Analyzing this data, we see that BW is better than WB, but not in a dramatic way. If we compare values at the ends of loops, we see that BW is about two loops ahead of WB. Using this two loop difference, BW is behind to start with, but there is almost identical results after 5 loops for BW versus 7 loops for WB. After that BW gets closer to 3 loops ahead before they both converge.

Note that both algorithms have convergence close to linear for about 10 loops and then shift to quadratic. We know that if roots are on the unit circle, we have linear convergence. So we can interpret this change in convergence as due to the filters acting as if the zeros were actually on the unit circle, but finally getting close enough to them to see that they are not on the unit circle.

Theorem: Ratio of autocorrelations

Suppose $B(z)$ and $A(z)$ are physically realizable polynomials whose leading terms (coefficients of z^0) are both unity and that $A(z)$ is also a prediction error filter. Then the zero lag term of $B^*(z^{-1})B(z)/[A^*(z^{-1})A(z)]$ is equal to or greater than unity. It is equal to unity only if $B(z) = A(z)$.

Proof:

$$B(z)/A(z) = 1 + \alpha_1 z^1 + \alpha_2 z^2 + ...$$

and thus the zero lag term of

$$\frac{B^*(z^{-1})B(z)}{A^*(z^{-1})A(z)} = 1 + \sum_{n=1}^{\infty} \alpha_1^* \alpha_1 \geqslant 1$$

Equality can happen if and only if $\alpha_n = 0$ for $n = 1$ to ∞, in which case $B(z) = A(z)$.

Q.E.D.

Burg-Wilson Algorithm continued

The basic equation of the algorithm is

$$\frac{SA^*(z^{-1})A(z)}{S_{n+1}A_n^*(z^{-1})A_n(z)} + 1 = \frac{A_{n+1}^*(z^{-1})}{A_n^*(z^{-1})} + \frac{A_{n+1}(z)}{A_n(z)} \tag{1}$$

where S_{n+1} is determined by making the zero lag value of the left most expression equal to unity. Thus

$$\frac{S_{n+1}}{S} = \text{zero lag of } \frac{A^*(z^{-1})A(z)}{A_n^*(z^{-1})A_n(z)} \geqslant 1 \tag{2}$$

according to the theorem in ratio of autocorrelations, Thus, $S_n \geqslant S$ and equal only if $A_n(z) = A(z)$. Once S_n is found ,then $A_{n+1}(z)$ is determined uniquely.

An earlier equation is

$$\frac{S}{S_n}A^*(z^{-1})A(z) + A_n^*(z^{-1})A_n(z) = A_{n+1}^*(z^{-1})A_n(z) + A_{n+1}(z)A_n^*(z^{-1}) \tag{3}$$

Dividing by $A_{n+1}^*(z^{-1})A_{n+1}(z)$, we have

$$\frac{SA^*(z^{-1})A(z)}{S_nA_{n+1}^*(z^{-1})A_{n+1}(z)} + \frac{A_n^*(z^{-1})A_n(z)}{A_{n+1}^*(z^{-1})A_{n+1}(z)} = \frac{A_n^*(z^{-1})}{A_{n+1}^*(z^{-1})} + \frac{A_n(z)}{A_{n+1}(z)} \tag{4}$$

Looking at the zero lag terms, and noting that zero lag of

$$\frac{SA^*(z^{-1})A(z)}{A_{n+1}^*(z^{-1})A_{n+1}(z)} \equiv S_{n+1} \tag{5}$$

and

$$\frac{A_n^*(z^{-1})A_n(z)}{A_{n+1}^*(z^{-1})A_{n+1}(z)} = 1 + \varepsilon, \qquad \text{with } \varepsilon \geqslant 0, \tag{6}$$

we have

$$\frac{S_{n+1}}{S_n} + 1 + \varepsilon = 1 + 1$$

or

$$\frac{S_{n+1}}{S_n} = 1 - \varepsilon \leqslant 1$$

Note from Eq (1), if $A_n(z) \neq A(z)$, then $A_{n+1}(z) \neq A_n(z)$. But then Eq (4) shows that $A_{n+1}(z) \neq A(z)$. Thus the iteration never stops.

If $\varepsilon = 0$, then $A_n(z) = A_{n+1}(z)$, which leads to $A(z) = A_n(z)$ and $S_n = S$, our solution. This is seen since the Z transforms of Eq (1) are just constants. If $\varepsilon \geqslant 1$, then $S_{n+1} \leqslant 0$, which could happen only if our original "autocorrelation" is not an autocorrelation. Thus we have shown that $S < S_{n+1} < S_n$ as long as the algorithm continues. Thus S_n is monotonically decreasing and bounded below by S. So S_n goes to some limit $\geqslant S$. But since $S > 0$, then S_{n+1}/S_n has unity for its limit and $\varepsilon \to 0$, and thus $A_n \to A_{n+1}(z) \to A(z)$ and $S_n \to S$. So the algorithm converges if we actually start with an autocorrelation. This is true even if $A(z)$ has zeros on the unit circle. We now study the rate of convergence.

We start with the identity which is valid if $A_n(z) \neq 0$.

$$\begin{aligned} &\frac{[A_n^*(z^{-1}) - A^*(z^{-1})][A_n(z) - A(z)]}{A_n^*(z^{-1})A_n(z)} \\ &= [1 - \frac{A^*(z^{-1})}{A_n^*(z^{-1})}][1 - \frac{A(z)}{A_n(z)}] \\ &= 1 - \frac{A^*(z^{-1})}{A_n^*(z^{-1})} - \frac{A(z)}{A_n(z)} + \frac{A^*(z^{-1})}{A_n^*(z^{-1})}\frac{A(z)}{A_n(z)} \end{aligned} \tag{7}$$

Adding this to Eq (1), we have

$$\frac{[A_n^*(z^{-1}) - A^*(z^{-1})][A_n(z) - A(z)]}{A_n^*(z^{-1})A_n(z)} = \frac{A_{n+1}^*(z^{-1}) - A^*(z^{-1})}{A_n^*(z^{-1})} + \frac{A_{n+1}(z) - A(z)}{A_n(z)} + \frac{A^*(z^{-1})A(z)}{A_n^*(z^{-1})A_n(z)}(1 - \frac{S}{S_{n+1}}) \tag{8}$$

Let

$$\varepsilon_n(z) = A_n(z) - A(z)$$

be the nth error polynomial. Note that it has zero for its z^0 coefficient.

$$\varepsilon_{n+1}(z) = A_{n+1}(z) - A(z)$$

also has no constant term, so

$$\frac{A_{n+1}(z) - A(z)}{A_n(z)}$$

also starts with z.

Looking at the zero lag terms, we have zero lag of

$$\frac{\varepsilon_n^*(z^{-1})\varepsilon_n(z)}{A_n^*(z^{-1})A_n(z)} = 0 + 0 + \frac{S_{n+1}}{S}(1 - \frac{S}{S_{n+1}}) = \frac{S_{n+1} - S}{S}$$

Thus the error in $S_{n+1}(S_{n+1} - S)$ is a homogeneous quadratic function of $\varepsilon_n(z)$ and thus has quadratic convergence. But then

$$\frac{A_{n+1}^*(z^{-1}) - A^*(z^{-1})}{A_n^*(z^{-1})} + \frac{A_{n+1}(z) - A(z)}{A_n(z)} = \frac{\varepsilon_{n+1}^*(z^{-1})}{A_{n+1}^*(z^{-1})} + \frac{\varepsilon_{n+1}(z)}{A_n(z)}$$

is also a second order homogeneous function of $\varepsilon_n(z)$ and thus has quadratic convergence.

Thus BW has second order convergence like the WB algorithm.

Professor Wilson was the first to make me aware of the Symmetrized Euclidean algorithm. See Demeure and Mullis. Here is my description of this excellent algorithm.

Let $D_n(z)$ be a complex conjugate symmetric z transform running from $D_N^{N*}z^{-N} + D_N^{N-1*}z^{-N+1} + ... + D_N^{N-1}z^{N-1} + D_N^N z^N$.

Let $A_M(z)$ be a prediction error filter of degree M. This means that the z^0 coefficient of this Mth degree polynomial is unity and that all of its reflection coefficients have a magnitude less than unity, making $A_M(z)$ to be minimum phase.

We wish to find a polynomial $B(z)$ such that $D_N(z) = A_M(z)B^*(z^{-1}) + A_M^*(z^{-1})B(z)$. This allows us to write

$$\frac{D_N(z)}{A_M^*(z^{-1})A_M(z)} = \frac{B^*(z^{-1})}{A_M^*(z^{-1})} + \frac{B(z)}{A_M(z)}$$

Now the left hand term is an infinity long complex conjugate symmetric z transform that converges on the unit circle. Sharing in half, the real coefficient of z^0, this term can be split into positive time and negative time pieces. The positive half is equal to $B(z)/A_M(z)$. Since the positive half is unique, and multiplying through by $A_M(z)$ is okay, we see that $B(z)$ is unique.

What can we say about the degree of the polynomial $B(z)$? Let $0 \leqslant s < \infty$be the degree of $B_S(z)$. Then

$$\begin{aligned} D_N(z) &= A_M(z)B_S^*(z^{-1}) + A_M^*(z^{-1})B_S(z) \\ &= \sum_{n=-S}^{M} X_n z^n + \sum_{n=-M}^{S} X_n^* z^n \end{aligned}$$

Now if $N > M$ then $S = N$ and

if $N < M$ then $S = M$ and

if $N = M$ then $0 \leqslant S \leqslant N = M$

In describing the algorithm, we assume $N = M$. This can be made true by augmenting the shorter term with zeros. Let

$$\begin{aligned} A_n(z) &= A_{n-1}(z) + C_n z^n A_{n-1}^*(z^{-1}), \text{ and} \\ B_n(z) &= [E_n(z) - C_n z^n E_n^*(z^{-1})]/(1 - C_n C_n^*) \end{aligned}$$

where $E_n(z) = B_{n-1}(z) + D_n^n z^n$.

Then

$$\begin{aligned} &D_n(z) = A_n(z)B_n^*(z^{-1}) + A_n^*(z^{-1})B_n(z) = \\ &\left\{ \left[A_{n-1}(z) + C_n z^n A_{n-1}^*(z^{-1})\right] \left[E_n^*(z^{-1}) - C_n^* z^{-n} E_n(z)\right] + \right. \\ &\left. \left[A_{n-1}^*(z^{-1}) + C_n^* z^{-n} A_{n-1}(z)\right] \left[E_n(z) - C_n z^n E_n^*(z^{-1})\right] \right\} / (1 - C_n C^*) \\ &= A_{n-1}(z)E_n^*(z^{-1}) + A_{n-1}^*(z^{-1})E_n(z) \\ &= A_{n-1}(z)B_{n-1}^*(z^{-1}) + A_{n-1}^*(z^{-1})B_n(z) \\ &+ A_{n-1}(z)D_n^* z^{-1} + A_{n-1}^*(z^{-1})D_n^n z^n. \end{aligned}$$

Thus, defining $D_{n-1}(z)$ as

$$\begin{aligned} &D_{n-1}(z) = D_n(z) - z^{-n}A_{n-1}(z)D_n^{n*} - z^n A_{n-1}^*(z^{-1})D_n^n \\ &= A_{n-1}(z)B_{n-1}^*(z^{-1}) + A_{n-1}^*(z^{-1})B_{n-1}(z). \end{aligned}$$

This last equation has the reduced order of $n-1$. From this, we see how the algorithm is designed.

Using the equations

$$A_{n-1}(z) = (A_n(z) - C_n z^n A_n^*(z^{-1}))/(1 - C_n C_n^*)$$

and

$$D_{n-1}(z) = D_n(z) - z^{-n} A_{n-1}(z) D_n^{n*} - z^n A_{n-1}^*(z^{-1}) D_n^n$$

One obtains the numbers $D_n^n, D_{n-1}^{n-1}, ...D_0^0$ Note that D_0^0 is the center term of $D_N(z)$, that is , the center term is not changed.

Then, starting with $B_0(z) = D_0^0/2$, and using also $[B_{n-1}(z) - C_n z^n B_{n-1}^*(z^{-1}) - C_n D_n^{n*} + z^n D_n^n]/(1 - C_N C_N^*)$

$$B_n(z) = [E_n(z) - C_n z^n E_n^*(z^{-1})]/(1 - C_n C_N^*)$$

where $E_n(z) = B_{n-1}(z) + D_n^n z^n$. For $n = 1$ to N, we finally find the $B_N(z)$ that goes with $A_N(z)$.

Actually, for the complex case, $B(z)$ is not unique since [see also above]

$$\frac{B^*(z^{-1})}{A^*(z^{-1})} + \frac{B(z)}{A(z)} = \frac{B^*(z^{-1})}{A^*(z^{-1})} - \alpha i + \frac{B(z)}{A(z)} + \alpha i$$
$$= \frac{B^*(z^{-1}) - \alpha i A^*(z^{-1})}{A^*(z^{-1})} + \frac{B(z) - \alpha i A(z)}{A^*(z)}$$

Even if we start with $B_0(z) = a$ real $D_0^0/2$, the leading terms of $B_n(z)$ will not stay real. However, we can make $B_n(z)$ unique by adding an $\alpha i A_n(z)$ to it to remove the imaginary part of the z^0 coefficient, that is , insisting that the z^0 coefficient be real, we can wait until the algorithm gives a final $B_N(z)$ to make the leading coefficient real.

This algorithm is about 40% faster than the other two algorithms.

????? Is the non-sysmmetric case section needed?

THE NON-SYMMETRIC CASE

Suppose $A(z)$ and $B(z)$ are minimum phase polynomials and $D(z)$ is a finite Z-transform $d^{-M}Z^{-M} + d^{-M+1}Z^{-M+1} + ... + d^{N-1}Z^{N-1} + d^{N}Z^{N}$ where the superscript on the ds are labels and not power terms like they are on the ZS .Let S be the order of $A(z)$ and T be the order of $B(z)$, and let K be the maximum value of M, N, S and T . Let $A(z)$ and $B(z)$ be extended with zeros until they are of order K and $D(z)$ be extended on both ends with zeros until it starts with Z^{-K} and ends with Z^{K}. These extensions simplify the following description of the algorithm to being order in both directions, we shall derive the algorithm for the complex case.

We want to find polynomials $E(z)$and $F(z)$ such that $D(Z) = A(Z) * E^{*}(Z^{-1}) + B(Z^{-1}) * F(Z)$. Knowing $E(z)$and $F(z)$, we can write

$$\frac{D(Z)}{A(Z)B^{*}(Z^{-1})} = \frac{F(Z)}{A(Z)} + \frac{E^{*}(Z^{-1})}{B^{*}(Z^{-1})}$$

Which means that we have separated $D(Z)/\left[A(Z) * B^{*}(Z^{-1})\right]$ into time separated terms.

Let $A_n(z)$ and $B_n(z)$ be nth order polynomials with leading coefficients of unity. Starting with $A_n(z) = 1$ and $B_n(z) = 1$, we can use the recursions

$$A_n(z) = A_{n-1}(z) + a_n^n z^n B_{n-1}^{*}(z^{-1})$$

and

$$B_n(z) = B_{n-1}(z) + b_n^n z^n A_{n-1}^{*}(z^{-1})$$

where we have been given the complex numbers a_m^m and b_m^m for $m = 1$ to n. Again, the superscripts are for labeling and are not power terms. Note that the leading terms of the $A_n(z)$ and $B_n(z)$ are unity and their last terms are a_n^n and b_n^n respectively.

Starting with $A_n(z)$ and $B_n(z)$, the inverse recursion is

$$A_{n-1}(z) = [A_n(z) - a_n^n z^n B_n^{*}(z^{-1})]/(1 - a_n^n b_n^{n*})$$

and

$$B_{n-1}(z) = [B_n(z) - b_n^n z^n A_n^{*}(z^{-1})]/(1 - a_n^{n*} b_n^n)$$

Our initial polynomials are $A_k(z)$ and $B_k(z)$ since we have made them kth order even if they were shorter to begin with. Using the inverse algorithm, we find the $A_n(z)$ and $B_n(z)$ for $n = k$ to 1, which also gives us the a_n^n and b_n^n for $n = k$ to 1. We next consider the pair of equations

$$E_n(z) = [G_n(z) - b_n^n z^n H_n^{*}(z^{-1})]/(1 - a_n^{n*} b_n^n)$$
$$F_n(z) = [H_n(z) - a_n^n z^n G_n^{*}(z^{-1})]/(1 - a_n^n b_n^{n*})$$

These equations are similar to an earlier pair, but this is not a recursion since all polynomials are of nth order. However, there are inverse equations, also,

$$G_n(z) = E_n(z) + b_n^n z^n F_n^*(z^{-1})$$
$$H_n(z) = F_n(z) + a_n^n z^n E_n^*(z^{-1})$$

Now we have

$$D_n(z) = A_n(z)E_n^*(z^{-1}) + B_n^*(z^{-1})F_n(z)$$
$$= [A_{n-1}(z) + a_n^n z^n B_{n-1}^*(z^{-1})][G_n(z^{-1}) - b_n^{n*} z^{-n} H_n(z)]$$
$$+[B_{n-1}(z^{-1}) + b_n^{n*} z^{-n} A_{n-1}(z)][H_n(z) - a_n^n z^n G_n^*(z^{-1})]$$

divided by $(1 - a_n^n b_n^{n*})$

$$= A_{n-1}(z)G_n^*(z^{-1}) + B_{n-1}^*(z^{-1})H_n(z)$$
$$= A_{n-1}(z)[E_{n-1}^*(z^{-1}) + z^{-n} D_n^{-n}]$$
$$+B_{n-1}^*(z^{-1})[F_{n-1}(z) + z^n D_n^n]$$

where we define

$$E_{n-1}(z) = G_n(z) + z^n D_n^{-n*}$$

and

$$F_{n-1}(z) = H_n(z) + z^n D_n^n$$

Defining $D_{n-1}(z)$ as

$$D_{n-1}(z) = D_n(z) - z^{-n} A_{n-1}(z) D_n^{-n} - z^n B_{n-1}^*(z^{-1}) D_n^n$$

we have the reduction from n to $n-1$ of

$$D_{n-1}(z) = A_{n-1}(z)E_{n-1}^*(z^{-1}) + B_{n-1}^*(z^{-1})F_{n-1}(z)$$

Note that $E_{n-1}(z)$, $F_{n-1}(z)$ and $D_{n-1}(z)$ are indeed $n-1$th order.
The reduction ends with $D_0 = E_0^* + F_0$. It appears that any choice for E_0^* and F_0 satisfying this equation will do. Choose $F_0 = D_0/2$ and $E_0 = D_0^*/2$.
Taking this start for E_0 and F_0 and the D_n^ns, a_n^ns and b_n^ns, we use

$$G_n(z) = E_{n-1}(z) + z^n D_n^{-n*}$$
$$H_n(z) = F_{n-1}(z) + z^n D_n^n$$

and then

$$E_n(z) = [G_n(z) - b_n^n z^n H_n^*(z^{-1})]/(1 - a_n^{n*} b_n^n)$$
$$F_n(z) = [H_n(z) - a_n^n z^n G_n^*(z^{-1})]/(1 - a_n^n b_n^{n*})$$

until $n = k$.

The logical basis of the algorithm is as follow.
We are given two polynomials, $A(z)$ and $B(z)$ with their coefficients of z^0 being unity. This can be achieved by multiplying by z^{-n} and normalization without changing the basis problem. Then as long as $a_n^n b_n^{n*} \neq 1$, the polynomials can be reduced to constants. Then, for a given $D(z)$, the algorithm can reduce $D(z)$ down to the constant D_0.
The basic assumption is that there are polynomials $E(z)$ and $F(z)$ such that

$$D(z) = A(z) * E^*(z^{-1}) + B^*(z^{-1}) * F(z)$$

Interruption, please!
We start with $D_k(z)$ and use

$$D_{n-1}(z) = D_n(z) - z^{-n} A_{n-1}(z) D_n^{-n} - z^n B_{n-1}^*(z^{-1}) D_n^n$$

to get reduced Z-transforms $D_{k-1}(z), ..., D_0$. The z^0 coefficients of the $D_n(z)$ stays constant, but the other coefficients change. However, in doing the reductions, we need only to save the end values, D_n^{-n} and D_n^n, to find the $E_n(z)$ and $F_n(z)$ polynomials.

Continuing, we note that the a_n, b_n, D_n^{-n} and D_n^n are calculated without knowledge of the $E(2\,or\,z?)$ or $F(z)$. And as long as no $a_n^n b_n^{n*} = 1$, these saved values are calculated and unique. Then, starting with $E_0 = D_0^*/(2\,or\,z?)$ and $F_0 = D_0/(2\,or\,z?)$, we calculate the $G_n(z)$ and $H_n(z)$, and then the $E_n(z)$ and $F_n(z)$ sequentially until we have the $E_k(z)$ and $F_k(z)$ companion polynomials corresponding to $D(z)$.

DOING THE BURG-WILSON ALGORITHM WITH THE SYMMETRIC EUCLIDEAN ALGORITHM

$R(z)$ is the finite complex conjugate autocorrelation and $A_n(z)$ is the nth guess at the corresponding prediction error filter.

Solve for $B(z)$ from

$$\frac{R(z)}{A_n^*(z^{-1})A_n(z)} = \frac{B^*(z^{-1})}{A_n^*(z^{-1})} + \frac{B(z)}{A_n(z)} \tag{9}$$

The coefficient of z^0 is $b_0^* + b_0 = x$. b_0 may be complex. Let $i\alpha$ be the imaginary part of b_0.

Then $B(z) - i\alpha A_n(z)$ has a real leading term $= x/2$ and Eq (9) equals

$$\frac{R(z)}{A_n^*(z^{-1})A_n(z)} = \frac{B^*(z^{-1}) + i\alpha A_n^*(z^{-1})}{A_n^*(z^{-1})} + \frac{B(z) - i\alpha A_n(z)}{A_n(z)}$$

Dividing through by x and adding one to both sides,

$$\frac{R(z)}{xA_n^*(z^{-1})A_n(z)} + 1 = \frac{\frac{B^*(z)+i\alpha A_n^*(z)}{x} + \frac{A_n^*(z)}{2}}{A_n^*(z)} + \frac{\frac{B(z)-i\alpha A_n(z)}{x} + \frac{A_n(z)}{2}}{A_n(z)}.$$

Thus

$$A_{n+1}(z) = \frac{B(z) - i\alpha A_n(z)}{x} + \frac{A_n(z)}{2}$$

and $S_{n+1} = X$.

??????? Is this non-sysmmetric section needed

NON-SYMMETRIC BURG-WILSON USING ALGORITHM ON PAGES 138-144 (???wrong page number in print version)

The polynomial $D(z)$ has been analyzed and the number of zeros inside and outside the unit circle determined. It is assumed that there are no zeros on the unit circle.
$A(z)$ and $B(z)$ are prediction error filters with $A(z)$ having the zeros outside of the unite circle and $B^*(z^{-1})$ having the ones inside, we have

$$D(z) = sA(z)B^*(z^{-1}) \tag{10}$$

where s is a complex scaler. Given $D(z)$, we wish to solve for s, $A(z)$ and $B(z)$. Let S_N, $A_N(z)$ and $B_N(z)$ be our Nth guess of the solution in the iteration. Then

$$D_N(z) = s_N A_N(z)B^*(z^{-1})$$

and

$$\delta D_N(z) = \delta s_N A_N(z)B^*(z^{-1}) + s_N \delta A_N(z)B^*(z^{-1}) + s_N A_N(z)\delta B^*(z^{-1})$$

We want $\delta D_N(z)$ to satisfy the equation

$$D(z) = D_N(z) + \delta D_N(z)$$

putting equations together and dividing by $D_N(z)$,

$$\frac{D(z)}{s_N A_N(z)B_N^*(z^{-1})} = 1 + \frac{\delta S_N}{S_N} + \frac{\delta A_N(z)}{A_N(z)} + \frac{\delta B_N^*(z^{-1})}{B_N^*(z^{-1})} \tag{11}$$

We now solve for an $E(z)$ and $F(z)$ such that

$$\frac{D(z)}{A_N(z)B_N^*(z^{-1})} = \frac{F(z)}{A_N(z)} + \frac{E^*(z^{-1})}{B_N^*(z^{-1})}$$

We also take our new guesses for $A(z)$ and $B(z)$ as

$$A_{N+1}(z) = A_N(z) + \delta A_N(z), B_{N+1}^*(z^{-1}) = B_N^*(z^{-1}) + \delta B_N^*(z^{-1})$$

Equation (11) then becomes

$$\frac{1}{s_N}[\frac{F(z)}{A_N(z)} + \frac{E^*(z^{-1})}{B_N^*(z^{-1})}] + 1 = \frac{\delta s_N}{s_N} + \frac{A_{N+1}(z)}{A_N(z)} + \frac{B_{N+1}^*(z^{-1})}{B_N^*(z^{-1})} \tag{12}$$

Looking at the coefficient of z^0 in this equation, we have $\frac{1}{s_N}[f_0 + e_0^*] + 1 = \frac{\delta s_N}{s_N} + 2$, where f_0 and e_0 are the z^0 coefficients of $F(z)$ and $E(z)$.

Now, if it so happened that our s_N guess was $s_N = f_0 + e_0^*$, then $\delta s_N = 0$ solves the z^0 equation. We should note that $F(z)$ and $E(z)$ are not unique, but $f_0 + e_0^*$ is. With this value for s_N, we can write Equation (12) as

$$\frac{1}{s_N}[\frac{F(z)}{A_N(z)} + \alpha + \frac{E^*(z^{-1})}{B_N^*(z^{-1})} - \alpha] + 1 = \frac{A_{N+1}(z)}{A_N(z)} + \frac{B_{N+1}^*(z^{-1})}{B_N^*(z^{-1})}$$

where α is a complex scalar so that their z^0 coefficients

$$\frac{1}{s_N}[\frac{F(z)}{A_N(z)} + \alpha] + \frac{1}{2} = \frac{A_{N+1}(z)}{A_N(z)}$$

are equal.

$$\frac{1}{f_0 + e_0^*}[f_0 + \alpha] = \frac{1}{2}$$
$$\alpha = \frac{f_0 + e_0^*}{2} - f_0 = \frac{e_0^* - f_0}{2}$$

Then

$$A_{N+1}(z) = \frac{1}{f_0 + e_0^*}[F(z) + \alpha A_N(z)] + A_N(z)/2$$
$$= \frac{1}{f_0 + e_0^*}[F(z) + \frac{e_0^* - f_0}{2} A_N(z) + \frac{e_0^* + f_0}{2} A_N(z)]$$
$$= \frac{F(z) + e_0^* A_N(z)}{f_0 + e_0^*}$$

Likewise

$$B_{N+1}(z) = \frac{E(z) + f_0 B_N(z)}{f_0 + e_0^*}$$