NOV 28,1984

FINDING THE Z-TRANSFORM OF THE INVERSE OF A MINIMUM PHASE FILTER TIMES A MAXIMUM PHASE FILTER

We begin looking at this problem on pages 27 thru 33 of this notebook. We discovered that the basic problem is a finite operation problem in that as much of the numerator coefficients as one wishes to calculate only requires a finite number of elementary operations, i.e., addition, subtraction, multiplication and division. An example where this finiteness is not true is finding the maximum and minimum phase factors of an arbitrary polynomial.

Let us first study again the development of forward and backward prediction error filters for the non-symmetric toeplitz matrix. We first look at a second order to third order example.

$$\begin{bmatrix} R(0) & R(-1) & R(-2) & R(-3) \\ R(1) & R(0) & R(-1) & R(-1) \\ R(2) & R(1) & R(0) & R(-1) \\ R(3) & R(2) & R(1) & R(0) \end{bmatrix} \left\{ \begin{Bmatrix} 1 \\ a_1^2 \\ a_2^2 \\ 0 \end{Bmatrix} + a_3^3 \begin{Bmatrix} 0 \\ b_2^2 \\ b_1^2 \\ 1 \end{Bmatrix} \right\} = \left\{ \begin{Bmatrix} \rho_2 \\ 0 \\ 0 \\ \Delta_3 \end{Bmatrix} + a_3^3 \begin{Bmatrix} \Delta_3^1 \\ 0 \\ 0 \\ \rho_2 \end{Bmatrix} \right\}$$

a_3^3 is determined by

$a_3^3 \rho_2 + \Delta_3 = 0$ Eq 86.1

or

$a_3^3 = -\Delta_3 / \rho_2$ Eq86.2

If $1 - a_n^n b_n^n = 0$, then $a_n = 1 / b_n^n$ and the forward and backward filters are the same with scaling, the righthand vector is all zeros so the matrix determinant is zero.

Likewise
$b_3^3 = -\Delta_3^1 / \rho_2$ (Eq 86.3), and $\rho_3 = \rho_2 (1 - a_3^3 b_3^3)$ Eq 86.4

We see that

$$\begin{bmatrix} & & R & & \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1^3 & 1 & 0 & 0 \\ a_2^3 & a_1^2 & 1 & 0 \\ a_3^3 & a_2^2 & a_1^1 & 1 \end{bmatrix} = \begin{bmatrix} \rho_3 & ? & ? & ? \\ 0 & \rho_2 & ? & ? \\ 0 & 0 & \rho_1 & ? \\ 0 & 0 & 0 & \rho_0 \end{bmatrix} \quad Eq86.5$$

and

$$\begin{bmatrix} & & R & & \end{bmatrix}\begin{bmatrix} 1 & \mathrm{b}_1^1 & \mathrm{b}_2^2 & \mathrm{b}_3^3 \\ 0 & 1 & \mathrm{b}_1^2 & \mathrm{b}_2^3 \\ 0 & 0 & 1 & \mathrm{b}_1^3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \rho_0 & 0 & 0 & 0 \\ ? & \rho_1 & 0 & 0 \\ ? & ? & \rho_2 & 0 \\ ? & ? & ? & \rho_3 \end{bmatrix} \quad Eq86.6$$

Defining

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_1^3 & 1 & 0 & 0 \\ a_2^3 & a_1^2 & 1 & 0 \\ a_3^3 & a_2^2 & a_1^1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & \mathrm{b}_1^1 & \mathrm{b}_2^2 & \mathrm{b}_3^3 \\ 0 & 1 & \mathrm{b}_1^2 & \mathrm{b}_2^3 \\ 0 & 0 & 1 & \mathrm{b}_1^3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We see that

Since $R = {}^{T}R$, where the pre-upper script T means transpose about the minor diagonal, RA is upper triangular and $R({}^{T}B)$ is lower triangular, then RB is lower triangular with diagonal elements, $\rho_3\ \rho_2\ \rho_1\ \rho_0$ and thus BRA is lower triangular with the same diagonal elements because A is lower triangular with unity diagonal elements.

But RA is upper triangular with diagonal elements $\rho_3\ \rho_2\ \rho_1$ and ρ_0 and thus BRA is also upper triangular with the same diagonal thus,

$$BRA = \begin{bmatrix} \rho_3 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_1 & 0 \\ 0 & 0 & 0 & \rho_0 \end{bmatrix} \equiv P$$

Thus

$R = B^{-1}PA^{-1}$ and $R^{-1} = AP^{-1}B$

Thus $|R| = |P| = \rho_3\ \rho_2\ \ \rho_1\ \rho_0$ and R is singular only if some ρ_n is zero. [Wrong!

If say p_2=0, then Eqs 86.5 and 86.6 don't exist. So R can be non-singular. May 22,2011.]

Now $R(0) = \rho_0$ and from Eq 86.5 , we have

$$R(1) + a_1^1 R(0) = 0$$
$$R(2) + a_1^2 R(1) + a_2^2 R(0) = 0$$

etc,

Thus $R(1)$, $R(2)$, etc, is determined by ρ_0 and A.

Likewise, $R(-1)$, $R(-2)$, etc, is determined by ρ_0 and B?(some letter may be missing by photocopying)

The filters in A and B are used in a feed back form and if $R(n)$ is extended using the highest order such filters, then we will probably want these filters to be minimum phase. When will they be minimum phase? This needs research . 6-23-07

Reverse algorithm, look at third order example

$$\begin{bmatrix} 1 \\ a_1^3 \\ a_2^3 \\ a_3^3 \end{bmatrix} = \begin{bmatrix} 1 \\ a_1^2 \\ a_2^2 \\ 0 \end{bmatrix} + a_3^3 \begin{bmatrix} 0 \\ b_2^2 \\ b_1^2 \\ 1 \end{bmatrix} \quad \text{Eq 88.1}$$

$$\begin{bmatrix} b_3^3 \\ b_2^3 \\ b_1^3 \\ 1 \end{bmatrix} = b_3^3 \begin{bmatrix} 1 \\ a_1^2 \\ a_2^2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2^2 \\ b_1^2 \\ 1 \end{bmatrix} \quad \text{Eq 88.2}$$

Meltiply 88.2 by $-a_3^3 + 88.1$

multiply 88.1 by $-b_3^3 + 88.2$

$$\begin{bmatrix} 1 \\ a_1^3 \\ a_2^3 \\ a_3^3 \end{bmatrix} - a_3^3 \begin{bmatrix} b_3^3 \\ b_2^3 \\ b_1^3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ a_1^2 \\ a_2^2 \\ 0 \end{bmatrix} (1 - a_3^3 b_3^3)$$

$$\begin{bmatrix} b_3^3 \\ b_2^3 \\ b_1^3 \\ 1 \end{bmatrix} - b_3^3 \begin{bmatrix} 1 \\ a_1^3 \\ a_2^3 \\ a_3^3 \end{bmatrix} = \begin{bmatrix} 0 \\ b_2^2 \\ b_1^2 \\ 1 \end{bmatrix} (1 - a_3^3 b_3^3)$$

So $a_n^{m-1} = (a_n^m - a_m^m b_{m-n}^m) / \rho_3$ $\qquad$ $b_n^{m-1} = (b_n^m - b_m^m a_{m-n}^m) / \rho_3$

$\rho_3 = 1 - a_3^3 b_3^3$

Suppose $R(n)$, n>3, is extended by using $A(Z)=1+a_1^3 Z + a_2^3 Z^2 + a_3^3 Z^3$ in a feedback loop and that this third order filter is minimum phase, Then we see that

$$R(Z)A(Z) = P_3 \sum_{N=0}^{\infty} q_n Z^{-n} \quad \text{with} \quad q_0 = 1$$

Then $R(Z)A(Z)B(Z^{-1}) = P_3 \sum_{n=0}^{\infty} s_n Z^{-n}$ with $s_0 = 1$.

If $R(n)$, n<-3, were extended by using $B(Z)=1+b_1^3 Z + b_2^3 Z^2 + b_3^3 Z^3$ in a feedback loop and $B(Z)$ were minimum phase, then

$$R(Z)B(Z^{-1}) = P_3 \sum_{n=0}^{\infty} u_n Z^n \quad , u_0 = 1$$

And $R(Z)B(Z^{-1})A(Z) = P_3 \sum_{n=0}^{\infty} v_n Z^n$, $v_0 = 1$.

If $R(n)$, $|n| > 3$ is extended in both directions as discussed above ,then

$$R(Z)A(Z)B(Z^{-1}) = P_3$$

and thus

$$R(Z) = \frac{P_3}{A(Z)B(Z^{-1})}$$

For $R(Z)$ to exist on the unit circle and be a valid Z-transtorm, $A(Z)$ and $B(Z)$ would have to be ?????(some words are missing by photocopying) Otherwise, the extension blows up.

Let $R^+(Z) = \frac{1}{2} R(0) + R(1)Z + R(2)Z^2 + ...$

And $R^-(Z^{-1}) = \frac{1}{2} R(0) + R(-1)Z^{-1} + R(-2)Z^{-2} + ...$

Then $R(Z) = R^+(Z) + R^-(Z^{-1})$.

Now if $R(Z) = \frac{P_n}{A(Z)B(Z^{-1})}$, then we see that since $R(Z)$ is extended by using $A(Z)$ in a feedback mode for n>N, we must have

$$R^{+}(Z)A(Z) = \frac{R(0)}{2}C(Z)$$

Where $C(Z) = \sum_{n=0}^{N} c_n Z^n, c_0 = 1$

Likewise

$$R^{-}(Z^{-1})B(Z^{-1}) = \frac{R(0)}{2}D(Z^{-1})$$

Where

$$D(Z^{-1}) = \sum_{n=0}^{N} d_n Z^{-n} \qquad \text{with} \quad d_0 = 1$$

This gives us our time separated form for $R(Z)$ of

$$R(Z) = R^{+}(Z) + R^{-}(Z^{-1}) = \frac{R(0)}{2}\left[\frac{C(Z)}{A(Z)} + \frac{D(Z^{-1})}{B(Z^{-1})}\right]$$

We now derive an algorithm for calculating $C(Z)$ and $D(Z^{-1})$. We shall go from a second order case to a third order case to illustrate the recursion.

We have

$$\begin{bmatrix} R(0) & R(-1) & R(-2) \\ R(1) & R(0) & R(-1) \\ R(2) & R(1) & R(0) \end{bmatrix}\begin{bmatrix} 1 \\ a_1^2 \\ a_2^2 \end{bmatrix} = \begin{bmatrix} \rho_2 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(Eq 91.1)} \qquad \text{and}$$

$$\begin{bmatrix} R(0) & R(-1) & R(-2) \\ R(1) & R(0) & R(-1) \\ R(2) & R(1) & R(0) \end{bmatrix}\begin{bmatrix} b_2^2 \\ b_1^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \rho_2 \end{bmatrix} \qquad \text{(Eq91.2)}$$

Also

$$\begin{bmatrix} \frac{R(0)}{2} & 0 & 0 \\ R(1) & \frac{R(0)}{2} & 0 \\ R(2) & R(1) & \frac{R(0)}{2} \end{bmatrix} \begin{bmatrix} 1 \\ a_1^2 \\ a_2^2 \end{bmatrix} = \frac{R(0)}{2} \begin{bmatrix} 1 \\ c_1^2 \\ c_2^2 \end{bmatrix} \qquad \text{(Eq 91.3)} \qquad \text{and}$$

$$\begin{bmatrix} \frac{R(0)}{2} & R(-1) & R(-2) \\ 0 & \frac{R(0)}{2} & R(-1) \\ 0 & 0 & \frac{R(0)}{2} \end{bmatrix} \begin{bmatrix} b_2^2 \\ b_1^2 \\ 1 \end{bmatrix} = \frac{R(0)}{2} \begin{bmatrix} d_2^2 \\ d_1^2 \\ 1 \end{bmatrix} \qquad \text{(Eq91.4)}$$

Subtracting Eq 91.4 from Eq 91.2 , we have

$$\begin{bmatrix} \frac{R(0)}{2} & 0 & 0 \\ R(1) & \frac{R(0)}{2} & 0 \\ R(2) & R(1) & \frac{R(0)}{2} \end{bmatrix} \begin{bmatrix} b_2^2 \\ b_1^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \rho_2 \end{bmatrix} - \begin{bmatrix} d_2^2 \\ d_1^2 \\ 1 \end{bmatrix} \frac{R(0)}{2} \qquad \text{(Eq 91.5)}$$

We now go to the third order case assuming we know a_3^3 and b_3^3. We need the right hand side of

$$\begin{bmatrix} \frac{R(0)}{2} & 0 & 0 & 0 \\ R(1) & \frac{R(0)}{2} & 0 & 0 \\ R(2) & R(1) & \frac{R(0)}{2} & 0 \\ R(3) & R(2) & R(1) & \frac{R(0)}{2} \end{bmatrix} \left[\begin{Bmatrix} 1 \\ a_1^2 \\ a_2^2 \\ 0 \end{Bmatrix} + a_3^3 \begin{Bmatrix} 0 \\ b_2^2 \\ b_1^2 \\ 1 \end{Bmatrix} \right] = \left[\begin{Bmatrix} \frac{R(0)}{2} \\ \frac{R(0)}{2} c_1^2 \\ \frac{R(0)}{2} c_2^2 \\ \Delta_3 \end{Bmatrix} + a_3^3 \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \rho_2 \end{Bmatrix} - a_3^3 \begin{Bmatrix} 0 \\ d_2^2 \\ d_1^2 \\ 1 \end{Bmatrix} \frac{R(0)}{2} \right]$$

$$= \frac{R(0)}{2} \left[\begin{Bmatrix} 1 \\ c_1^2 \\ c_2^2 \\ 0 \end{Bmatrix} - a_3^3 \begin{Bmatrix} 0 \\ d_2^2 \\ d_1^2 \\ 1 \end{Bmatrix} \right] = \frac{R(0)}{2} \begin{Bmatrix} 1 \\ c_1^3 \\ c_2^3 \\ c_3^3 \end{Bmatrix} ; a_3^3 \rho_2 + \Delta_3 = 0$$

Thus

$$\begin{bmatrix} 1 \\ c_1^3 \\ c_2^3 \\ c_3^3 \end{bmatrix} = \begin{bmatrix} 1 \\ c_1^2 \\ c_2^2 \\ 0 \end{bmatrix} - a_3^3 \begin{bmatrix} 0 \\ d_3^2 \\ d_2^2 \\ 1 \end{bmatrix}$$

and likewise

$$\begin{bmatrix} 1 \\ d_1^3 \\ d_2^3 \\ d_3^3 \end{bmatrix} = \begin{bmatrix} 1 \\ d_1^2 \\ d_2^2 \\ 0 \end{bmatrix} - b_3^3 \begin{bmatrix} 0 \\ c_2^2 \\ c_1^2 \\ 1 \end{bmatrix}$$

Thus, given a_1^1, a_2^2, a_3^3 to a_N^N and b_1^1, b_2^2 to b_N^N, we can construct $A_N(Z), B_N(Z)$ by $A_0(Z) = 1$, $B_0(Z) = 1$

$$A_{m+1}(Z) = A_m(Z) + a_{m+1}^{m+1} Z^{m+1} B_m(Z^{-1})$$
$$B_{m+1}(Z) = B_m(Z) + b_{m+1}^{m+1} Z^{m+1} A_m(Z^{-1})$$

$C_N(Z)$ and $D_N(Z)$ are likewise constructed but with the signs of the a_m^m and b_m^m reversed.