

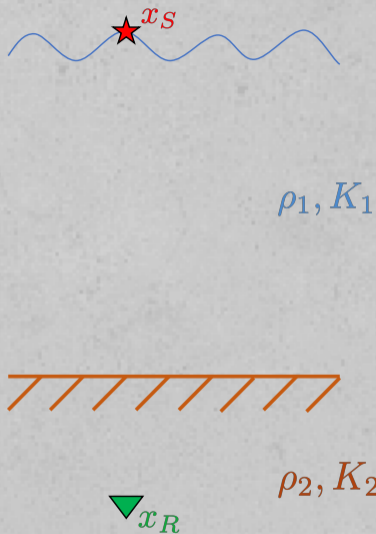
Multiparameter Full-Waveform Inversion with Near-Interface Sources using Staggered-grid Finite Differences

Joseph Jennings, Martin Almquist and Eric Dunham

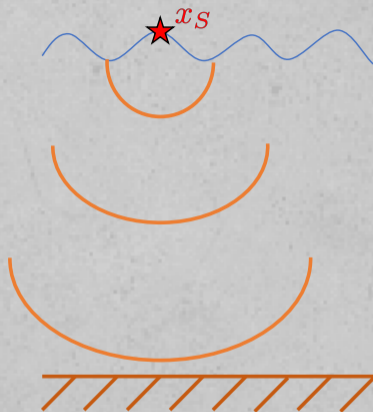
Department of Geophysics, Stanford University

March 12, 2019

Problem setup



Forward problem

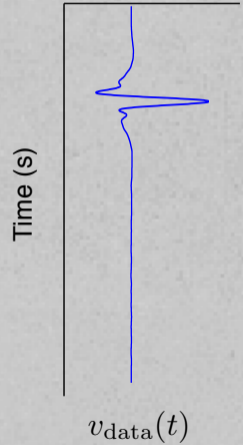
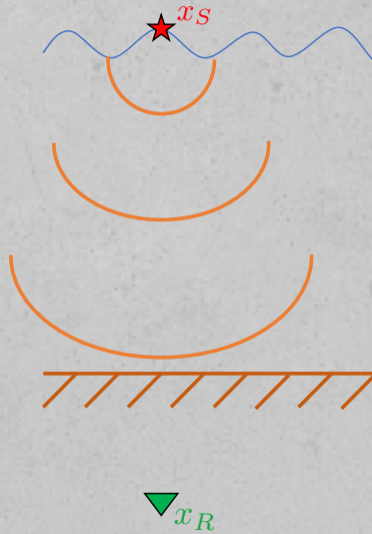


$$\rho \dot{v} + p_x = 0$$

$$\frac{1}{K} \dot{p} + v_x = \delta(x - x_S) f(t)$$

∇x_R

Recorded data



$p_{\text{data}}(t)$ can also exist

Full-Waveform Inversion

minimize $F(v, p)$, where
 $\rho(x), K(x), f(t)$

$$F(v, p) = \frac{w_1}{2} \int_0^T (v(x_R, t) - v_{\text{data}}(t))^2 dt$$
$$+ \frac{w_2}{2} \int_0^T (p(x_R, t) - p_{\text{data}}(t))^2 dt$$

subject to $\rho \dot{v} + p_x = 0$,

$$\frac{1}{K} \dot{p} + v_x - f(t) \delta(x - x_S) = 0.$$

Full-Waveform Inversion

minimize $F(v, p)$, where
 $\rho(x), K(x), f(t)$

$$F(v, p) = \frac{w_1}{2} \int_0^T (v(x_R, t) - v_{\text{data}}(t))^2 dt$$

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Loss function
(least squares)

subject to

$$\rho \dot{v} + p_x = 0,$$
$$\frac{1}{K} \dot{p} + v_x - f(t) \delta(x - x_S) = 0.$$

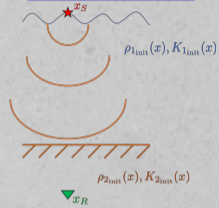
PDE constraint

Optimizing the functional

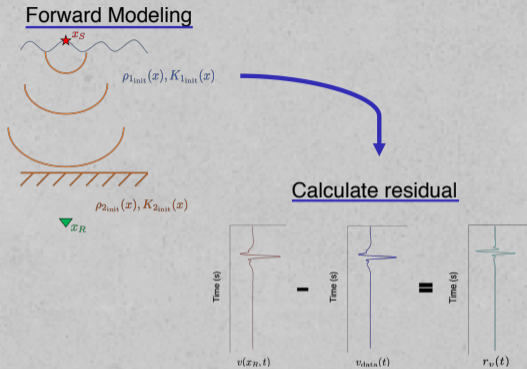
- Global optimization \rightarrow too many PDE solves
- Local/gradient-based optimization is preferred
 - **Requires use of the adjoint method**

Adjoint method: Forward modeling

Forward Modeling

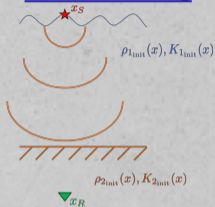


Adjoint method: Calculate residual

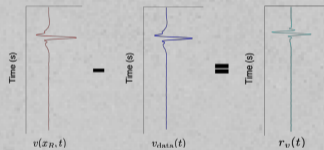


Adjoint method: Adjoint modeling

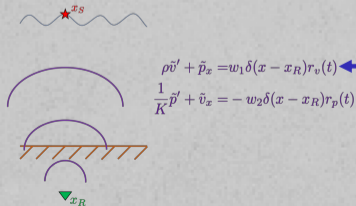
Forward Modeling



Calculate residual

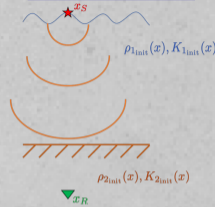


Adjoint modeling

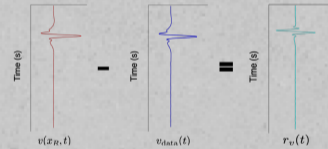


Adjoint method: Calculate gradient

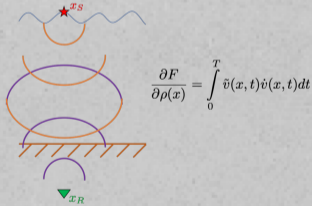
Forward Modeling



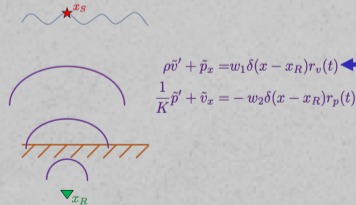
Calculate residual



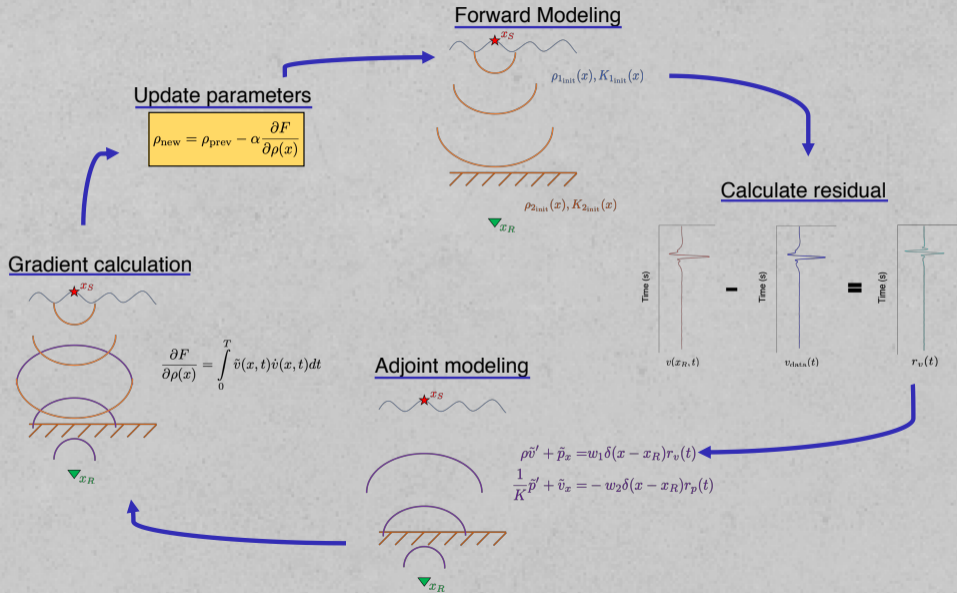
Gradient calculation



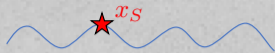
Adjoint modeling



Adjoint method: Update parameters



Problem setup

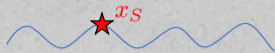


ρ_1, K_1

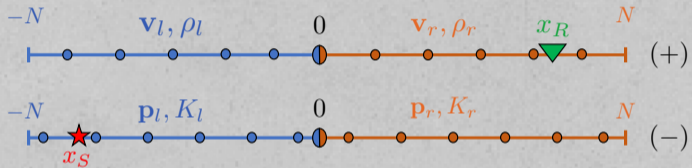


ρ_2, K_2
 x_R

Finite difference discretization

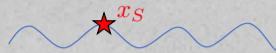


ρ_1, K_1



ρ_2, K_2
 x_R

Reduced accuracy at boundaries



ρ_1, K_1

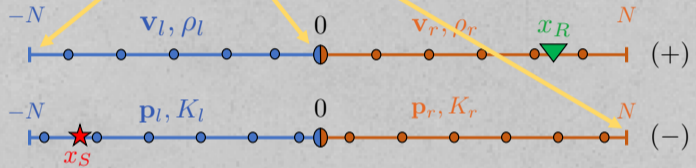
Discretization



ρ_2, K_2



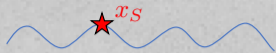
Reduced accuracy



Issues with finite differences

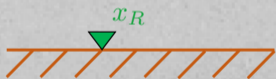
1. Reduced accuracy at interfaces and boundaries

Ocean-bottom node acquisition (OBN)



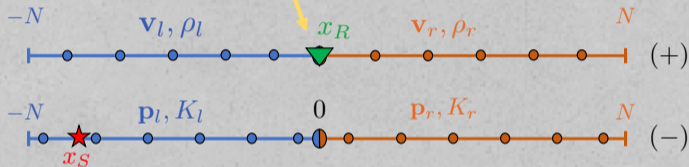
ρ_1, K_1

Discretization

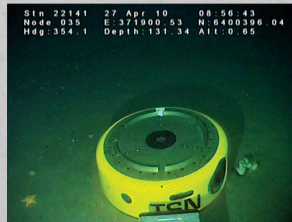


ρ_2, K_2

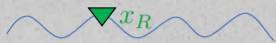
Point receiver at interface



Ocean bottom node
(Ronen et al., 2012)



OBN: Adjoint problem



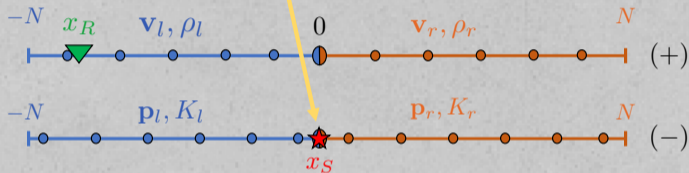
ρ_1, K_1

Discretization



ρ_2, K_2

Point source at interface



Issues with finite differences

1. Reduced accuracy at interfaces and boundaries
2. Sources can be on/near interfaces

Shifting leads to gradient errors



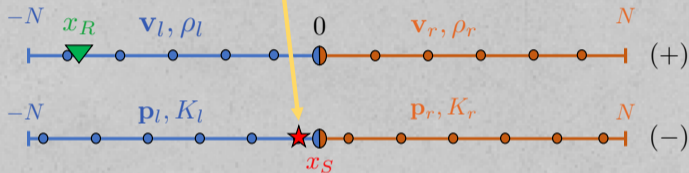
ρ_1, K_1

Discretization



ρ_2, K_2

Shift by arbitrary amount



Issues with finite differences

1. Reduced accuracy at interfaces and boundaries
2. Sources can be on/near interfaces
 - Shifting of sources leads to **gradient errors**

Presentation takeaways

- 1. High-order accurate modeling of the forward problem**

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- Accurate finite difference (FD) stencils at boundaries and interfaces
- Modeling of point sources at interfaces

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2. Immediate availability of the discrete adjoint

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1. High-order accurate modeling of the forward problem

- Accurate finite difference (FD) stencils at boundaries and interfaces
- Modeling of point sources at interfaces

2. Immediate availability of the discrete adjoint

- Semi-discrete adjoint equations are the same as the forward with different source

Dual consistency (Berg and Nordstrom, 2012)

Continuous

- Forward:

$$\rho \dot{v} + p_x = 0$$

$$\frac{1}{K} \dot{p} + v_x = \delta(x - x_S) f(t)$$

- Adjoint:

Discrete

- Forward:

- Adjoint:

Dual consistency (Berg and Nordstrom, 2012)

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Discretize



Discrete

- Forward:

$$\rho \dot{\mathbf{v}} + \mathbf{D}_+ \mathbf{p} = \mathbf{0}$$

$$K^{-1} \dot{\mathbf{p}} + \mathbf{D}_- \mathbf{v} = f(t) \mathbf{d}_{S-}$$

- Adjoint:

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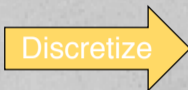
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- Adjoint:



Compute adjoint

$$\rho \tilde{v}' + \tilde{p}_x = w_1 \delta(x - x_R) r_v(t)$$

$$\frac{1}{K} \tilde{p}' + \tilde{v}_x = -w_2 \delta(x - x_R) r_p(t)$$

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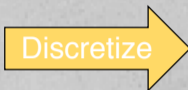
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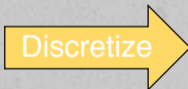


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Compute adjoint

- Adjoint:

$$\rho \tilde{\mathbf{v}}' + \mathbf{D}_+ \tilde{\mathbf{p}} = w_1 \mathbf{d}_{R-} r_v(t)$$

$$K^{-1} \tilde{\mathbf{p}} + \mathbf{D}_- \tilde{\mathbf{v}} = -w_2 \mathbf{d}_{R+} r_p(t)$$

Outline for presentation

1. Summation-by-parts with simultaneous approximation term method (SBP-SAT)
2. Adjoint optimization with SBP-SAT
3. Numerical examples

Summation-by-parts (SBP) operator

$$\frac{\partial p}{\partial x} \approx \mathbf{D} \mathbf{p},$$

Summation-by-parts (SBP) operator

$$\frac{\partial p}{\partial x} \approx \mathbf{D} \mathbf{p},$$

$$\mathbf{D} = \frac{1}{h} \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} & & & & & \\ \theta_{21} & 0 & \theta_{23} & \theta_{24} & & & & & \\ \theta_{31} & \theta_{32} & 0 & \theta_{33} & \theta_{34} & & & & \\ \theta_{41} & \theta_{42} & \theta_{43} & 0 & \theta_{45} & \theta_{46} & & & \\ 0 & 0 & \frac{1}{12} & -\frac{8}{12} & 0 & \frac{8}{12} & -\frac{1}{12} & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & & & & \ddots \end{bmatrix}$$

(Del Rey Fernández et al., 2014)

Discrete form of integration-by-parts

1. Inner-product

- $(u, v) = \int_a^b u(x)v(x)dx$

- $(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T \mathbf{H} \mathbf{v}$, (\mathbf{H} : discrete quadrature)

Discrete form of integration-by-parts

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- $(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T \mathbf{H} \mathbf{v}$, (\mathbf{H} : discrete quadrature)

2. Integration-by-parts and SBP

- $(u, \frac{dv}{dx}) = -(\frac{du}{dx}, v) + u(x)v(x)|_a^b$
- $(\mathbf{u}, \mathbf{D}\mathbf{v}) = -(\mathbf{D}\mathbf{u}, \mathbf{v}) + u_N v_N - u_0 v_0$

Why is this useful?

1. Discrete energy balance \Rightarrow provable stability
2. Consistent approximation of continuous adjoint problem
 - Dual consistency

Simultaneous approximation term method

Simultaneous approximation term (SAT) method
with SBP (SBP-SAT):

→ Weak imposition of boundary conditions (BC)

Simultaneous approximation term method

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$$\begin{aligned}\rho \frac{\partial \mathbf{v}}{\partial t} &= \dots - c \mathbf{H}^{-1} [\mathbf{e}_0(v_0 - \hat{v}_0) + \mathbf{e}_n(v_N - \hat{v}_N)], \\ \frac{1}{K} \frac{\partial \mathbf{p}}{\partial t} &= \dots - c \mathbf{H}^{-1} [\mathbf{e}_0(p_0 - \hat{p}_0) + \mathbf{e}_n(p_N - \hat{p}_N)].\end{aligned}$$

- \hat{p}_0, \hat{v}_0 : “target” variables that satisfy BC

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Continuous optimization problem

minimize $F(v, p)$, where
 $\rho(x), K(x), f(t)$

$$F(v, p) = \frac{w_1}{2} \int_0^T (v(x_R, t) - v_{\text{data}}(t))^2 dt$$

$$+ \frac{w_2}{2} \int_0^T (p(x_R, t) - p_{\text{data}}(t))^2 dt$$

} Loss function
(least squares)

subject to $\rho \dot{v} + p_x = 0$,

$$\frac{1}{K} \dot{p} + v_x - f(t) \delta(x - x_S) = 0.$$

} PDE constraint

Semi-discrete (SD) optimization problem

minimize $F(\mathbf{v}, \mathbf{p})$, where
 $\rho, \mathbf{K}, f(t)$

$$F(\mathbf{v}, \mathbf{p}) = \frac{w_1}{2} \int_0^T ((\mathbf{H}_+ \mathbf{d}_{R+})^T \mathbf{v} - v_{\text{data}}(t))^2 dt$$
$$+ \frac{w_2}{2} \int_0^T ((\mathbf{H}_- \mathbf{d}_{R-})^T \mathbf{p} - p_{\text{data}}(t))^2 dt$$

subject to $\rho \dot{\mathbf{v}} + \mathbf{D}_+ \mathbf{p} = \mathbf{0}$,
 $K^{-1} \dot{\mathbf{p}} + \mathbf{D}_- \mathbf{v} - f(t) \mathbf{d}_{S-} = \mathbf{0}$.

Point source/receiver operators

minimize $F(\mathbf{v}, \mathbf{p})$, where
 $\rho, \mathbf{K}, f(t)$

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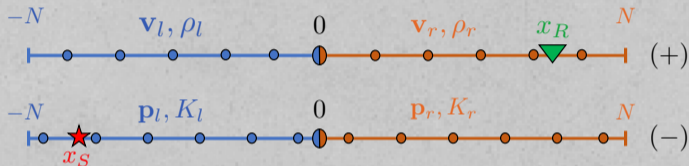
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Finite difference discretization



ρ_1, K_1



ρ_2, K_2
 x_R

Continuous

$$\rho \dot{v} + p_x = 0$$

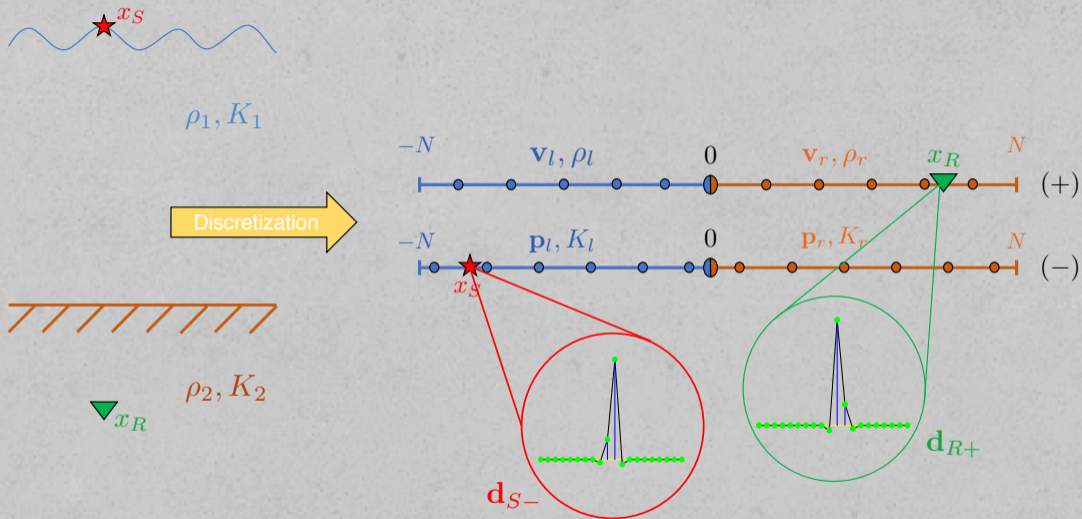
$$\frac{1}{K} \dot{p} + v_x = \delta(x - x_S) f(t)$$

Semi-Discrete

$$\rho \dot{\mathbf{v}} + \mathbf{D}_+ \mathbf{p} = 0$$

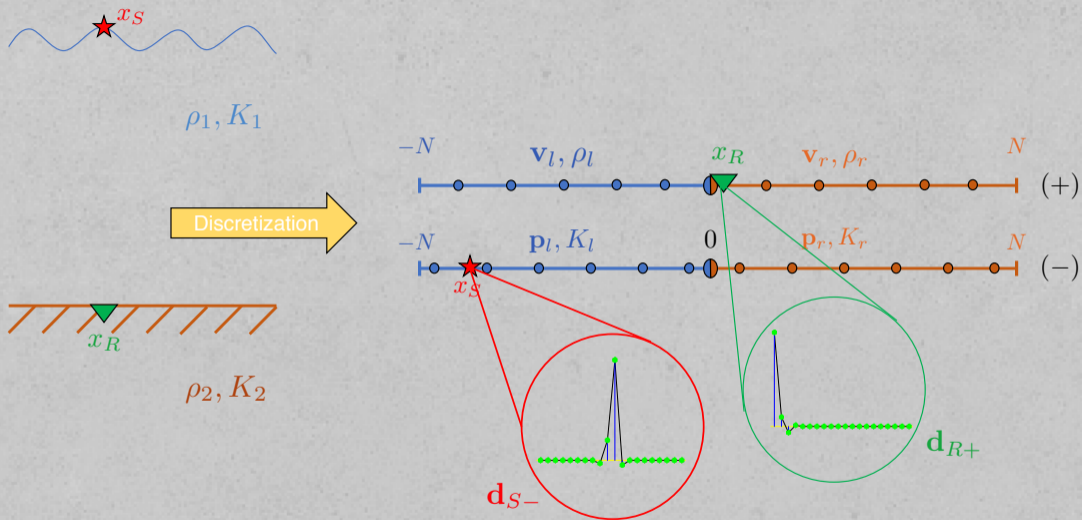
$$K^{-1} \dot{\mathbf{p}} + \mathbf{D}_- \mathbf{v} = f(t) \mathbf{d}_{S-}$$

Point source/receiver discretization



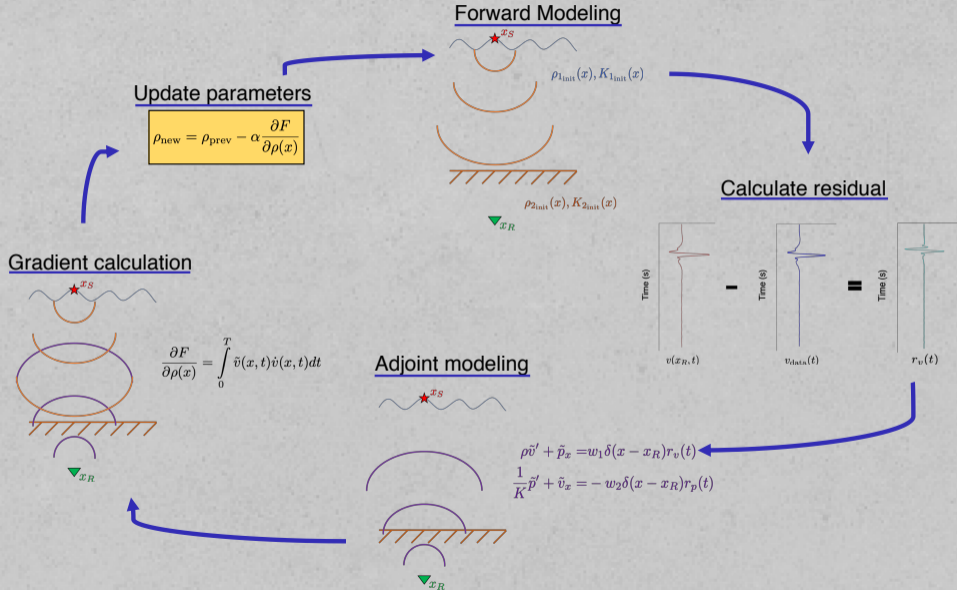
(Petersson et al., 2016)

Point receiver near interface



(Petersson et al., 2016)

Residual becomes adjoint source



Source-receiver dual consistency

- SD adjoint equations

$$\begin{aligned}\rho \tilde{\mathbf{v}}' + \mathbf{D}_+ \tilde{\mathbf{p}} &= w_1 \mathbf{d}_{R+} \left(\mathbf{d}_{R+}^T \mathbf{H}_+ \mathbf{v} - v_{\text{data}}(t) \right), \\ K^{-1} \tilde{\mathbf{p}}' + \mathbf{D}_- \tilde{\mathbf{v}} &= -w_2 \mathbf{d}_{R-} \left(\mathbf{d}_{R-}^T \mathbf{H}_- \mathbf{p} - p_{\text{data}}(t) \right).\end{aligned}$$

- $\mathbf{d}_{R\pm}$: receiver restriction in functional
- $\mathbf{d}_{R\pm}$: delta function (point source) in adjoint

Easy to obtain adjoint equations

- SD adjoint equations

$$\begin{aligned}\rho \tilde{\mathbf{v}}' + \mathbf{D}_+ \tilde{\mathbf{p}} &= w_1 \mathbf{d}_{R+} r_p(t), \\ K^{-1} \tilde{\mathbf{p}}' + \mathbf{D}_- \tilde{\mathbf{v}} &= -w_2 \mathbf{d}_{R-} r_v(t).\end{aligned}$$

- SD forward equations

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- SD forward equations

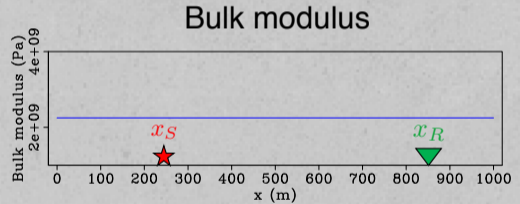
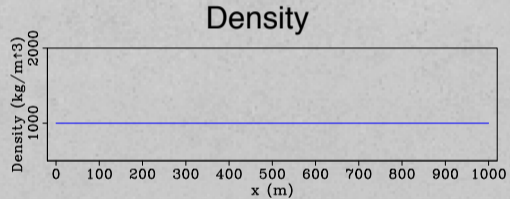
$$\begin{aligned}\rho \dot{\mathbf{v}} + \mathbf{D}_+ \mathbf{p} &= \mathbf{0}, \\ K^{-1} \dot{\mathbf{p}} + \mathbf{D}_- \mathbf{v} &= \mathbf{d}_{s-} f(t).\end{aligned}$$

- We can use same code for forward and adjoint!

Outline for presentation

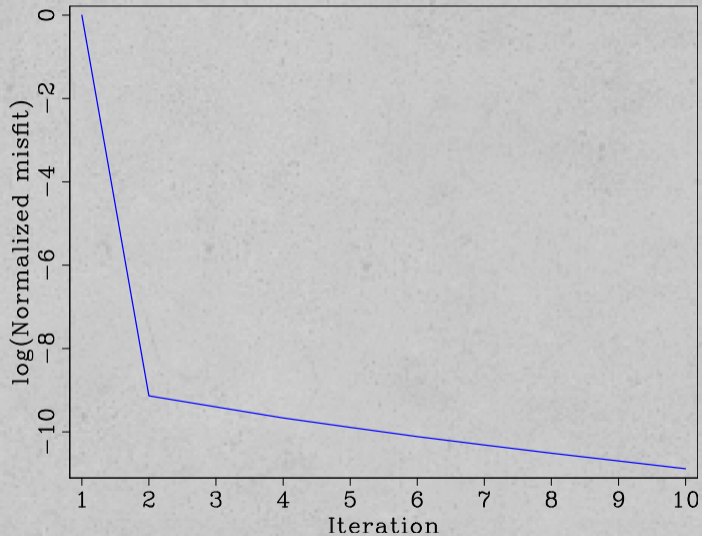
1. Summation-by-parts with simultaneous approximation term method (SBP-SAT)
 - High-order accurate, energy stable, dual consistency
2. Adjoint optimization with SBP-SAT
 - Source and receiver discretization
 - Source-receiver dual consistency
 - Same code for forward and adjoint
4. Numerical examples

Setup for source inversion



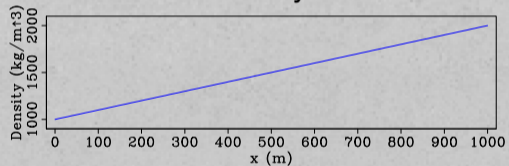
Inversion for $f(t)$: iteration movie

Inversion for $f(t)$: misfit reduction

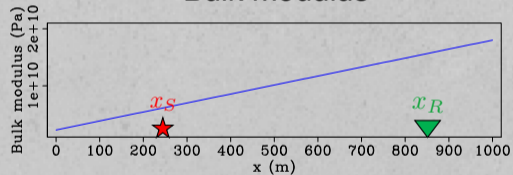


Setup for medium parameter inversion

Density

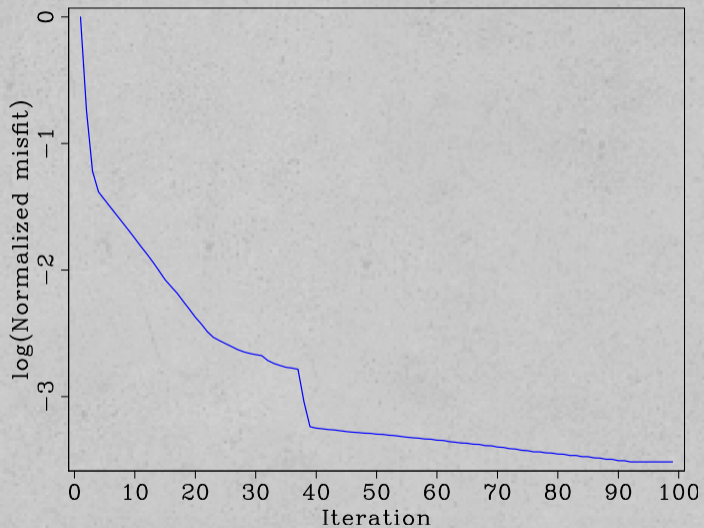


Bulk modulus



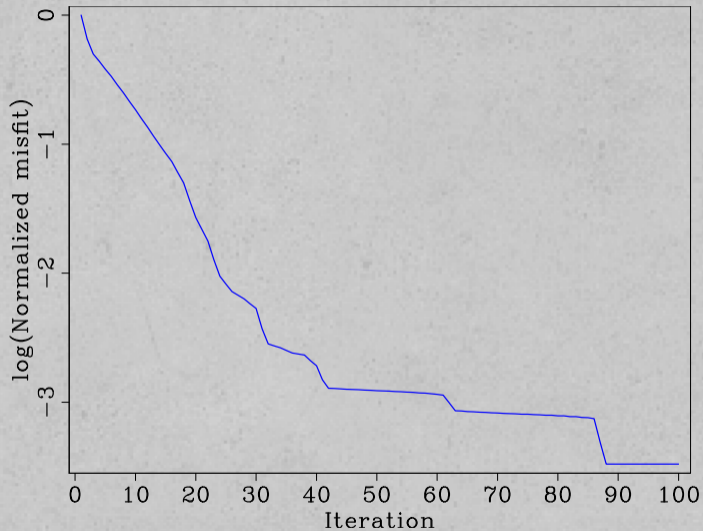
Inversion for $\rho(x)$: iteration movie

Inversion for $\rho(x)$: misfit reduction



Inversion for $K(x)$: results

Inversion for $K(x)$: misfit reduction



Conclusions and road ahead

1. SBP-SAT framework is ideal for adjoint optimization

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 - (a) **Forward problem:** High-order accurate, energy-stable

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 - (a) **Forward problem:** High-order accurate, energy-stable
 - Boundaries and interfaces
 - Point sources

Conclusions and road ahead

1. SBP-SAT framework is ideal for adjoint optimization
 - (a) **Forward problem:** High-order accurate, energy-stable
 - Boundaries and interfaces
 - Point sources
 - (b) **Adjoint problem:** dual-consistent

Conclusions and road ahead

1. SBP-SAT framework is ideal for adjoint optimization
 - (a) **Forward problem:** High-order accurate, energy-stable
 - Boundaries and interfaces
 - Point sources
 - (b) **Adjoint problem:** dual-consistent
 - Source-receiver dual consistency
 - SD adjoint equations are immediately available

Conclusions and road ahead

1. SBP-SAT framework is ideal for adjoint optimization
 - (a) **Forward problem:** High-order accurate, energy-stable
 - Boundaries and interfaces
 - Point sources
 - (b) **Adjoint problem:** dual-consistent
 - Source-receiver dual consistency
 - SD adjoint equations are immediately available
2. Apply to more complicated numerical examples

Questions?

Specifying SATs

1. Target variables $(\hat{v}_0, \hat{v}_N, \hat{p}_0, \hat{p}_N)$ must satisfy boundary conditions

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2. Outgoing characteristic is preserved

- $\hat{w}^- = w^-$