{\partial t}{\partial h_{\xi_1}} \frac{\partial h_{x_3}}{\partial h_{\xi_3}} - \frac{\partial t}{\partial h_{\xi_3}} \frac{\partial h_{x_3}}{\partial h_{\xi_1}} \right) \\ \frac{\partial h_{x_3}}{\partial h_{\xi_1}} \left( \frac{\partial t}{\partial h_{\xi_3}} \frac{\partial h_{x_1}}{\partial h_{\xi_1}} - \frac{\partial t}{\partial h_{\xi_1}} \frac{\partial h_{x_1}}{\partial h_{\xi_3}} \right) \end{bmatrix} = 2 J_{\mathbf{h}} s \sin \gamma \begin{bmatrix} \frac{\partial h_{x_1}}{\partial h_{\xi_1}} \cos \alpha \\ \frac{\partial h_{x_3}}{\partial h_{\xi_1}} \sin \alpha \end{bmatrix}, \quad (\text{E-8})$$

and subtracting the two expressions above yields

$$\frac{\partial t}{\partial h_{\xi_1}} \left( \frac{\partial h_{x_1}}{\partial h_{\xi_1}} \frac{\partial h_{x_3}}{\partial h_{\xi_3}} - \frac{\partial h_{x_1}}{\partial h_{\xi_3}} \frac{\partial h_{x_3}}{\partial h_{\xi_1}} \right) = 2 J_{\mathbf{h}} s \sin \gamma \left( \frac{\partial h_{x_1}}{\partial h_{\xi_1}} \cos \alpha - \frac{\partial h_{x_3}}{\partial h_{\xi_1}} \sin \alpha \right). \quad (\text{E-9})$$

An expression for ADCIGs can be obtained by dividing equation E-9 by equation E-5

$$\frac{\frac{\partial t}{\partial h_{\xi_1}} \left( \frac{\partial h_{x_1}}{\partial h_{\xi_1}} \frac{\partial h_{x_3}}{\partial h_{\xi_3}} - \frac{\partial h_{x_1}}{\partial h_{\xi_3}} \frac{\partial h_{x_3}}{\partial h_{\xi_1}} \right)}{\frac{\partial t}{\partial \xi_3} \left( \frac{\partial x_3}{\partial \xi_3} \frac{\partial x_1}{\partial \xi_1} - \frac{\partial x_1}{\partial \xi_3} \frac{\partial x_3}{\partial \xi_1} \right)} = \tan \gamma \frac{J_{\mathbf{h}} \left( \frac{\partial h_{x_1}}{\partial h_{\xi_1}} \cos \alpha - \frac{\partial h_{x_3}}{\partial h_{\xi_1}} \sin \alpha \right)}{J_{\xi} \left( \frac{\partial x_3}{\partial \xi_3} \cos \alpha + \frac{\partial x_1}{\partial \xi_3} \sin \alpha \right)}. \quad (\text{E-10})$$

One question arising from the geometric factors in equation E-10 is what do the terms  $\frac{\partial h_{x_1}}{\partial h_{\xi_1}}$ ,  $\frac{\partial h_{x_3}}{\partial h_{\xi_1}}$ ,  $\frac{\partial h_{x_1}}{\partial h_{\xi_3}}$  and  $\frac{\partial h_{x_3}}{\partial h_{\xi_3}}$  represent? I assume that the subsurface offset axes are generated by uniform wavefield shifting such that the following equations are valid:

$$\begin{bmatrix} h_{x_1} \\ h_{x_3} \\ h_{\xi_1} \\ h_{\xi_1} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ \xi_1 \\ \xi_3 \end{bmatrix} \quad \text{such that} \quad \begin{bmatrix} \frac{\partial h_{x_1}}{\partial h_{\xi_1}} \\ \frac{\partial h_{x_3}}{\partial h_{\xi_1}} \\ \frac{\partial h_{x_1}}{\partial h_{\xi_3}} \\ \frac{\partial h_{x_3}}{\partial h_{\xi_3}} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} \\ \frac{\partial x_3}{\partial \xi_1} \\ \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_3} \end{bmatrix}. \quad (\text{E-11})$$

If the subsurface offset axes were generated by anything other than uniform shifting (e.g.  $h_{x_1} = x_1^2$ ), then the assumptions behind equations E-11 would not be honored.

Using these identities in equation E-5 reduces equation E-10 to

$$-\left. \frac{\partial \xi_3}{\partial h_{\xi_1}} \right|_{\xi_1, t} = \frac{\partial t}{\partial h_{\xi_1}} \bigg/ \frac{\partial t}{\partial \xi_3} = \tan \gamma \frac{\left( \frac{\partial x_1}{\partial \xi_1} \cos \alpha - \frac{\partial x_3}{\partial \xi_1} \sin \alpha \right)}{\left( \frac{\partial x_3}{\partial \xi_3} \cos \alpha + \frac{\partial x_1}{\partial \xi_3} \sin \alpha \right)}, \quad (\text{E-12})$$

where the two Jacobian transformations are equivalent (i.e.  $J_{\xi} = J_{\mathbf{h}}$ ). This completes the derivation of equation ??.